Interpolating between Li-Yau and Chow-Hamilton Harnack inequalities along the Yamabe flow

Chang-Jun Li and Xiang Gao

Abstract. In this paper, we establish a one-parameter family of Harnack inequalities connecting the constrained trace Li-Yau differential Harnack inequality to the constrained trace Chow-Hamilton Harnack inequality for a nonlinear parabolic equation with respect to evolving metrics related to the Yamabe flow on the n-dimensional complete manifold.

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1 Introduction

Let $(M^n, g(t))$, $t \in [0, T)$, be a solution to the ε-Yamabe flow on the n-dimensional complete manifold M^n as follows:

$$\frac{\partial}{\partial t} g_{ij} = -\varepsilon R \cdot g_{ij}, \tag{1.1}$$

where ε is a nonnegative constant and R is the scalar curvature of $g(t)$. It is obvious that when $\varepsilon = 1$, the ε-Yamabe flow becomes the Yamabe flow. Recall that along the ε-Yamabe flow, we have

$$\frac{\partial R}{\partial t} = (n-1)\, \varepsilon \Delta R + \varepsilon R^2. \tag{1.2}$$

Using the maximum principle, one can see that $R \geq c$ for some $c \in \mathbb{R}$ is preserved along the ε-Yamabe flow.

In this paper, we will establish an interpolation between the constrained trace Li-Yau differential Harnack inequality for a nonlinear parabolic equation with respect to static metrics and the constrained trace Chow-Hamilton Harnack inequality for the nonlinear parabolic equation with respect to evolving metrics related to Yamabe flow.

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Recall that the research of Harnack estimates for parabolic equations originated in Moser's work [11], in which he treated the case of linear divergence-form equations. In his paper, the inequality estimates a solution from below, in terms of the values it attains on an earlier region of the parabolic domain. Inequalities of this type have recently appeared for many geometric evolution equations. These new developments began with the work of Li and Yau [9], in which they obtained a Harnack inequality for the heat equation on a Riemannian manifold. Their proof relies only on the parabolic maximum principle. From then on, their Harnack inequalities are often called Li-Yau differential Harnack inequalities. Surprisingly, similar techniques were employed by R. Hamilton, who proved Harnack inequalities for the Ricci flow [6] and [7], the mean curvature flow [5] and a matrix Harnack inequality for the heat equation [4]. Moreover, Perelman [12] proved a Harnack estimate for the fundamental solution of the conjugate heat equation under the Ricci flow without any curvature assumption.

On the other hand, differential Harnack inequalities for (backward) heat equations coupled with the Ricci flow have become an important object, which were first studied by R. Hamilton [6]. One of the excellent important work is that G. Perelman [12] derived differential Harnack inequalities for the fundamental solution to the conjugate heat equation coupled with the Ricci flow without any curvature assumption. Later X. Cao [1], and S.-L. Kuang and Qi S. Zhang [10] both extended Perelman's result to the case of all positive solutions to the conjugate heat equation under the Ricci flow on closed manifolds with nonnegative scalar curvature.

In order to make a clear statement of our Harnack inequalities, we need to recall some known results. In [4], B. Chow and R. Hamilton extended Li-Yau differential Harnack inequality [9] for the heat equation on a closed manifold, which they called a constrained trace Harnack inequality.

Theorem 1.1 (Chow-Hamilton [4]). *Let (M^n, g) be a closed manifold with nonnegative Ricci curvature. If S and T are two solutions to the heat equations*

$$\frac{\partial S}{\partial t} = \Delta S \qquad \text{and} \qquad \frac{\partial T}{\partial t} = \Delta T$$

with $|T| < S$, then

$$\frac{\partial}{\partial t} \ln S - |\nabla \ln S|^2 + \frac{n}{2t} = \Delta \ln S + \frac{n}{2t} > \frac{|\nabla h|^2}{1 - h^2},$$

where $h := T/S$.

Furthermore they generalized Hamilton's trace Harnack inequality [6] for the Ricci flow on surfaces with positive scalar curvature, and proved the following constrained linear trace Harnack inequality.

Theorem 1.2 (Chow-Hamilton [4]). *Let $\left(M^2, g(t)\right)$ be a solution to the Ricci flow*

$$\frac{\partial}{\partial t} g_{ij} = -R \cdot g_{ij},$$

on a closed surface with scalar curvature $R > 0$. If S and T are two solutions to

$$\frac{\partial S}{\partial t} = \Delta S + RS \qquad \text{and} \qquad \frac{\partial T}{\partial t} = \Delta T + RT$$

with $|T| < S$*, then*

$$\frac{\partial}{\partial t}\ln S - |\nabla \ln S|^2 + \frac{1}{t} = \Delta \ln S + R + \frac{1}{t} > \frac{|\nabla h|^2}{1-h^2},$$

where $h := T/S$*.*

Recently, J.-Y. Wu and Y. Zheng [14] generalized Theorem 1.2 and Chow's interpolated Harnack inequality [3] and proved the interpolated and constrained linear trace Harnack inequality.

