Warped product slant lightlike submanifolds of indefinite Sasakian manifolds

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Abstract. We obtain some characterization theorems for the nonexistence of warped product slant lightlike submanifolds of indefinite Sasakian manifolds.

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1 Introduction

Duggal and Bejancu [4] introduced the geometry of lightlike hypersurfaces and submanifolds of semi-Riemannian manifolds and further studied by many authors. The theory of lightlike geometry have significant applications in general relativity, particularly in black hole theory. On the other hand significant uses of the contact geometry in differential equations, optics, and phase spaces of a dynamical system [1, 8, 9] and very limited specific information available on its lightlike case motivated the authors to do work on the geometry of lightlike submanifolds of indefinite Sasakian manifolds.

Let $\bar{M}$ be a Sasakian manifold with almost contact structure (ϕ, η, V) and M be a Riemannian manifold isometrically immersed in $\bar{M}$ such that the characteristic vector field V is tangent to M. Then M is called an invariant submanifold if $\phi(T_pM) = T_pM$, for every $p \in M$, where T_pM denotes the tangent space to M at the point p. M is called an anti-invariant submanifold if $\phi(T_pM) \subset (T_pM)^{\perp}$ for every $p \in M$, where $(T_pM)^{\perp}$ denotes the normal space to M at the point p. As a generalization of invariant and totally real (anti-invariant) submanifolds of almost contact metric manifolds, slant submanifolds of Sasakian manifolds were introduced by Lotta [7] and further studied by Cabrerizo et al. [3]. Recently Sahin and Yildirim [13] introduced slant lightlike submanifolds of indefinite Sasakian manifolds and we in [11] studied the non existence of totally umbilical proper slant lightlike submanifold of indefinite Sasakian manifolds. In this paper, we derive some characterization theorems on minimal hemi-slant lightlike submanifolds. Finally, we obtain some characterization theorems for the nonexistence of warped product slant lightlike submanifolds of indefinite Sasakian manifolds.

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2 Lightlike submanifolds

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g on M, $TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $RadT_xM = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping $RadTM : x \in M \longrightarrow RadT_xM$, defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $RadTM$ is called the radical distribution on M, (for detail see [4]).

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is,

$$TM = RadTM \perp S(TM), \tag{2.1}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $RadTM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $RadTM$ in $S(TM^{\perp})^{\perp}$ respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \tag{2.2}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{2.3}$$

For quasi-orthonormal fields of frames, we have

Theorem 2.1. *([4]). Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $RadTM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)\mid_u)$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_u$, where u is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \quad \textit{for any} \quad i, j \in \{1, 2, .., r\}, \tag{2.4}$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then, according to the decomposition (2.3), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{2.5}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called the second fundamental form, A_U is linear a operator on M, known as the shape operator. According to (2.2), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively then (2.5) becomes

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.6}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{2.7}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{2.8}$$

where we put $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$, $\nabla^l_X N = L(\nabla^\perp_X N)$, $\nabla^s_X W = S(\nabla^\perp_X W)$, $D^s(X,N) = S(\nabla^\perp_X N)$ and $D^l(X,W) = L(\nabla^\perp_X W)$, for any $X \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$.

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^\perp))$-valued respectively, therefore they are called as the lightlike second fundamental form and the screen second fundamental form on M. Using (2.2)-(2.3) and (2.6)-(2.8), we obtain

$$\bar{g}(h^s(X,Y), W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.9}$$

for any $X, Y \in \Gamma(TM)$, $W \in \Gamma(S(TM^\perp))$ and $N \in \Gamma(ltr(TM))$.

Let $\bar{P}$ is a projection of TM on $S(TM)$. Now, we consider the decomposition (2.1), we can write

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \quad \nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where $\{\nabla^*_X \bar{P}Y,\ A^*_\xi X\}$ and $\{h^*(X, \bar{P}Y), \nabla^{*t}_X \xi\}$ belongs to $\Gamma(S(TM))$ and $\Gamma(RadTM)$, respectively. Here ∇^* and ∇^{*t}_X are linear connections on $S(TM)$ and $RadTM$, respectively. By using (2.7), (2.8) and (2.10), we obtain

$$\bar{g}(h^l(X, \bar{P}Y), \xi) = g(A^*_\xi X, \bar{P}Y), \quad \bar{g}(h^*(X, \bar{P}Y), N) = \bar{g}(A_N X, \bar{P}Y).$$

