On some classes of foliations

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Abstract. The goal of the paper is to present in a unitary way some conditions that a foliation be Riemannian, involving general conditions on higher order normal bundles (jets or accelerations).

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1 Introduction

Various conditions that a foliation be Riemannian are studied in many papers, for example [3, 4, 10, 11, 12, 14].

The conditions studied in this paper have initially the origin in a special case of a problem presented by E. Ghys in Appendix E of P. Molino's book [6], i.e. asking if the existence of a foliated Finsler metric assure that a foliation is Riemannian (Ghys conjecture). We proved the answer is affirmative in a more general case of a transverse Lagrangian fulfilling a natural regularity condition, automatically fulfilled by a transverse Finslerian (see [10]).

Our goal below is to present in a unitary way, following [11, 12], some conditions that a foliation be Riemannian, involving general conditions on higher order normal bundles (jets or accelerations). Some other aspects of the problem can be stressed. For example, if the leaves of a Riemannian foliation $\mathcal{F}$ are compact, then the leaf spaces $M/\mathcal{F}$ is a Satake manifold (or a V-manifold, in the original terminology of Satake), one of the first known non-trivial orbifold. The existence of a transverse Lagrangian or Hamiltonian is worth to be studied on such generalized manifolds, together with their physical properties; it is also the case of the normal bundle (of first order) of a foliation.

In the sequel we study the real case, but it can also be developed a study in a complex setting for foliations, as in [1, 2].

Let M be an n-dimensional manifold and $\mathcal{F}$ be a k-dimensional foliation on M. We denote by $\tau\mathcal{F}$ and $\nu\mathcal{F}$ the tangent plane field and the normal bundle respectively.

A bundle E over M is called *foliated* if there is a bundle atlas on E such that all the components of the structural functions are basic ones. In this case a canonical

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foliation $\mathcal{F}_E$ on E is induced, having the same dimension k, such that p restricted to leaves is a local diffeomorphism. In particular, we consider affine and vector bundles that are foliated. For example, $\nu\mathcal{F}$ is a foliated bundle and a natural foliation on $\nu\mathcal{F}$ can be considered.

According to [11], a *positively admissible Lagrangian* on a foliated vector bundle $p : E \to M$ is a continuous map $L : E \to I\!R$ that is asked to be differentiable at least when it is restricted to the total space of the slashed bundle $E_* = E\backslash\{\bar{0}\} \to M$, where $\{\bar{0}\}$ is the image of the null section, such that the following conditions hold: 1) L is positively defined (i.e. its vertical Hessian is positively defined) and $L(x, y) \geq 0 = L(x, 0)$, $(\forall)x \in M$ and $y \in E_x = p^{-1}(x)$; 2) L is locally projectable on a transverse Lagrangian; 3) there is a basic function $\varphi : M \to (0, \infty)$, such that for every $x \in M$ there is $y \in E_x$ such that $L(x, y) = \varphi(x)$.

If a positively transverse Lagrangian F is 2–homogeneous (i.e. $F(x, \lambda y) = \lambda^2 F(x, y)$, $(\forall)\lambda > 0$), then F is called a *Finslerian*; it is also a positively admissible Lagrangian, taking $\varphi \equiv 1$, or any positive constant. For a foliated bundle, we can see the vertical bundle $VTE = \ker p_* \to E$ as a vector subbundle of $\nu\mathcal{F}_E \to E$ by mean of the canonical projection $TE \to \nu F_E$, since VTE is transverse to τF_E. We say that an invariant Riemannian metric G' on νF_E is *vertically exact* if its restriction to the vertical foliated sections is the transverse vertical Hessian of a positively admissible Lagrangian $L : E \to I\!R$; in this case, we say that the foliation $\mathcal{F}_E$ is *vertically exact*. Notice that if $p : E \to M$ is an affine bundle, then the *vertical Hessian* Hess L of a Lagrangian $L : E \to I\!R$ is a symmetric bilinear form on the fibers of the vertical bundle VTE, given by the second order derivatives of L, using the fiber coordinates (see [10, 13] for more details using coordinates).

