On some criterion of conformality

A. Kaźmierczak, A. Pierzchalski

Abstract. In the paper it is shown that in order to prove the conformality of a given diffeomorphism between Riemannian manifolds it is enough to limit investigation to the conformal modules of some special families of curves. The main result (Theorem 1.1) asserts that a sufficient condition for the conformality of a diffeomorphism is the conservation of n-modules of a family of mutually orthogonal 1-dimensional foliations.

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1 Introduction

The module is a geometric quantity whose origins lie in physics (can be treated as a generalization of the capacity of a condenser). It is the inverse of the extremal length, that was introduced by L.Ahlfors and H.Berling [2] at the beginning of 50s. During the following years, it was subjected to several generalizations. Briefly, owing to the notion of the module of a family of k-dimensional surfaces in $\mathbb{R}^n$ [5], it was possible to define the p-module of a family of submanifolds of a Riemannian manifold (see: [9] and Definition 2.2 in this paper). In particular, one can consider the p-module of a k-dimensional foliation.

The module of a family of curves or, more generally, hypersurfaces, is a conformal invariant that played the fundamental role in the study of the geometric properties of diffeomorphisms. In particular, it was used by several authors for characterisation of conformal and quasiconformal mappings ([1], [11]). The conformality criterions (see: [2, 6, 8, 11]), either local or global, demanded that modules of all such families be conserved. However, one can notice that in order to ensure the conformality of a mapping, it is enough to control its distortion in a sufficient number of directions. Thus, a natural candidate for a tool for verification of conformality are the modules of mutually orthogonal foliations. We proved that the sufficient condition for the conformality of a diffeomorphism is the conservation of n-modules of a family of mutually orthogonal 1-dimensional foliations.

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Theorem 1.1. *Let M and N be n-dimensional Riemannian manifolds, and $f : M \to N$ - a diffeomorphism. Assume that for every $x \in M$ there exists a neighborhood $U \ni x$ and mutually orthogonal 1-dimensional foliations $\mathcal{F}_1, ..., \mathcal{F}_n$ on U, such that the foliations $f(\mathcal{F}_i)$ $(i = 1, ..., n)$ on $f(U)$ are also mutually orthogonal and f locally preserves n-modules of $\mathcal{F}_i$. Then f is conformal.*

The assumption of the existence of these foliations is not so restrictive. It is fulfilled, e.g., on every smooth, n-dimensional Riemannian manifold that locally can be isometrically embedded in $\mathbb{R}^{n+1}$ in such a way, that it consists of strongly unumbilical points (see: Corollary 3.3).

2 Preliminaries

Module of a family of submanifolds

Let us consider a n-dimensional, smooth Riemannian manifold (M, g) with the Lebesgue measure μ_M.

Definition 2.1. We say that a family $\mathcal{M}$ of submanifolds of M is **of measure zero** if $\mu_M(\underset{L\in\mathcal{M}}{\cup} L) = 0$. We write that a given property holds **for almost every** element of a family $\mathcal{M}$ if the set of elements for which it does not hold has measure zero.

Set $k \in N$, $0 < k < n$.

Definition 2.2. (Compare also [5]).Denote by $\mathcal{M}$ a family of smooth, k-dimensional submanifolds of M. We call the function f **p-admissible** ($p \geq 1$) for $\mathcal{M}$ with respect to M (writing: $f \in \mathrm{adm}_p(\mathcal{M}, M)$) if

1. $f \in L^p(M)$
2. $f \geq 0$ almost everywhere on M
3. $\int_L f d\mu_L \geq 1$ for almost every element $L \in \mathcal{M}$.

The **p-module** of $\mathcal{M}$ is the number:

$$\mathrm{mod}_p(\mathcal{M}, M) = \inf_{f\in\mathrm{adm}_p(\mathcal{M},M)} ||f||_{L^p(M)},$$

(setting: $\mathrm{mod}_p(\mathcal{M}, M) = \infty$ if $\mathrm{adm}_p(\mathcal{M}, M) = \varnothing$).

Definition 2.3. A p-admissible function f_0 is called *p-extremal* if

$$||f_0||_{L^p(M)} = \mathrm{mod}_p(\mathcal{M}, M).$$

It is a direct consequence of the above definition that if N is an open submanifold of M, such that $\bigcup \mathcal{M} \subset N$, then

$$\mathrm{mod}_p(\mathcal{N}, M) = \mathrm{mod}_p(\mathcal{N}, N).$$

So we will write $\mathrm{mod}_p(\mathcal{M})$ instead of $\mathrm{mod}_p(\mathcal{M}, M)$ and $\mathrm{adm}_p(\mathcal{M})$ instead of $\mathrm{adm}_p(\mathcal{M}, M)$.