Definition 2.1. An odd-dimensional semi-Riemannian manifold $\bar{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $\{\phi, V, \eta, \bar{g}\}$, where ϕ is a $(1,1)$ tensor field, V a vector field, called characteristic vector field, η a 1-form and $\bar{g}$ is the semi-Riemannian metric on $\bar{M}$ satisfying (see [2])

$$\phi^2 X = -X + \eta(X)V, \quad \eta \circ \phi = 0, \quad \phi V = 0, \quad \eta(V) = 1. \tag{2.11}$$

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X,Y) - \eta(X)\eta(Y), \quad \bar{g}(X, V) = \eta(X). \tag{2.12}$$

for $X, Y \in \Gamma(T\bar{M})$, where $T\bar{M}$ denotes the Lie algebra of vector fields on $\bar{M}$.

An indefinite almost contact metric manifold $\bar{M}$ is called an indefinite Sasakian manifold if (see [6]),

$$(\bar{\nabla}_X \phi)Y = -\bar{g}(X,Y)V + \eta(Y)X, \ \textit{and}\ \bar{\nabla}_X V = \phi X, \tag{2.13}$$

for any $X, Y \in \Gamma(T\bar{M})$, where $\bar{\nabla}$ denote the Levi-Civita connection on $\bar{M}$.

3 Slant lightlike submanifolds

A lightlike submanifold has two distributions, namely the radical distribution and the screen distribution. The radical distribution is totally lightlike and it is not possible to define angle between two vector fields of the radical distribution where the screen distribution is non-degenerate. There are some definitions for angle between two vector fields in Lorentzian setup [10], but not appropriate for our goal. Therefore to introduce the notion of slant lightlike submanifolds one needs a Riemannian distribution. For such distribution Sahin and Yildirim [13] proved the following lemmas.

Lemma 3.1. *Let M be an r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$. Suppose that $\phi RadTM$ is a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$. Then $\phi ltr(TM)$ is a subbundle of the screen distribution $S(TM)$ and $\phi ltr(TM) \cap \phi RadTM = \{0\}$.*

Lemma 3.2. *Let M be an r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2r$. Suppose that $\phi RadTM$ is a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$. Then any complementary distribution to $\phi ltr(TM) \oplus \phi(RadTM)$ in screen distribution S(TM) is Riemannian.*

Definition 3.1. ([13]) Let M be an r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2r$. Then we say that M is a slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(A) $RadTM$ is a distribution on M such that $\phi RadTM \cap RadTM = \{0\}$.

(B) For each non zero vector field X tangent to $\bar{D} = D \perp \{V\}$ at $x \in U \subset M$, if X and V are linearly independent, then the angle $\theta(X)$ between ϕX and the vector space $\bar{D}_x$ is constant, that is, it is independent of the choice of $x \in U \subset M$ and $X \in \bar{D}_x$, where $\bar{D}$ is complementary distribution to $\phi ltr(TM) \oplus \phi RadTM$ in screen distribution $S(TM)$.

The constant angle $\theta(X)$ is called the slant angle of the distribution $\bar{D}$. A slant lightlike submanifold M is said to be proper if $\bar{D} \neq \{0\}$, and $\theta \neq 0, \frac{\pi}{2}$.

Then the tangent bundle TM of M is decomposed as

$$TM = RadTM \perp S(TM) = RadTM \perp (\phi RadTM \oplus \phi ltr(TM)) \perp \bar{D}, \tag{3.1}$$

where $\bar{D} = D \perp \{V\}$. Therefore for any $X \in \Gamma(TM)$, we write

$$\phi X = TX + FX, \tag{3.2}$$

where TX is the tangential component of ϕX and FX is the transversal component of ϕX. Similarly for any $U \in \Gamma(tr(TM))$, we write

$$\phi U = BU + CU, \tag{3.3}$$

where BU is the tangential component of ϕU and CU is the transversal component of ϕU. Using the decomposition in (3.1), we denote by P_1, P_2, Q_1, Q_2 and $\bar{Q}_2$ be the projections on the distributions $RadTM$, $\phi RadTM$, $\phi ltr(TM)$, D and $\bar{D} = D \perp V$, respectively. Then for any $X \in \Gamma(TM)$, we can write

$$X = P_1X + P_2X + Q_1X + \bar{Q}_2X, \tag{3.4}$$

where $\bar{Q}_2X = Q_2X + \eta(X)V$. Applying ϕ to (3.4), we obtain

$$\phi X = \phi P_1X + \phi P_2X + FQ_1X + TQ_2X + FQ_2X. \tag{3.5}$$

Then using (3.2) and (3.3), we get

$$\phi P_1X = TP_1X \in \Gamma(\phi RadTM), \quad \phi P_2X = TP_2X \in \Gamma(RadTM),$$

$$FP_1X = FP_2X = 0, \quad TQ_2X \in \Gamma(D), \quad FQ_1X \in \Gamma(ltr(TM)).$$