2 The jet bundle case

If $p : E \to M$ is a foliated bundle, then $\mathcal{J}^1 E \to M$ is a foliated bundle of 1-jets of foliated sections of E; a canonical foliation $\mathcal{F}^1_E$ on $\mathcal{J}^1 E$ can be considered. For $r \geq 1$, the canonical projection $\pi^r_{r-1} : \mathcal{J}^r E \to \mathcal{J}^{r-1} E$ is also an affine bundle, with the director vector bundle $\mathrm{Hom}((\nu F)^r, E))$. For $r = 0$ one obtain a bundle $\pi^r_{-1} : \mathcal{J}^r E \to M$. If $p : E \to M$ is a *foliated vector bundle*, then $\pi^r_{-1} : \mathcal{J}^r E \to M$ is also a foliated vector bundle and a natural vector subbundle of $\mathcal{J}^1\mathcal{J}^{r-1} E \to M$, the first jet bundle of $\pi^{r-1}_{-1} : \mathcal{J}^{r-1} E \to M$.

Theorem 2.1. *The lifted foliation $\mathcal{F}^r$ is Riemannian for some $r \geq 1$ iff $\mathcal{F}$ is Riemannian.*

Considering the induced foliation $\mathcal{F}^r_0$ on the slashed vector bundle $\mathcal{J}^r_* = \mathcal{J}^r\backslash\{\bar{0}\}$, then Theorem 2.1 can not give any answer to the following question: *when is $\mathcal{F}$ Riemannian if $\mathcal{F}^r_0$ is Riemannian for some $r \geq 1$?*

Theorem 2.2. *Let $\mathcal{F}$ be a foliation on a manifold M and $\mathcal{F}^r_0$ be the lifted foliation on the slashed bundle of r-jets of sections of the normal bundle $\nu\mathcal{F}$. Then $\mathcal{F}^r_0$ is Riemannian and vertically exact for some $r \geq 1$ iff $\mathcal{F}$ is Riemannian.*

In particular, it follows that any invariant metric g on νF gives rise to a canonical Lagrangian on $\mathcal{J}^r$, coming from the vertical part of the vertically exact invariant

Riemannian metric on νF^r. So, it is natural to ask for the converse: does the existence of a Lagrangian on $\mathcal{J}^r$ give guaranties that $\mathcal{F}$ is Riemannian?

Theorem 2.3. *Let $p : E \to M$ be a foliated vector bundle over a foliated manifold $(M, \mathcal{F})$. There is a positively admissible Lagrangian on $\mathcal{J}^r E$ for some $r \geq 1$ iff the foliation $\mathcal{F}$ is Riemannian.*

The key result to prove the above Theorems, as well as the main results is the following statement.

Proposition 2.4. *Let $p_1 : E_1 \to M$ and $p_2 : E_2 \to M$ be foliated vector bundles over a foliated manifold $(M, \mathcal{F})$ and $q_2 : E_{2*} \to M$ be the slashed bundle. If there are a positively admissible Lagrangian $L : E_2 \to I\!R$ and a metric b on the pull back bundle $q_2^* E_1 \to E_{2*}$, foliated with respect to $\mathcal{F}_{E_{2*}}$, then there is a foliated metric on E_1, with respect to $\mathcal{F}$.*