Theorem 1.3 (Wu-Zheng [14]). *Let* $\big(M^2, g(t)\big)$ *be a solution to the* ε*-Ricci flow*

$$\frac{\partial}{\partial t} g_{ij} = -\varepsilon R \cdot g_{ij}, \tag{1.3}$$

on a closed surface with $R > 0$*. If* S *and* T *are solutions to the following equations*

$$\frac{\partial S}{\partial t} = \Delta S + \varepsilon RS \qquad \text{and} \qquad \frac{\partial T}{\partial t} = \Delta T + \varepsilon RT$$

with $|T| < S$*, then*

$$\frac{\partial}{\partial t}\ln S - |\nabla \ln S|^2 + \frac{1}{t} = \Delta \ln S + \varepsilon R + \frac{1}{t} > \frac{|\nabla h|^2}{1-h^2},$$

where $h := T/S$*.*

In Theorem 1.3, if let $T \equiv 0$, then the result of J.-Y. Wu and Y. Zheng recovers the Chow's interpolated Harnack inequality [3]. Very recently, J.-Y. Wu in [13] also generalized Theorem 1.3, and established an interpolated phenomenon for the nonlinear parabolic equation

$$\frac{\partial f}{\partial t} = \Delta f - f \ln f + \varepsilon R f \tag{1.4}$$

under the ε-Ricci flow.

Theorem 1.4 (Wu [13]). *Let* $\big(M^2, g(t)\big)$ *be a solution to the* ε*-Ricci flow* (1.4) *on a closed surface with the initial scalar curvature satisfying*

$$R(g(0)) \geq -\frac{2\ln c_0}{1-c_0^2} - 1 > 0,$$

where c_0 *is a free parameter satisfying* $0 < c_0 < 1$*. If* S *and* T *are solutions to the following nonlinear parabolic equations*

$$\frac{\partial S}{\partial t} = \Delta S - S\ln S + \varepsilon RS \qquad \text{and} \qquad \frac{\partial T}{\partial t} = \Delta T - T\ln T + \varepsilon RT,$$

respectively with $0 < c_0 S < T < S$ *(this condition preserved by the* ε*-Ricci flow), where* c_0 *is a free parameter satisfying* $0 < c_0 < 1$*, then*

$$\frac{\partial}{\partial t}\ln S - |\nabla \ln S|^2 + \ln S + \frac{1}{t} = \Delta \ln S + \varepsilon R + \frac{1}{t} > \frac{|\nabla h|^2}{1-h^2}, \tag{1.5}$$

where $h := T/S$*.*

In this paper, using the approach of Chow, J.-Y. Wu and Y. Zheng, we will establish the following one-parameter family of interpolated Harnack inequalities connecting the constrained trace Li-Yau differential Harnack inequality to the constrained trace Chow-Hamilton Harnack inequality for the nonlinear parabolic equation

$$\frac{\partial S}{\partial t} = \Delta S - S \ln S + \varepsilon R S, \tag{1.6}$$

$$\frac{\partial T}{\partial t} = \Delta T - T \ln T + \varepsilon R T, \tag{1.7}$$

with respect to evolving metrics related to the ε-Yamabe flow (1.1). To establish our main results, we need the following curvature condition used by Huisken in [8] and by B.-L. Chen and X.-P. Zhu in [2]. Firstly recall the following facts.

Let M^n be an n-dimensional ($n \geq 3$) smooth complete Riemannian manifold. It is well known that the curvature tensor $Rm = \{R_{ijkl}\}$ can be decomposed into the orthogonal components which have the same symmetries as

$$Rm = W + V + U,$$

where $W = \{W_{ijkl}\}$ is the Weyl conformal curvature tensor, and $V = \{V_{ijkl}\}$ and $U = \{U_{ijkl}\}$ denote the traceless Ricci part and the scalar curvature part respectively. When the following curvature condition is satisfied

$$|W|^2 + |V|^2 \leq \delta_n (1-\varepsilon)^2 |U|^2, \tag{1.8}$$

where $\varepsilon > 0, \delta_4 = \frac{1}{5}, \delta_5 = \frac{1}{10}$, and $\delta_n = \frac{2}{(n-2)(n+1)}, n \geq 6$, B.-L. Chen and X.-P. Zhu in [2] proved that

Theorem 1.5 (Chen-Zhu [2]). *Suppose M^n, where $n \geq 4$, is a smooth complete n-dimensional manifold with positive and bounded scalar curvature and satisfies the pointwise pinching condition* (1.8)*, then M^n is compact. Moreover, let M^3 be a 3-dimensional complete noncompact Riemannian manifold with bounded and nonnegative sectional curvature, suppose M^3 satisfies the following Ricci pinching condition*

$$R_{ij} \geq \varepsilon R g_{ij}$$

for some $\varepsilon > 0$, then M^3 is flat.