Lemma 3.3. *Let M be a slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ then $FQ_2X \in \Gamma(S(TM^{\perp}))$, for any $X \in \Gamma(TM)$.*

Proof. Using (2.2) and (2.4), it is clear that $FQ_2X \in \Gamma(S(TM^{\perp}))$, if and only if, $\bar{g}(FQ_2X, \xi) = 0$, for any $\xi \in \Gamma(RadTM)$. Using (2.11), (2.12) and (3.5), we have $\bar{g}(FQ_2X, \xi) = \bar{g}(\phi X - \phi P_1X - \phi P_2X - FQ_1X - TQ_2X, \xi) == -\bar{g}(P_1X + P_2X, \phi\xi) = 0$, hence the result follows. □

Thus from the Lemma (3.3) it follows that $F(D_p)$ is a subspace of $S(TM^{\perp})$. Therefore there exists an invariant subspace μ_p of $T_p\bar{M}$ such that

$$S(T_pM^{\perp}) = F(D_p) \perp \mu_p,$$

therefore

$$T_p\bar{M} = S(T_pM) \perp \{Rad(T_pM) \oplus ltr(T_pM)\} \perp \{F(D_p) \perp \mu_p\}.$$

In [13], Sahin and Yildirim proved the following theorem.

Theorem 3.4. *Let M be a q lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then M is a slant lightlike submanifold, if and only if,*

(i) $\phi(RadTM)$ is a distribution on M such that $\phi RadTM \cap Rad(TM) = \{0\}$.

(ii) For any tangent vector field X tangent to $\bar{D}$, there exists a constant $\lambda \in [-1, 0]$ such that

$$T^2X = -\lambda(X - \eta(X)V), \tag{3.6}$$

where $\bar{D}$ is a complementary distribution such to $\phi ltr(TM) \oplus \phi RadTM$ in TM and $\lambda = -cos^2\theta$.

Lemma 3.5. *Let M be a slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then we have*

$$g(T\bar{Q}_2X, T\bar{Q}_2Y) = cos^2\theta[g(\bar{Q}_2X, \bar{Q}_2Y) - \eta(\bar{Q}_2X)\eta(\bar{Q}_2Y)] \tag{3.7}$$

and

$$g(F\bar{Q}_2X, F\bar{Q}_2Y) = sin^2\theta[g(\bar{Q}_2X, \bar{Q}_2Y) - \eta(\bar{Q}_2X)\eta(\bar{Q}_2Y)] \tag{3.8}$$

for any $X, Y \in \Gamma(TM)$.

Proof. From (2.1) and (3.2), we obtain

$$g(T\bar{Q}_2X, T\bar{Q}_2Y) = -g(\bar{Q}_2X, T^2\bar{Q}_2Y), \quad \forall \quad X, Y \in \Gamma(TM).$$

Then from Theorem (3.4), we obtain (3.7) and (3.8). This completes the proof. □

Similar to the proof of the Theorem (3.4), we have the following theorem.

Theorem 3.6. *Let M be a q lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then M is a slant lightlike submanifold, if and only if,* *(i)* $\phi(RadTM)$ *is a distribution on* M *such that* $\phi RadTM \cap Rad(TM) = \{0\}$.

(ii) *For any tangent vector field* X *tangent to* $\bar{D}$, *there exists a constant* $\mu \in [-1, 0]$ *such that*

$$BFX = \mu(X - \eta(X)V), \tag{3.9}$$

where $\bar{D}$ *is a complementary distribution such to* $\phi ltr(TM) \oplus \phi RadTM$ *in* TM *and* $\mu = -sin^2\theta$.

Definition 3.2. ([5]) Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold, tangent to characteristic vector field V, of an indefinite Sasakian manifold $(\bar{M}, \bar{g})$. Then M is said to be a contact Screen Cauchy Riemann (SCR) lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(i) There exist real non-null distributions $D \subset S(TM)$ and $D^{\perp}$ such that

$$S(TM) = D \oplus D^{\perp} \perp \{V\}, \quad \phi D^{\perp} \subset (S(TM^{\perp})), \quad D \cap D^{\perp} = \{0\},$$

where $D^{\perp}$ is orthogonal complementary to $D \perp \{V\}$ in $S(TM)$.