Module of a foliation

Definition 2.4. A k-dimensional foliation is a decomposition of M into a family $\mathcal{F}$ of disjoint, connected submanifolds of dimension k with the property that for every point $x \in M$ there exists a neighborhood D of x and a chart $\varphi = (\varphi^1, \varphi^2, \ldots, \varphi^n) : D \to \mathbb{R}^n$, such that $\varphi(D)$ is an open cube and for every $L \in \mathcal{F}$, satisfying: $L \cap D \neq 0$,

$$\begin{array}{lcl} \varphi^1_{|L} = const, & & \varphi^{j+k+1}_{|L} = const, \\ \vdots & and & \vdots \\ \varphi^j_{|L} = const, & & \varphi^n_{|L} = const, \end{array}$$

for a given $j \in \{0, ..., n-k\}$. The elements of $\mathcal{F}$ are called *leaves* and φ is named a *foliated chart* for $\mathcal{F}$.

Definition 2.5. Two families of submanifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ are called *transversal*, if for arbitrary two elements $L_1 \in \mathcal{M}_1$, $L_2 \in \mathcal{M}_2$ and for every point $x \in L_1 \cap L_2$, $T_x L_1 \cap T_x L_2 = 0$.

Definition 2.6. Let $\mathcal{F}_1, ..., \mathcal{F}_k$ be mutually transversal foliations of M, of dimensions $n_1, .., n_k$ such that $n_1 + ... + n_k = n$. By a *n-(foliated) chart* we will understand a chart which is a foliated chart of every foliation $\mathcal{F}_i$.

The following fact is a straightforward generalization of the Theorem 5.1.4 from [4]:

Theorem 2.1. *Let $\mathcal{F}_1, ..., \mathcal{F}_n$ be mutually transversal, 1-dimensional foliations of M. Then for every $x \in M$ there exists an n-chart (defined) in the neighborhood of this point.*

We will use the following

Definition 2.7. Let M, N be smooth manifolds and $\phi : M \to N$ - such a submersion, that for every $y \in N$ the preimage $\phi^{-1}(y)$ is connected. We will call a foliation whose leaves are: $L_y = \phi^{-1}(y)$, $y \in N$ *a foliation given by the submersion.*

Definition 2.8. If M and N are Riemannian, *the Jacobian J_ϕ of the submersion ϕ* is a function that assigns to every $x \in M$ the Jacobian of the isomorphism: $\phi_*(x)|_{\ker(\phi_*(x))^\perp}$.

The next theorem, being a more exact version of that one in [9], specifies the formula for the module of a foliation given by the submersion.

Theorem 2.2. *If a foliation $\mathcal{F}$ of M is given by the submersion ϕ and for almost every $y \in \phi(M)$, $\int_{L_y} J_\phi^{\frac{1}{p-1}}(x) d\mu_{L_y} < \infty$, then for $p > 1$*

$$\operatorname{mod}_p^p(\mathcal{F}) = \int_{\phi(M)} \Big(\int_{L_y} J_\phi^{\frac{1}{p-1}}(x) d\mu_{L_y} \Big)^{1-p} d\mu_{\phi(M)}. \tag{2.1}$$

If $\operatorname{mod}_p(\mathcal{F}) < \infty$ then there exists an extremal function $f_0 : M \to \mathbb{R}$,

$$f_0(x) = \frac{J_\phi(x)^{\frac{1}{p-1}}}{\int_{L_y} J_\phi^{\frac{1}{p-1}}(x) d\mu_{L_y}}, \tag{2.2}$$

where $y \in \phi(M)$ is the only point such that $x \in \phi^{-1}(y)$.

Proof. Take an arbitrary $f \in \mathrm{adm}_p(\mathcal{F})$. According to the Fubini's theorem (see for example [10]),

$$\int_M |f(x)|^p d\mu_M = \int_{\phi(M)} (\int_{L_y} |f(x)|^p \frac{1}{J_\phi(x)} d\mu_{L_y}) d\mu_{\phi(M)}. \tag{2.3}$$