Theorem 1.6 (Main result I). *Let $(M^n, g(t))$, where $n \geq 4$, $t \in [0, T)$ be a solution to the ε-Yamabe flow* (1.1) *on a complete manifold M^n with the scalar curvature satisfying*

$$R(g(0)) \geq -\frac{2 \ln c_0}{1 - c_0^2} - 1 > 0 \tag{1.9}$$

and the curvature tensor $Rm\,(g\,(0))$ satisfying the pointwise pinching condition (1.8)*, where c_0 is a free parameter satisfying $0 < c_0 < 1$. If S and T are solutions to the nonlinear parabolic equations* (1.6) *and* (1.7) *respectively with $0 < c_0 S < T < S$ (this condition preserved by the ε-Yamabe flow), then*

$$\frac{\partial}{\partial t} \ln S - |\nabla \ln S|^2 + \ln S + \frac{1}{t} = \Delta \ln S + \varepsilon R + \frac{1}{t} > \frac{|\nabla h|^2}{1 - h^2}, \tag{1.10}$$

where $h := T/S$.

In the two and three-dimensional cases, we can weaken the curvature operator pinching condition (1.8) to an arbitrary Ricci curvature pinching condition as follows.

Theorem 1.7 (Main result II). *Let* $(M^n, g(t))$, *where* $2 \leq n \leq 3$, $t \in [0, T)$ *be a solution to the* ε*-Yamabe flow* (1.1) *on a complete manifold* M^n *with the scalar curvature satisfying* (1.9) *and the Ricci curvature* $Rc\,(g\,(0))$ *satisfying the pointwise pinching condition*

$$Rc\,(g\,(0)) \geq \lambda R\,(g\,(0))\,g\,(0) \tag{1.11}$$

for some $\lambda > 0$,*where* c_0 *is a free parameter satisfying* $0 < c_0 < 1$. *If* S *and* T *are solutions to the nonlinear parabolic equations* (1.6) *and* (1.7) *respectively with* $0 < c_0 S < T < S$ (*this condition preserved by the* ε*-Yamabe flow*), *then we also have the Harnack inequality*(1.10).

Remark 1.1. Recall the facts that when $n = 2$, the ε-Yamabe flow is the same as ε-Ricci flow and 2-dim manifold is an Einstein manifold satisfying (1.11) clearly, thus Theorem 1.7 is actually a generalization of Theorem 1.4 of J.-Y. Wu.

As the consequences of Theorem 1.6 and 1.7, we have classical Harnack inequalities as follows.

Theorem 1.8 (Harnack inequality I). *Let* $(M^n, g(t))$, *where* $n \geq 4$, $t \in [0, T)$ *be a solution to the* ε*-Yamabe flow* (1.1) *on a complete manifold* M^n *with the initial scalar curvature satisfying* (1.9) *and the initial curvature tensor* $Rm\,(g\,(0))$ *satisfying the pointwise pinching condition* (1.8). *Let* S *and* T *be two solutions to* (1.6) *and* (1.7) *respectively with* $0 < c_0 S < T < S$, *and assume that* (x_1, t_1) *and* (x_2, t_2), $0 < t_1 < t_2$, *are two points in* $M^n \times (0, T)$, *then we have*

$$e^{t_1} \ln S(x_1, t_1) < e^{t_2} \ln S(x_2, t_2) + \frac{1}{4} \inf_{\gamma} \int_{t_1}^{t_2} e^t \left(\left| \frac{d\gamma}{dt}(t) \right|^2 + \frac{4}{t} \right) dt, \tag{1.12}$$

where γ *is any space-time path joining* (x_1, t_1) *and* (x_2, t_2).

Theorem 1.9 (Harnack inequality II). *Let* $(M^n, g(t))$, *where* $2 \leq n \leq 3$, $t \in [0, T)$ *be a solution to the* ε*-Yamabe flow* (1.1) *on a complete manifold* M^n *with the initial scalar curvature satisfying* (1.9) *and the initial Ricci curvature satisfying the pointwise pinching condition* (1.11),. *Let* S *and* T *be two solutions to* (1.6) *and* (1.7) *respectively with* $0 < c_0 S < T < S$, *then we also have the classical Harnack inequality* (1.12).

The paper is organized as follows. In Section 2, we will prove Theorem 1.6 and 1.7 following the approach in [13], which needs a lengthy but straight-forward computation and makes use of the parabolic maximum principle. In Section 3, using Theorem 1.6 and 1.7, we will prove Theorem 1.8 and 1.9 by the standard arguments.