(ii) The distributions D and $Rad(TM)$ are invariant with respect to ϕ.

Theorem 3.7. *A contact* SCR*-lightlike submanifold* M, *of an indefinite Sasakian manifold* $\bar{M}$, *is a holomorphic or complex (resp. screen real) lightlike submanifold, if and only if,* $D^{\perp} = \{0\}$ *(resp.* $D = \{0\}$*).*

Definition 3.3. ([12]) Let M be a lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is said to be a transversal lightlike submanifold if the following conditions are satisfied:

(i) $Rad(TM)$ is transversal with respect to $\bar{J}$, that is, $\bar{J}(Rad(TM)) = ltr(TM)$.

(ii) $S(TM)$ is transversal with respect to $\bar{J}$, that is, $\bar{J}(S(TM)) \subseteq S(TM^{\perp})$.

4 Non-existence of warped product slant lightlike submanifolds

Let B and F be two Riemannian manifolds with Riemannian metrics g_B and g_F respectively and $f > 0$ a differentiable function on B. Assume the product manifold $B \times F$ with its projection $\pi : B \times F \to B$ and $\eta : B \times F \to F$. The warped product $M = B \times_f F$ is the manifold $B \times F$ equipped with the Riemannian metric g where

$$g = g_B + f^2 g_F.$$

If X is tangent to $M = B \times_f F$ at (p, q) then we have

$$\|X\|^2 = \|\pi_* X\|^2 + f^2(\pi(X))\|\eta_* X\|^2.$$

The function f is called the warping function of the warped product. For differentiable function f on M, the gradient ∇f is defined by $g(\nabla f, X) = Xf$, for all $X \in T(M)$.

Lemma 4.1. *([10]) Let $M = B \times_f F$ be a warped product manifold. If $X, Y \in T(B)$ and $U, Z \in T(F)$ then*

$$\nabla_X Y \in T(B).$$

$$\nabla_X U = \nabla_U X = \frac{Xf}{f} U = X(ln\ f)U. \tag{4.1}$$

$$\nabla_U Z = -\frac{g(U, Z)}{f} \nabla f.$$

Corollary 4.2. *On a warped product manifold $M = B \times_f F$ we have*

- *(i) B is totally geodesic in M.*
- *(ii) F is totally umbilical in M.*

Theorem 4.3. *Let $\bar{M}$ be an indefinite Sasakian manifold. Then there does not exist warped product submanifold $M = M_\theta \times_f M_T$ of $\bar{M}$ such that M_θ is a proper slant lightlike submanifold of $\bar{M}$ and M_T is a holomorphic Screen Cauchy-Riemann (SCR) lightlike submanifold of $\bar{M}$.*

Proof. Let X, linearly independent of V, be tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ. Then using (4.1), we have

$$g(\nabla_{\phi X} Z, X) = Z(ln\ f)g(\phi X, X) = 0.$$

Therefore using (2.6), (2.11) to (2.13) and (3.2), we get

$$\begin{aligned} 0 &= \bar{g}(\bar{\nabla}_{\phi X} Z, X) = -\bar{g}(\phi Z, \bar{\nabla}_{\phi X} \phi X) \\ &= \bar{g}(\bar{\nabla}_{\phi X} TZ, \phi X) - \bar{g}(FZ, \bar{\nabla}_{\phi X} \phi X) \\ &= \bar{g}(\nabla_{\phi X} TZ, \phi X) - \bar{g}(FZ, h^s(\phi X, \phi X)). \end{aligned}$$

Further by virtue of (4.1), we obtain

$$TZ(ln\ f)g(X, X) = \bar{g}(h^s(\phi X, \phi X), FZ).$$

Thus, using polarization identity we get

$$TZ(ln\ f)g(X, Y) = \bar{g}(h^s(\phi X, \phi Y), FZ), \tag{4.2}$$

for any X, Y, linearly independent of V, tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ. On the other hand, using (2.8) and (4.1), we have

$$\begin{aligned} g(A_{FZ} \phi X, \phi Y) &= -g(\bar{\nabla}_{\phi X} FZ, \phi Y) = \bar{g}(Z, \bar{\nabla}_{\phi X} Y) - \bar{g}(TZ, \bar{\nabla}_{\phi X} \phi Y) \\ &= -\bar{g}(\bar{\nabla}_{\phi X} Z, Y) + \bar{g}(\bar{\nabla}_{\phi X} TZ, \phi Y) \\ &= -Z(ln\ f)g(\phi X, Y) + TZ(ln\ f)g(X, Y). \end{aligned}$$