Since $f \in \mathrm{adm}_p(\mathcal{F})$,

$$(\int_{L_y} |f(x)| d\mu_{L_y})^p \geq 1$$

for a.e. $y \in \phi(M)$. Applying Hölder's inequality to the left side of the latter inequality, we get:

$$\int_{L_y} |f(x)|^p \frac{1}{J_\phi(x)} d\mu_{L_y} \cdot (\int_{L_y} J_\phi^{\frac{1}{p-1}}(x) d\mu_{L_y})^{p-1} \geq 1,$$

so, from (2.3),

$$\mathrm{mod}_p^p(\mathcal{F}) \geq \int_{\phi(M)} (\int_{L_y} J_\phi^{\frac{1}{p-1}}(x) d\mu_{L_y})^{1-p} d\mu_{\phi(M)}. \tag{2.4}$$

On the other hand, one can notice that if the right-hand side of (2.3) is finite, then

$$f_0(x) = \frac{J_\phi^{\frac{1}{p-1}}(x)}{\int_{L_y} J_\phi^{\frac{1}{p-1}}(x) d\mu_{L_y}}$$

is admissible for $\mathcal{F}$. Indeed, the positivity of the Jacobian J_ϕ (ϕ is a submersion) and the assumption that $\int_{L_y} J_\phi^{\frac{1}{p-1}}(x) d\mu_{L_y} < \infty$ for a.e. $y \in \phi(M)$, imply that $f_0 > 0$ almost everywhere and

$$\int_{L_y} f_0 d\mu_{L_y} = \int_{L_y} \frac{J_\phi^{\frac{1}{p-1}}(x)}{\int_{L_y} J_\phi^{\frac{1}{p-1}}(x) d\mu_{L_y}} = 1$$

for a.e. $y \in \phi(M)$. Moreover,

$$\| f_0 \|_{L_p(M)}^p = \int_M |f_0(x)|^p d\mu_m = \int_{\phi(M)} (\int_{L_y} |f_0(x)|^p \frac{1}{J_\phi(x)} d\mu_{L_y}) d\mu_{\phi(M)} =$$
$$\int_{\phi(M)} (\int_{L_y} \frac{J_\phi^{\frac{1}{p-1}}(x)}{(\int_{L_y} J_\phi^{\frac{1}{p-1}}(x) d\mu_{L_y})^p} d\mu_{L_y}) d\mu_{\phi(M)} = \int_{\phi(M)} (\int_{L_y} J_\phi^{\frac{1}{p-1}}(x) d\mu_{L_y})^{1-p} d\mu_{\phi(M)} < \infty.$$

Therefore $f_0 \in \mathrm{adm}_p(\mathcal{F})$ and

$$\mathrm{mod}_p^p(\mathcal{F}) \leq \int_{\phi(M)} (\int_{L_y} J_\phi^{\frac{1}{p-1}}(x) d\mu_{L_y})^{1-p} d\mu_{\phi(M)}. \tag{2.5}$$

The above inequality together with (2.4) gives the thesis. □

3 Module and conformality

Module as a conformal invariant

Definition 3.1. Let (M, g) and (N, h) be Riemannian manifolds. A diffeomorphism $f : M \to N$ is called *conformal* if there exists a function $\lambda : M \to \mathbb{R}$, such that $f^*h = \lambda^2 g$ (where $(f^*h)(x)(X, Y) = h(f(x))(f_*(X), f_*(Y))$ for arbitrary $x \in M$ and $X, Y \in T_xM$).

The following fact specifies a well known necessary condition for conformality (see: [6],[7], etc.)

Theorem 3.1. *Let $\mathcal{M}$ be a family of k-dimensional submanifolds of M. Assume that $f : M \to N$ is a conformal diffeomorphism. Then for $p = \frac{n}{k}$,*

$$\mathrm{mod}_p(\mathcal{M}) = \mathrm{mod}_p(f(\mathcal{M})).$$

On the other hand, our Thorem 1.1 specifies a sufficient condition for conformality.

Proof of Theorem 1.1

Before presenting the proof, we need an auxiliary

Definition 3.2. Let M and N be n-dimensional Riemannian manifolds, and $f : M \to N$ - a diffeomorphism. Denote by $\mathcal{F}_1, ..., \mathcal{F}_n$ mutually transversal, one dimensional foliations of M. We write that f *locally preserves p-modules* of these foliations, if for every point $x \in M$ and every neighborhood $U \ni x$, there exists $D \subset U$, $x \in D$, such that D is the domain of some n-chart and for every i, $\mathrm{mod}_p(\mathcal{F}_i|_D) = \mathrm{mod}_p(f(\mathcal{F}_i|_D))$.