2 Proof of Theorem 1.6 and 1.7

Under the ε-Yamabe flow (1.1), we can compute that

$$\frac{\partial}{\partial t} \ln S = \frac{1}{S} \frac{\partial S}{\partial t} = \frac{1}{S} \left(\Delta S - S \ln S + \varepsilon R S \right)$$

and

$$\begin{aligned}\Delta \ln S + |\nabla \ln S|^2 - \ln S + \varepsilon R &= \nabla\left(\frac{\nabla S}{S}\right) + \left|\frac{\nabla S}{S}\right|^2 - \ln S + \varepsilon R \\ &= \frac{\Delta S}{S} - \frac{\nabla S \cdot \nabla S}{S^2} + \left|\frac{\nabla S}{S}\right|^2 - \ln S + \varepsilon R.\end{aligned}$$

Thus we have

$$\frac{\partial}{\partial t} \ln S = \Delta \ln S + |\nabla \ln S|^2 - \ln S + \varepsilon R, \tag{2.1}$$

$$\frac{\partial}{\partial t}(\Delta) = \varepsilon R \Delta \tag{2.2}$$

and

$$\frac{\partial}{\partial t} \ln R = (n-1)\,\varepsilon \frac{\Delta R}{R} + \varepsilon R = (n-1)\,\varepsilon\left(\Delta \ln R + |\nabla \ln R|^2\right) + \varepsilon R, \tag{2.3}$$

where the Laplacian Δ is acting on smooth functions. Now we can complete the proof of Theorem 1.6.

Proof of Theorem 1.6. Firstly, by using Theorem 1.5, the pointwise pinching condition (1.8) which the initial curvature tensor $Rm\,(g\,(0))$ satisfies implies that the manifold $(M^n, g(0))$ is compact, then $Rc(g(0)) \geq \lambda R(g(0))g(0)$, for some free parameter $\lambda \leq \frac{1}{n}$. Thus the positive Ricci pinching condition is preserved along the ε-Yamabe flow (1.1), which is well-understood by R. Ye's work in [15]. Hence for any $0 \leq t < T$, we also have $Rc(g(t)) \geq \lambda R(g(t))g(t)$ for the free parameter $\lambda \leq \frac{1}{n}$, and the rest proof follows from a direct computation and the parabolic maximum principle. Here we mainly follow the arguments of [14]. Let

$$Q := \Delta \ln S + \varepsilon R = \frac{\partial}{\partial t} \ln S - |\nabla \ln S|^2 + \ln S,$$

where S is a positive solution to the equation (1.6). Following [13], using (1.2), (2.1), (2.2), (2.3) and Bochner formula, we compute that

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \Delta\left(\frac{\partial}{\partial t}\ln S\right) + \left(\frac{\partial}{\partial t}\Delta\right)\ln S + \varepsilon\frac{\partial R}{\partial t} \\ &= \Delta\left(\Delta \ln S + |\nabla \ln S|^2 - \ln S + \varepsilon R\right) + \varepsilon R \Delta \ln S + \varepsilon \frac{\partial R}{\partial t} \\ &= \Delta Q + \Delta|\nabla \ln S|^2 + (\varepsilon R - 1)\,Q + \varepsilon R - \varepsilon^2 R^2 + \varepsilon\frac{\partial R}{\partial t} \\ &= \Delta Q + 2|\nabla\nabla \ln S|^2 + 2\nabla\Delta \ln S \cdot \nabla \ln S + Rc\,(\nabla \ln S, \nabla \ln S) \\ &\quad + (\varepsilon R - 1)\,Q + \varepsilon R - \varepsilon^2 R^2 + \varepsilon\frac{\partial R}{\partial t} \\ &\geq \Delta Q + 2|\nabla\nabla \ln S|^2 + 2\nabla Q \cdot \nabla \ln S + \lambda R|\nabla \ln S|^2 - 2\varepsilon\nabla R \cdot \nabla \ln S \\ &\quad + (\varepsilon R - 1)\,Q + \varepsilon R - \varepsilon^2 R^2 + \varepsilon\frac{\partial R}{\partial t} \\ &= \Delta Q + 2\nabla Q \cdot \nabla \ln S + 2\varepsilon R \Delta \ln S + 2|\nabla\nabla \ln S|^2 + \frac{n\varepsilon^2}{2}R^2\end{aligned}$$

$$
\begin{aligned}
&+\lambda R\left|\nabla \ln S\right|^2+\frac{\varepsilon^2}{\lambda}R\left|\nabla \ln R\right|^2-2\varepsilon\nabla R\cdot\nabla\ln S-2\varepsilon R\Delta\ln S-\frac{n\varepsilon^2}{2}R^2\\
&+\left(\varepsilon R-1\right)Q+\varepsilon R-\varepsilon^2R^2+\varepsilon R\left(\frac{\partial}{\partial t}\ln R-\frac{\varepsilon}{\lambda}\left|\nabla\ln R\right|^2\right)\\
=&\,\Delta Q+2\nabla Q\cdot\nabla\ln S-\left(\varepsilon R+1\right)Q+2\left|\nabla\nabla\ln S+\frac{\varepsilon}{2}Rg\right|^2+R\left|\sqrt{\lambda}\nabla\ln S-\varepsilon\nabla\ln R\right|^2\\
&+\varepsilon R\left(\left(n-1\right)\varepsilon\Delta\ln R-\left(\frac{1}{\lambda}-n+1\right)\varepsilon\left|\nabla\ln R\right|^2+1-\frac{\left(n-4\right)\varepsilon}{2}R\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{\partial Q}{\partial t}\geq\,&\Delta Q+2\nabla Q\cdot\nabla\ln S-\left(\varepsilon R+1\right)Q+2\left|\nabla\nabla\ln S+\frac{\varepsilon}{2}Rg\right|^2\\
&+\varepsilon R\left(\left(n-1\right)\varepsilon\Delta\ln R-\left(\frac{1}{\lambda}-n+1\right)\varepsilon\left|\nabla\ln R\right|^2+1-\frac{\left(n-4\right)\varepsilon}{2}R\right).
\end{aligned}
\tag{2.4}
$$