3 The acceleration bundle case

We consider now the higher order transverse foliated bundle of order $r \geq 1$ of a foliation $\mathcal{F}$ on M, denoted by $\nu^r \mathcal{F}$, as spaces of classes of transverse curves having a transverse contact of order $r \geq 0$. Notice that in the foliate case the transverse $\nu \mathcal{F}^r$ play a role of a tangent space for $\nu^r \mathcal{F}$, as the tangent space $\tau T^r M$ is for $T^r M$ in the non-foliate case in [5]. We denote by $\mathcal{F}^r$ the foliation on $\nu^r \mathcal{F}$. In a similar way as in the non-foliate case in [5, Sect. 6.1], some constructions can be performed. For example, various bundle structures can be considered over a $\nu^r \mathcal{F}$; for example, for $0 \leq r' \leq r$, the canonical projection $\pi^r_{r'} : \nu^r \mathcal{F} \to \nu^{r'} \mathcal{F}$ is a foliated bundle. In particular, for $r \geq 1$, $\pi^r_{r-1} : \nu^r \mathcal{F} \to \nu^{r-1} \mathcal{F}$ is a (foliated) affine bundle for $r > 1$ and $\pi^1_0 : \nu \mathcal{F} \to \nu^0 \mathcal{F} = M$ is a (foliated) vector bundle (for $r = 1$).

Proposition 3.1. *For $1 \leq r' \leq r$, there is an inclusion of foliated submanifolds (in fact of foliated subbundles over M), $I^r_{r'} : \nu^{r'} \mathcal{F} \to \nu^r \mathcal{F}$, where the inclusion assigns to an equivalence class in $[\gamma] \in \nu^{r'}_{;m} \mathcal{F}$ an equivalence class in $\nu^r_{;m} \mathcal{F}$ that the first $r - r'$ derivatives vanish, then the next r' derivatives are the same as the first r' derivatives of γ.*

Thus we have $I^r_0(M) \subset I^r_1(\nu \mathcal{F}) \subset I^r_2(\nu^2 \mathcal{F}) \subset \cdots \subset I^r_{r-1}(\nu^{r-1} \mathcal{F}) \subset \nu^r \mathcal{F}$.

A transverse vector field $\bar{X} \in \Gamma(\nu \mathcal{F})$ lifts in this way to the transverse section $I^r_1(\bar{X}) : M \to \nu^r \mathcal{F}$ of the bundle $\pi^r_0 : \nu^r \mathcal{F} \to M$. An other lift can be constructed as it follows. Denoting by γ^X_t the one parameter group of local transformations of X, we consider $[\gamma^X_{t=0}(m)] \in \nu^r_{;m} \mathcal{F}$. The simplest case is when $\bar{X} = \bar{0}$ is the null vector field; its lift is the null section $\bar{0}^r : M \to \nu^r \mathcal{F}$, $\bar{0}^r(m) = I^r_0(m)$.

For every $r \geq 1$ and $0 \leq r' \leq r$, , the canonical projection $\pi^r_{r'} : \nu^r \mathcal{F} \to \nu^{r'} \mathcal{F}$ induces a transverse map $\bar{\pi}^r_{r'} : \nu \mathcal{F}^r \to \nu \mathcal{F}^{r'}$ that is a vector bundle map of foliated vector bundles; notice that $\pi^r_0 = \pi^r$, $\mathcal{F}^0 = \mathcal{F}$, $\nu^1 \mathcal{F} = \nu \mathcal{F}$, $\nu^0 \mathcal{F} = M$ and $\bar{\pi}^r_0 = \bar{\pi}^r$. We denote the kernel vector subbundle $\ker \bar{\pi}^r_{r'} \subset \nu \mathcal{F}^r$ by $\bar{V}^r_{r'}$; it is a foliate vector bundle as well. Since for $r_1 \leq r_2 \leq r_3$, one have $\pi^{r_3}_{r_1} = \pi^{r_3}_{r_2} \circ \pi^{r_2}_{r_1}$ and $\bar{\pi}^{r_3}_{r_1} = \bar{\pi}^{r_3}_{r_2} \circ \bar{\pi}^{r_2}_{r_1}$, it follows that there are foliated vector subbundles $\bar{V}^r_{r-1} \subset \bar{V}^r_{r-2} \subset \cdots \subset \bar{V}^r_0 \subset \nu \mathcal{F}^r$. Notice that $\nu^{r+1} \mathcal{F} \subset \nu \mathcal{F}^r$ is an affine subbundle over $\nu^r \mathcal{F}$, for $r \geq 1$, while $\nu^1 \mathcal{F} = \nu \mathcal{F}^0 = \nu \mathcal{F}$ for $r = 0$.