Now using (2.9), we have $\bar{g}(h^s(\phi X, \phi Y), FZ) = g(A_{FZ} \phi X, \phi Y)$, therefore we obtain

$$\bar{g}(h^s(\phi X, \phi Y), FZ) = -Z(ln\ f)g(\phi X, Y) + TZ(ln\ f)g(X, Y). \tag{4.3}$$

Thus (4.2) and (4.3) imply that $Z(ln\ f)g(\phi X, Y) = 0$ for any X, Y, linearly independent of V, tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ. Since $M_T \neq \{0\}$ is a Riemannian and invariant therefore we obtain

$$Zln\ f = 0,$$

this shows that f is constant. Hence the proof is complete. □

Theorem 4.4. *Let $\bar{M}$ be an indefinite Sasakian manifold. Then there does not exist warped product submanifold $M = M_T \times_f M_\theta$ in $\bar{M}$ such that M_T is a holomorphic SCR-lightlike submanifold and M_θ is a proper slant lightlike submanifold of $\bar{M}$.*

Proof. Let X, linearly independent of V, be tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ. Then using (4.1) we have $g(\nabla_{TZ}X, Z) = X(ln\ f)g(TZ, Z) = 0$. This further using with (2.8), (2.9) and (3.7) implies that

$$\begin{aligned} 0 &= \bar{g}(\bar{\nabla}_{TZ}X, Z) = -\bar{g}(\phi X, \bar{\nabla}_{TZ}TZ) - \bar{g}(\phi X, \bar{\nabla}_{TZ}FZ) \\ &= g(\nabla_{TZ}\phi X, TZ) + g(\phi X, A_{FZ}TZ) \\ &= g(\nabla_{TZ}\phi X, TZ) + g(h^s(\phi X, TZ), FZ) \\ &= \phi X(ln\ f)g(TZ, TZ) + \bar{g}(h^s(\phi X, TZ), FZ) \\ &= \phi X(ln\ f).cos^2\theta g(Z, Z) + \bar{g}(h^s(\phi X, TZ), FZ). \end{aligned}$$

Replace X by ϕX, we get

$$X(ln\ f).cos^2\theta g(Z, Z) + \bar{g}(h^s(X, TZ), FZ) = 0. \tag{4.4}$$

After replacing Z by TZ and then using (3.6), (3.7), we obtain

$$\bar{g}(h^s(X, Z), FTZ) = X(ln\ f).cos^2\theta g(Z, Z).$$

Next, on the other hand using (2.6), (3.2), (3.6), (3.7) and (4.1), for any X, linearly independent of V, tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Y, Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ, we have

$$\begin{aligned} \bar{g}(h^s(TZ, X), FY) &= -\bar{g}(TZ, \bar{\nabla}_X\phi Y) + \bar{g}(TZ, \bar{\nabla}_X TY) \\ &= \bar{g}(T^2Z, \bar{\nabla}_X Y) + \bar{g}(FTZ, \bar{\nabla}_X Y) + \bar{g}(TZ, \nabla_X TY) \\ &= -cos^2\theta X(ln\ f)g(Z, Y) + \bar{g}(FTZ, h^s(X, Y)) \\ &\quad + X(ln\ f)g(TZ, TY) \\ &= \bar{g}(FTZ, h^s(X, Y)). \end{aligned}$$

Put $Y = Z$, we get

$$\bar{g}(h^s(TZ, X), FZ) = \bar{g}(FTZ, h^s(X, Z)). \tag{4.5}$$

Thus from (4.4) to (4.5), we have

$$X(ln\ f)cos^2\theta g(Z, Z) = 0.$$

Since D^θ is a proper slant and Z is non-null, we obtain $X(ln\ f) = 0$. This proves our assertion. □

Theorem 4.5. *Let $\bar{M}$ be an indefinite Sasakian manifold. Then there does not exist warped product submanifold $M = M_\perp \times_f M_\theta$ of $\bar{M}$ such that $M_\perp$ is a transversal lightlike submanifold and M_θ is a proper slant lightlike submanifold of $\bar{M}$.*