On the other hand, by (1.6) and (1.7), we can calculate that

$$
\frac{\partial h}{\partial t}=\Delta h+2\nabla h\cdot\nabla\ln S-h\ln h,
$$

which leads to the evolution equation of ∇h as follows

$$
\begin{aligned}
\frac{\partial}{\partial t}(\nabla h)&=\nabla\left(\frac{\partial h}{\partial t}\right)\\
&=\nabla\left(\Delta h+2\nabla h\cdot\nabla\ln S-h\ln h\right)\\
&=\Delta\nabla h+2\langle\nabla\nabla\ln S,\nabla h\rangle+2\langle\nabla\ln S,\nabla\nabla h\rangle-\frac{R\nabla h}{2}-(1+\ln h)\nabla h.
\end{aligned}
\tag{2.5}
$$

Thus under the ε-Yamabe flow, using (2.5), we have

$$
\begin{aligned}
\frac{\partial}{\partial t}|\nabla h|^2&=2\nabla h\left(\frac{\partial}{\partial t}\nabla h\right)-g^{ki}g^{lj}\frac{\partial}{\partial t}g_{kl}\nabla_i h\nabla_j h\\
&=2\nabla h\left(\Delta\nabla h+2\langle\nabla\nabla\ln S,\nabla h\rangle+2\langle\nabla\ln S,\nabla\nabla h\rangle-\frac{R\nabla h}{2}-(1+\ln h)\nabla h\right)\\
&\quad+\varepsilon R|\nabla h|^2\\
&=\Delta|\nabla h|^2-2|\nabla\nabla h|^2+4\langle\nabla\nabla\ln S,\nabla h\nabla h\rangle+2\langle\nabla\ln S,\nabla|\nabla h|^2\rangle\\
&\quad+\left((\varepsilon-1)R-2(1+\ln h)\right)|\nabla h|^2.
\end{aligned}
$$

We can also compute that

$$
\frac{\partial}{\partial t}(1-h^2)=\Delta(1-h^2)+2\langle\nabla\ln S,\nabla(1-h^2)\rangle+2|\nabla h|^2+2h^2\ln h.
$$

Then we shall compute the evolution equation of $\frac{|\nabla h|^2}{1-h^2}$. Recall the following general result that if two functions E and F satisfy the heat equations of the form

$$
\frac{\partial E}{\partial t}=\Delta E+A\qquad\text{and}\qquad\frac{\partial F}{\partial t}=\Delta F+B,
$$

where A and B are some functions, then

$$\frac{\partial}{\partial t}\left(\frac{E}{F}\right) = \Delta\left(\frac{E}{F}\right) + \frac{2}{F^2}\langle \nabla E, \nabla F\rangle - \frac{2E}{F^3}|\nabla F|^2 + \frac{A}{F} - \frac{EB}{F^2}.$$

Applying this result to

$$E := |\nabla h|^2, \quad F := 1 - h^2,$$
$$B := 2\langle \nabla \ln S, \nabla(1-h^2)\rangle + 2|\nabla h|^2 + 2h^2 \ln h$$

and

$$\begin{aligned} A :&= -2|\nabla\nabla h|^2 + 4\langle \nabla\nabla \ln S, \nabla h \nabla h\rangle + 2\langle \nabla \ln S, \nabla|\nabla h|^2\rangle \\ &+ \left((\varepsilon-1)R - 2(1+\ln h)\right)|\nabla h|^2, \end{aligned}$$

we get that

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{|\nabla h|^2}{1-h^2}\right) &= \Delta\left(\frac{|\nabla h|^2}{1-h^2}\right) + \frac{2\langle \nabla(1-h^2), \nabla|\nabla h|^2\rangle}{(1-h^2)^2} - \frac{2|\nabla h|^2}{(1-h^2)^3}\left|\nabla(1-h^2)\right|^2 \\ &+ \frac{1}{1-h^2}\cdot\left(-2|\nabla\nabla h|^2 + 4\langle \nabla\nabla \ln S, \nabla h\nabla h\rangle\right) \\ &+ \frac{2}{1-h^2}\cdot\langle \nabla \ln S, \nabla|\nabla h|^2\rangle + \frac{(\varepsilon-1)R - 2(1+\ln h)}{1-h^2}|\nabla h|^2 \\ &- \frac{2|\nabla h|^2}{(1-h^2)^2}\cdot\left(\langle \nabla \ln S, \nabla(1-h^2)\rangle + |\nabla h|^2 + h^2 \ln h\right). \end{aligned}$$