There is an r-transverse structure in the fibers of on $\nu\mathcal{F}^r$, i.e. a vector bundle map $J : \nu\mathcal{F}^r \to \nu\mathcal{F}^r$ (analogous of the r-tangent structures in the non-foliate case), and its dual $J^* : \nu^*\mathcal{F}^r \to \nu^*\mathcal{F}^r$.

A *transverse r-nonlinear connection* is a splitting of $\nu\mathcal{F}^r$ as a Whitney sum of transverse vector bundles

$$\nu\mathcal{F}^r = \bar{V}_0^r \oplus \bar{H}_0^r, \tag{3.1}$$

where $\bar{H}_0^r$ is the r-horizontal vector bundle, that is canonically isomorphic with $(\bar{\pi}^r)^*\nu\mathcal{F}$. We denote by $h : \nu\mathcal{F}^r \to \bar{H}_0^r$ the projector given by the above decomposition.

Given a transverse r-nonlinear connection by a splitting (3.1), the consecutive images by J in the fibers of $\nu\mathcal{F}^r$,

$$J\left(\bar{H}_0^r\right) = \bar{H}_1^r, \ldots, J\left(\bar{H}_{r-1}^r\right) = \bar{H}_r^r$$

define some transverse vector subbundles of $\nu\mathcal{F}^r$, all isomorphic with $\bar{H}_0^r$, such that there are the following Whitney sum decompositions

$$\bar{V}_0^r = \bar{H}_1^r \oplus \cdots \oplus \bar{H}_r^r,\ \nu\mathcal{F}^r = \bar{H}_0^r \oplus \bar{H}_1^r \oplus \cdots \oplus \bar{H}_r^r. \tag{3.2}$$

Notice that $\bar{H}_r^r = \bar{V}_{r-1}^r$ and we can prove the following result.

Proposition 3.2. *Any splitting $\nu\mathcal{F}^r = \bar{V}_{r-1}^r \oplus \bar{H}_{r-1}^r$ gives rise to a splitting (3.1).*

A *transverse r-semispray* is a foliate section $S : \nu^r\mathcal{F} \to \nu^{r+1}\mathcal{F}$ of the affine bundle $\pi_r^{r+1} : \nu^{r+1}\mathcal{F} \to \nu^r\mathcal{F}$. Since $\nu^{r+1}\mathcal{F} \subset \nu\mathcal{F}^r$, it follows that an r-semispray can be regarded as well as a transverse section $S : \nu^r\mathcal{F} \to \nu\mathcal{F}^r$.

Proposition 3.3. *Any transverse r-semispray gives rise to a transverse r-nonlinear connection, i.e. a splitting (3.1).*

A fact that we use latter is the following result.

Proposition 3.4. *A transverse r-nonlinear connection and a transverse Riemannian metric in the fibers of $\bar{V}_{r-1}^r$ give a transverse Riemannian metric on $\nu\mathcal{F}^r$. Conversely, a transverse Riemannian metric on $\nu\mathcal{F}^r$ gives a transverse r-nonlinear connection and a transverse Riemannian metric in the fibers of $\bar{V}_{r-1}^r$.*

4 The Lagrangian case

Some r-transverse non-linear connections, semi-sprays and Riemannian metrics are involved in the case of regular r-transverse Lagrangians that we consider in the sequel.