Proof of Theorem 1.1. Suppose that there exists a point $x \in M$, at which f is not conformal. Choose an orthonormal basis $e_1(x), ..., e_n(x)$ of T_xM, such that $e_i(x) \in T_xL_i$. Denote $y = f(x)$ and let $e_1^{'}(y), ..., e_n^{'}(y)$ be an orthonormal basis of T_yN, with the property that $e_i^{'}(y) \in T_yf(L_i)$ and such that the deformation coefficient $\lambda_i(x)$ of f $(f_*(e_i(x)) = \lambda_i(x)e_i^{'}(y))$ in the direction T_xL_i is positive.

Since it was supposed that f is not conformal at x, we can find an index $j \in \{1, ..., n\}$, such that $\lambda_j(x) < J_f(x)^{\frac{1}{n}}$ (where J_f denotes the Jacobian of f). Furthermore, it remains true also in some neighborhood $W \subset U$ of x.

Now let $D \subset W$ be such a neighborhood of x, that there exists an n-chart $\phi : D \to \mathbb{R}^n$. Thus, for every $j \in \{1, ..., n\}$, $\mathcal{F}_j|_D$ is given by a submersion related with ϕ (that is the projection of ϕ onto the coordinates $i = 1, .., j-1, j+1, .., n$), whereas $f(\mathcal{F}_j|_D)$ is given by an analogical submersion related with $\psi|_{f(D)}$. On the basis of Theorem 2.2, all the considered foliations have extremal functions on D. Let v_j and $v_j^{'}$ be extremal functions of $\mathcal{F}_j|_D$ and $f(\mathcal{F}_j|_D)$, respectively. Denote by L_j a leaf of $\mathcal{F}_j|_D$, and by $L_j^{'}$ - a leaf of $f(\mathcal{F}_j|_D)$. From the assumption: $\mathrm{mod}_n(\mathcal{F}_j|_D) = \mathrm{mod}_n(f(\mathcal{F}_j|_D))$ and by the change of variables, we get:

$$\int_D (v_j)^n d\mu_M = \int_{f(D)} (v_j^{'})^n d\mu_N = \int_D (v_j^{'})^n \circ f J_f d\mu_M.$$

Owing to this sequence equalities, we see that the function $(v_j^{'} \circ f) \cdot J_f^{\frac{1}{n}}$ realizes the n-module of $\mathcal{F}_j|_D$, so it would be an extremal function, if only it was n-admissible. Since

$$1 \le \int_{L_j^{'}} v_j^{'} d\mu_{L_j^{'}} = \int_{L_j} v_j^{'} \circ f \lambda_j d\mu_{L_j} < \int_{L_j} v_j^{'} \circ f J_f^{\frac{1}{n}} d\mu_{L_j}, \tag{3.1}$$

so $(v_j^{'} \circ f) \cdot J_f^{\frac{1}{n}}$ is n-admissible. On the other hand, we know that that the integral from the extremal function (see: [3]) over almost every leaf is equal to 1. Thus, from (3.1), the function $(v_j^{'} \circ f) \cdot J_f^{\frac{1}{n}}$ cannot be extremal. Contradiction. □

Conclusions

We will see that the assumption of the existence of the foliations specified in Theorem 1.1 is not very restrictive.

Definition 3.3. Let M be a smooth, n-dimensional submanifold embedded in R^{n+1}. Fix a smooth field N of unit normal vectors, defined in some neighborhood of $p \in M$. Recall that the *shape operator* at p is the operator $S_p : T_pM \to T_pM$ defined on $X \in T_pM$ as $S_p(X) = -(D_X N)_p$ We say that the point p is *strongly unumbilical* if the shape operator S_p has n different eignevalues.

Proposition 3.2. *Let M be a smooth n-dimensional Riemannian manifold isometrically embedded in $\mathbb{R}^{n+1}$, such that all its points are strongly unumbilical. Then there exist n mutually orthogonal 1-dimensional foliations on M.*

Proof. Take any $p \in M$. Since p is strongly unumbilical, the shape operator in p has n different eigenvalues, each of the multiplicity 1. Moreover, the eigenvalues can be ordered globally. Thus they generate a system of smooth 1-dimensional distributions that are defined globally and determine the desired system of foliations. □

Corollary 3.3. *If M is a smooth n-dimensional Riemannian manifold that locally can be isometrically embedded in $\mathbb{R}^{n+1}$ in such a way that this imbedding consists only of strongly unumbilical points, then around every point of M there exist n mutually orthogonal foliations.*

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Authors' address:

Anna Kaźmierczak *and* Antoni Pierzchalski
University of Lodz, Banacha 22, 90-238 Lodz, Poland.
E-mail: akaz@math.uni.lodz.pl & antoni@math.uni.lodz.pl