Rearranging terms yields

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{|\nabla h|^2}{1-h^2}\right) &= \Delta\left(\frac{|\nabla h|^2}{1-h^2}\right) + 2\left\langle \nabla\left(\frac{|\nabla h|^2}{1-h^2}\right), \nabla \ln S\right\rangle \\ &- \frac{2}{(1-h^2)^3}\left|2h\nabla h\nabla h + (1-h^2)\nabla\nabla h\right|^2 \\ &+ \frac{4}{1-h^2}\langle \nabla\nabla \ln S, \nabla h\nabla h\rangle - \frac{2|\nabla h|^4}{(1-h^2)^2} \\ &+ \frac{(\varepsilon-1)R - 2(1+\ln h)}{1-h^2}|\nabla h|^2 - \frac{2h^2\ln h}{(1-h^2)^2}|\nabla h|^2. \end{aligned} \tag{2.6}$$

Thus we define

$$P := Q - \frac{|\nabla h|^2}{1-h^2} = \Delta \ln S + \varepsilon R - \frac{|\nabla h|^2}{1-h^2}. \tag{2.7}$$

Combining (2.4) and (2.6), we conclude that

$$\begin{aligned} \frac{\partial P}{\partial t} &\geq \Delta P + 2\nabla P\cdot\nabla \ln S - (\varepsilon R + 1)\,Q + 2\left|\nabla\nabla \ln S + \frac{\varepsilon}{2}Rg\right|^2 \\ &+ \varepsilon R\left((n-1)\,\varepsilon\Delta \ln R - \left(\frac{1}{\lambda} - n + 1\right)\varepsilon\,|\nabla \ln R|^2 + 1 - \frac{(n-4)\,\varepsilon}{2}R\right) \\ &+ \frac{2}{(1-h^2)^3}\left|2h\nabla h\nabla h + (1-h^2)\nabla\nabla h\right|^2 - \frac{4}{1-h^2}\langle \nabla\nabla \ln S, \nabla h\nabla h\rangle \\ &+ \frac{2|\nabla h|^4}{(1-h^2)^2} + \frac{(1-\varepsilon)R + 2(1+\ln h)}{1-h^2}|\nabla h|^2 + \frac{2h^2\ln h}{(1-h^2)^2}|\nabla h|^2 \end{aligned}$$

$$
\begin{aligned}
&= \Delta P + 2\nabla P \cdot \nabla \ln S - (\varepsilon R + 1) Q + 2 \left| \nabla\nabla \ln S + \frac{\varepsilon}{2} R g - \frac{\nabla h \nabla h}{1-h^2} \right|^2 \\
&\quad + \varepsilon R \left((n-1)\varepsilon \Delta \ln R - \left(\frac{1}{\lambda} - n + 1\right) \varepsilon \left|\nabla \ln R\right|^2 + 1 - \frac{(n-4)\varepsilon}{2} R \right) \\
&\quad + \frac{(1+\varepsilon)R + 2(1+\ln h)}{1-h^2} |\nabla h|^2 + \frac{2h^2 \ln h}{(1-h^2)^2} |\nabla h|^2 \\
&\quad + \frac{2}{(1-h^2)^3} \left| 2h \nabla h \nabla h + (1-h^2) \nabla\nabla h \right|^2 .
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\frac{\partial P}{\partial t} &\geq \Delta P + 2\nabla P \cdot \nabla \ln S + \frac{2}{n} P^2 - (\varepsilon R + 1) P + \frac{|\nabla h|^2}{1-h^2} \left(R + 1 + \frac{2 \ln h}{1-h^2} \right) \\
&\quad + \varepsilon R \left((n-1)\varepsilon \Delta \ln R - \left(\frac{1}{\lambda} - n + 1\right) \varepsilon \left|\nabla \ln R\right|^2 + 1 - \frac{(n-4)\varepsilon}{2} R \right),
\end{aligned} \tag{2.8}
$$

where we used the following elementary inequality

$$
\left| \nabla\nabla \ln S + \frac{\varepsilon}{2} R g - \frac{\nabla h \nabla h}{1-h^2} \right|^2 \geq \frac{1}{n} \left(\Delta \ln S + \varepsilon R - \frac{|\nabla h|^2}{1-h^2} \right)^2 = \frac{P^2}{n}.
$$