An *r-transverse Lagrangian* (a transverse Lagrangian of order $r \geq 1$, i.e. locally projectable on an r-Lagrangian) is a continuous real map $L : \nu^r\mathcal{F} \to I\!R$, smooth on an open fibered submanifold $\nu_*^r\mathcal{F} \subset \nu^r\mathcal{F}$. The cases studied in the paper are when $\nu_*^r\mathcal{F} = \nu^r\mathcal{F}$, i.e. L is smooth, or when $\nu^r\mathcal{F}\backslash\nu_*^r\mathcal{F}$ contains $I_{r-1}^r(\nu^{r-1}\mathcal{F})$, i.e. L is slashed. For sake of simplicity, we perform the next constructions in the case of a smooth L, in the slashed case we must be care of domains where the objects are defined. As usually, the *vertical Hessian* of L is the bilinear form h in the fibers of $\bar{V}_{r-1}^r$, given in some generic coordinates by the second order derivatives. We say that L is *regular* if its vertical Hessian is non-degenerated. The fibers of the fibered manifold $\nu^r\mathcal{F} \to \nu^{r-1}\mathcal{F}$ are affine spaces.

Proposition 4.1. *1) If an r-Lagrangian L is regular, then it can define canonically a transverse r-semispray and a transverse r-nonlinear connection.*

2) If the vertical Hessian of an r-Lagrangian L is positively defined, then $\mathcal{F}^r$ is a Riemannian foliation.

As in the case of trivial foliation of M by points in [9], $\nu^{r-1}\mathcal{F}\times_M \nu^*\mathcal{F} \overset{not.}{=} \nu^{r*}\mathcal{F}$ play the role of the vectorial dual of the affine bundle $\nu^r\mathcal{F}\to\nu^{r-1}\mathcal{F}$. The usual partial derivatives of L in the highest order transverse coordinates define a well-defined Legendre map $\mathcal{L}:\nu^r\to\nu^{r*}\mathcal{F}$. If L is regular, then $\mathcal{L}$ is a local diffeomorphism; if $\mathcal{L}$ is a global diffeomorphism we say that L is *hyperregular*. We say that $H:\nu^{r*}\mathcal{F}\to I\!R$, $H=L\circ\mathcal{L}^{-1}$ is the pseudo-Hamiltonian associated with L. For $0\leq r'\leq r$, let us denote $\nu^{r',(r-r')*}\mathcal{F}=\nu^{r'}\mathcal{F}\times_M(\nu^*\mathcal{F})^{r-r'}$, where $(\nu^*\mathcal{F})^{r-r'}=\nu^*\mathcal{F}\times_M\cdots\times_M\nu^*\mathcal{F}$, with the fibered product of $(r-r')$-times. In particular, $\nu^{r*}=\nu^{r-1,r*}\mathcal{F}=\nu^{r-1}\mathcal{F}\times_M\nu^*\mathcal{F}$. A *transverse slashed Lagrangian* of order r is a continuous map $L^r:\nu^r\mathcal{F}\to I\!R$ that is differentiable on an open fibered submanifold $\nu^r_*\mathcal{F}\subset\nu^r\mathcal{F}$, called a *slashed bundle*. All the above constructions can be adapted for slashed Lagrangians.