Proof. Let $Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ and X independent of V and tangent to $S(TM)$ of a transversal lightlike submanifold $M_\perp$ then using (2.8), (2.11) to (2.13), (3.2), (3.7), and (4.1), we have

$$\begin{aligned} g(A_{\phi X}TZ, Z) &= \bar{g}(\bar{\nabla}_{TZ}X, \phi Z) = g(\nabla_{TZ}X, TZ) + \bar{g}(h^s(TZ, X), FZ) \\ &= X(ln\ f)g(TZ, TZ) + \bar{g}(h^s(TZ, X), FZ) \\ &= X(ln\ f)cos^2\theta g(Z, Z) + \bar{g}(h^s(TZ, X), FZ). \end{aligned}$$

Using (2.9) in the left hand side of above equation, we obtain

$$\bar{g}(h^s(TZ, Z), \phi X) = X(ln\ f)cos^2\theta g(Z, Z) + \bar{g}(h^s(TZ, X), FZ). \tag{4.6}$$

Replace Z by TZ in (4.6) and then using (3.6) and (3.7), we get

$$\bar{g}(h^s(Z, TZ), \phi X) = -X(ln\ f)cos^2\theta g(Z, Z) + \bar{g}(h^s(Z, X), FTZ).$$

On the other hand, using (2.8), (2.11) to (2.13), (3.2), (3.6) and (4.1), we have

$$\begin{aligned} g(A_{FZ}X, TZ) &= -\bar{g}(\bar{\nabla}_X FZ, TZ) = \bar{g}(\bar{\nabla}_X Z, \phi TZ) + \bar{g}(\bar{\nabla}_X TZ, TZ) \\ &= \bar{g}(\bar{\nabla}_X Z, T^2 Z) + \bar{g}(\bar{\nabla}_X Z, FTZ) + g(\nabla_X TZ, TZ) \\ &= -cos^2\theta g(\nabla_X Z, Z) + \bar{g}(h^s(X, Z), FTZ) + X(ln\ f)g(TZ, TZ) \\ &= -cos^2\theta X(ln\ f)g(Z, Z) + \bar{g}(h^s(X, Z), FTZ) \\ &\quad + X(ln\ f)cos^2\theta g(Z, Z) \\ &= \bar{g}(h^s(X, Z), FTZ). \end{aligned}$$

Hence using (2.9), we obtain

$$\bar{g}(h^s(TZ, X), FZ) = \bar{g}(h^s(X, Z), FTZ). \tag{4.7}$$

Thus using (4.6) to (4.7), we get

$$2X(ln\ f)cos^2\theta g(Z, Z) = 0.$$

Since M_θ is proper slant lightlike submanifold and D^θ is Riemannian therefore we obtain $X(ln\ f) = 0$. Hence f is constant, which proves our assertion. □

Thus using the Theorem (4.3), Theorem (4.4) and Theorem (4.5), now onwards, we call $M = M_\theta \times_f M_\perp$ as a warped product slant lightlike submanifold, where M_θ is a proper slant lightlike submanifold and $M_\perp$ is a transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$.

Theorem 4.6. *Let $M = M_\theta \times_f M_\perp$ be a warped product slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ such that $M_\perp$ is a transversal lightlike submanifold and M_θ is a proper slant lightlike submanifold of $\bar{M}$. Then*

$$g(h^s(X, Y), JZ) = -TX(ln\ f)g(Y, Z),$$

for any $X \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ and Y, Z, independent of V and tangent to $S(TM)$ of transversal lightlike submanifold $M_\perp$.

Proof. For any $X \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ and Y, Z independent of V and tangent to $S(TM)$ of transversal lightlike submanifold $M_\perp$, using (2.8), (2.11) to (2.13) and (3.2), we have

$$g(h^s(TX,Y),\phi Z) = g(\bar{\nabla}_Y TX,\phi Z) = g(\nabla_Y X, Z) + g(\bar{\nabla}_Y \phi FX, Z).$$

Since $F(D^\theta) \subset S(TM^\perp)$ and μ is invariant therefore using (3.3), we have $\phi FX = BFX$ and $CFX = 0$. Hence using (3.9) and (4.1), we obtain

$$g(h^s(TX,Y),\phi Z) = X(ln\ f)g(Y,Z) - sin^2\theta g(\bar{\nabla}_Y X, Z).$$

Again using (4.1), we have

$$g(h^s(TX,Y),\phi Z) = (1 - sin^2\theta)X(ln\ f)g(Y,Z) = cos^2\theta X(ln\ f)g(Y,Z).$$

Replacing X by TX and then using (3.6), the assertion follows. □

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