Since $0 < c_0 < h < 1$ and the function $\frac{2\ln h}{1-h^2}$ is increasing on $(0,1)$, then $\frac{2\ln h}{1-h^2} > \frac{2 \ln c_0}{1-c_0^2}$. By the assumption of the theorem, using the maximum principle, we can see that the inequality (1.9) still holds under the ε-Yamabe flow. Hence

$$
R + 1 + \frac{2\ln h}{1-h^2} > R + 1 + \frac{2\ln c_0}{1-c_0^2} > 0
$$

for all time t. Therefore, (2.8) becomes

$$
\begin{aligned}
\frac{\partial P}{\partial t} &\geq \Delta P + 2\nabla P \cdot \nabla \ln S + \frac{2}{n} P^2 - (\varepsilon R + 1) P \\
&\quad + \varepsilon R \left((n-1)\varepsilon \Delta \ln R - \left(\frac{1}{\lambda} - n + 1\right) \varepsilon \left|\nabla \ln R\right|^2 + 1 - \frac{(n-4)\varepsilon}{2} R \right).
\end{aligned}
$$

Adding $\frac{1}{t}$ to P yields

$$
\begin{aligned}
&\frac{\partial}{\partial t} \left(P + \frac{1}{t} \right) \\
\geq &\Delta \left(P + \frac{1}{t} \right) + 2\nabla \left(P + \frac{1}{t} \right) \cdot \nabla \ln S + \left(P + \frac{1}{t} \right) \left(P - \frac{1}{t} \right) - (\varepsilon R + 1) \left(P + \frac{1}{t} \right) \\
&+ \varepsilon R \left((n-1)\varepsilon \Delta \ln R - \left(\frac{1}{\lambda} - n + 1\right) \varepsilon \left|\nabla \ln R\right|^2 + 1 - \frac{(n-4)\varepsilon}{2} R + \frac{1}{t} \right).
\end{aligned} \tag{2.9}
$$

Noting that we have the following relation

$$
\begin{aligned}
&(n-1)\varepsilon \Delta \ln R - \left(\frac{1}{\lambda} - n + 1\right) \varepsilon \left|\nabla \ln R\right|^2 + 1 - \frac{(n-4)\varepsilon}{2} R \\
= &(n-1)\varepsilon \left(\Delta \ln R + \left|\nabla \ln R\right|^2 \right) + \varepsilon R - \frac{1}{\lambda} \varepsilon \left|\nabla \ln R\right|^2 + 1 - \frac{(n-2)\varepsilon}{2} R
\end{aligned}
$$

$$=\frac{\partial}{\partial t}\ln R-\frac{1}{\lambda}\varepsilon\left|\nabla\ln R\right|^2+1-\frac{(n-2)\,\varepsilon}{2}R,$$

then as the approach of the trace Harnack inequality for the ε-Ricci flow on a closed surface proved by B. Chow in [5] (see also Lemma 2.1 in [14]) implies that

$$(n-1)\,\varepsilon\Delta\ln R-\left(\frac{1}{\lambda}-n+1\right)\varepsilon\left|\nabla\ln R\right|^2+1-\frac{(n-4)\,\varepsilon}{2}R+\frac{1}{t}\geq 0.$$

Since $g(t)$ has positive scalar curvature, we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(P+\frac{1}{t}\right)\geq\Delta\left(P+\frac{1}{t}\right)+2\nabla\left(P+\frac{1}{t}\right)\cdot\nabla\ln S+\left(P+\frac{1}{t}\right)\left(P-\frac{1}{t}\right)\\ -\left(\varepsilon R+1\right)\left(P+\frac{1}{t}\right).\end{aligned}\tag{2.10}$$

It is clear to see that $P+1/t>0$ for very small positive t. Then applying the maximum principle to the above evolution formula, we conclude that $P+1/t>0$ for all positive time t, and hence the desired theorem follows. □

Proof of Theorem 1.7. Firstly, by using Theorem 1.5, the pointwise pinching condition (1.11) which the initial Ricci curvature tensor $Rc\,(g\,(0))$ satisfies implies that the manifold $(M^n,g(0))$ is compact, then the positive Ricci pinching condition is preserved along the ε-Yamabe flow (1.1), which is well-understood by R. Ye's work in [15]. Hence for any $0\leq t<T$, we also have $Rc(g(t))\geq\lambda R(g(t))g(t)$ for the free parameter $\lambda\leq\frac{1}{n}$, and the rest proof is the same as the proof of Theorem 1.6. □

For Theorem 1.6 and 1.7, if we let $\varepsilon=0$, then

Corollary 2.1. *Let (M^n,g), where $n\geq 4$, be a complete manifold M^n with the scalar curvature satisfying* (1.9) *and the curvature tensor $Rm\,(g)$ satisfying the pointwise pinching condition* (1.8)*, where c_0 is a free parameter satisfying* $0<c_0<1$*. If S and T are solutions to the nonlinear parabolic equations* (1.6) *and* (1.7) *respectively with $0<c_0S<T<S$, then*

$$\frac{\partial}{\partial t}\ln S-\left|\nabla\ln S\right|^2+\ln S+\frac{1}{t}=\Delta\ln S+\frac{1}{t}>\frac{|\nabla h|^2}{1-h^2},\tag{2.11}$$

where $h:=T/S$.

Corollary 2.2. *Let (M^n,g), where $2\leq n\leq 3$, be a complete manifold M^n with the scalar curvature satisfying* (1.9) *and the Ricci curvature $Rc\,(g)$ satisfying the pointwise pinching condition* (1.11) *for some $\lambda>0$,where c_0 is a free parameter satisfying $0<c_0<1$. If S and T are solutions to the nonlinear parabolic equations* (1.6) *and* (1.7) *respectively with $0<c_0S<T<S$, then we also have the Harnack inequality* (2.11)*.*

If we set

$$\bar{g}=\varepsilon^{-1}g\qquad\text{and}\qquad\alpha=\varepsilon^{-1}$$

in Theorem 1.6 and 1.7, then

$$\bar{\Delta}=\varepsilon\Delta\qquad\text{and}\qquad\bar{R}=\varepsilon R.$$

Hence Theorem 1.6 and 1.7 can be rephrased as follows:

Corollary 2.3. *Let $(M^n, \bar{g}(t))$, where $n \geq 4$, $t \in [0,T)$ be a solution to the Yamabe flow* (1.1) *on a complete manifold M^n with initial the scalar curvature satisfying $\alpha \bar{R}(\bar{g}(0)) \geq -\frac{2\ln c_0}{1-c_0^2} - 1 > 0$, and the curvature tensor $Rm\,(g\,(0))$ satisfying the pointwise pinching condition* (1.8)*, where α is a positive constant and c_0 is a free parameter satisfying $0 < c_0 < 1$. If S and T are solutions to the nonlinear parabolic equations* (1.6) *and* (1.7) *respectively with $0 < c_0 S < T < S$ (this condition preserved by the Yamabe flow), then*

$$\frac{\partial}{\partial t} \ln S - \alpha |\bar{\nabla} \ln S|^2 + \ln S + \frac{1}{t} = \alpha \bar{\Delta} \ln S + \bar{R} + \frac{1}{t} > \frac{\alpha |\bar{\nabla} h|^2}{1-h^2},$$

where $h := T/S$.

Corollary 2.4. *Let $(M^n, \bar{g}(t))$, where $2 \leq n \leq 3$, $t \in [0,T)$ be a solution to the Yamabe flow* (1.1) *on a complete manifold M^n with the scalar curvature satisfying $\alpha \bar{R}(\bar{g}(0)) \geq -\frac{2\ln c_0}{1-c_0^2} - 1 > 0$, and the Ricci curvature $Rc\,(g\,(0))$ satisfying the pointwise pinching condition* (1.11) *for some $\lambda > 0$, where α is a positive constant and c_0 is a free parameter satisfying $0 < c_0 < 1$. If S and T are solutions to the nonlinear parabolic equations* (1.6) *and* (1.7) *respectively with $0 < c_0 S < T < S$ (this condition preserved by the Yamabe flow), then we also have the Harnack inequality* (2.11)*.*

3 Proof of Theorem 1.8 and 1.9

In section 3, we prove Theorem 1.8 and 1.9 by using Theorem 1.6 and 1.7. Note that the proof of Theorem 1.9 is the same as Theorem 1.8, thus we only prove Theorem 1.8 which is quite standard by integrating the inequality (1.10).

Proof of Theorem 1.8. We pick a space-time path $\gamma(x,t)$ joining (x_1,t_1) and (x_2,t_2) with $t_2 > t_1 > 0$. Along γ, by Theorem 1.6 we have

$$\begin{aligned} \frac{d}{dt} \ln S(x,t) &= \frac{\partial}{\partial t} \ln S + \nabla \ln S \cdot \frac{d\gamma}{dt} \\ &> |\nabla \ln S|^2 - \ln S - \frac{1}{t} + \frac{|\nabla h|^2}{1-h^2} + \nabla \ln S \cdot \frac{d\gamma}{dt} \\ &\geq -\frac{1}{4} \left| \frac{d\gamma}{dt}(t) \right|^2 - \ln S - \frac{1}{t}. \end{aligned}$$

Hence $\frac{d}{dt}\left(e^t \ln S(x,t)\right) > -e^t \left(\frac{1}{4} \left| \frac{d\gamma}{dt}(t) \right|^2 + \frac{1}{t} \right)$. Integrating this inequality from the time t_1 to t_2 yields

$$e^{t_1} \ln S(x_1,t_1) - e^{t_2} \ln S(x_2,t_2) < \int_{t_1}^{t_2} e^t \left(\frac{1}{4} \left| \frac{d\gamma}{dt}(t) \right|^2 + \frac{1}{t} \right) dt,$$

which completes the proof of Theorem 1.8. □

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Author's address:

Chang-Jun Li
School of Mathematical Sciences, Ocean University of China,
Lane 238, Songling Road, Laoshan District, Qingdao City, Shandong Province, 266100,
People's Republic of China.
E-mail: licj@ouc.edu.cn.

Xiang Gao (The corresponding author)
School of Mathematical Sciences, Ocean University of China,
Lane 238, Songling Road, Laoshan District, Qingdao City, Shandong Province, 266100,
People's Republic of China.
E-mail: gaoxiangshuli@126.com.