Let us suppose that L^r is *hyperregular*, i.e. the Legendre map $\mathcal{L}^{(r)}:\nu^r_*\to\nu^{1,(r-1)*}\mathcal{F}=\nu^{r-1}\mathcal{F}\times_M\nu^*\mathcal{F}$ is an diffeomorphism on its image. Let us suppose also that $\mathcal{L}^{(r)}(\nu^r_*)=\nu_*^{1,(r-1)*}\mathcal{F}=\nu_*^{r-1}\mathcal{F}\times_M\nu^*_*\mathcal{F}$; here $\nu^*_*\mathcal{F}=\nu^*\mathcal{F}\backslash\{\bar{0}\}$ (where $\{\bar{0}\}$ is the zero section) and $\nu_*^{r-1}\mathcal{F}$ is a slashed subbundle of $\nu^{r-1}\mathcal{F}$. We denote by $H^{1,r-1}=L^r\circ\left(\mathcal{L}^{(r)}\right)^{-1}:\nu_*^{1,(r-1)*}\mathcal{F}\to I\!R$ its pseudo-Hamiltonian. (See [9] for its classical definition and [8] for a coordinate description of the whole construction in the non-foliate case). Analogous, for $0\leq j<r-1$, we suppose, step by step, backward from $r-1$ from 0, that the usual partial derivatives of $L^{(j+1)}:\nu_*^{j+1,(r-j-1)*}\mathcal{F}=\nu_*^{r-j-1}\mathcal{F}\times_M(\nu^*_*\mathcal{F})^{j+1}\to I\!R$ in the highest order transverse coordinates (of order $j+1$) define a well-defined Legendre map $\mathcal{L}^{(j+1)}:\nu_*^{j+1,(r-j-1)*}\mathcal{F}=\nu_*^{j+1}\mathcal{F}\times_M(\nu^*_*\mathcal{F})^{r-j-1}\to\nu^{j,(r-j)*}\mathcal{F}=\nu^j\mathcal{F}\times_M(\nu^*\mathcal{F})^{r-j}$. We suppose that $\mathcal{L}^{(j+1)}$ is a diffeomorphism on its image and the image is exactly $\mathcal{L}^{(j+1)}\left(\nu_*^{j+1,(r-j-1)*}\mathcal{F}\right)=\nu_*^{j,(r-j)*}\mathcal{F}=\nu_*^j\mathcal{F}\times_M(\nu^*_*\mathcal{F})^{r-j}$. Then the pseudo-Hamiltonian $L^{(j)}=L^{(j+1)}\circ\left(\mathcal{L}^{(j+1)}\right)^{-1}:\nu_*^{j,(r-j)*}\mathcal{F}\to I\!R$ can be considered. Finally, for $j=0$, we obtain a transverse slashed Lagrangian $L^{(0)}=L^1\circ\left(\mathcal{L}^{(1)}\right)^{-1}:\nu_*^{0,r*}\mathcal{F}=(\nu^*_*\mathcal{F})^r\to I\!R$ and we suppose that $\mathcal{L}^{(1)}:\nu_*^{1,(r-1)*}\mathcal{F}=\nu_*\mathcal{F}\times_M(\nu^*_*\mathcal{F})^{r-1}\to\nu_*^{0,r*}\mathcal{F}=(\nu^*_*\mathcal{F})^r\subset\nu^{0,r*}\mathcal{F}=(\nu^*\mathcal{F})^r$ is a diffeomorphism. It follows a diffeomorphism $\mathcal{L}=\mathcal{L}^{(1)}\circ\cdots\circ\mathcal{L}^{(r)}:\nu^r_*\to(\nu^*_*\mathcal{F})^r$ and a transverse slashed Lagrangian $L^{(0)}:(\nu^*_*\mathcal{F})^r\to I\!R$. The canonical diagonal inclusion $\nu^*\mathcal{F}\to(\nu^*\mathcal{F})^r$ sends $\nu^*_*\mathcal{F}\to(\nu^*_*\mathcal{F})^r$. We suppose that the restriction of $L^{(0)}$ to the diagonal is a positively admissible Lagrangian on $\nu^*\mathcal{F}$, in fact a transverse Hamiltonian $H:\nu^*_*\mathcal{F}\to I\!R$. If the given transverse Lagrangian $L^r:\nu^r\mathcal{F}\to I\!R$ fulfills all the above conditions, we say that L itself is a *positively admissible Lagrangian* (of order r) and H is its *diagonal Hamiltonian*. The existence of a lifted metric, from the base space to the higher order tangent bundle, is an well-known fact in the non-foliate case (see, for example [5, Sect. 9.2]); we have to consider a simpler construction in the foliated case, that it is also vertically exact, as in [7, 8, 9].

Proposition 4.2. *Any transverse metric g on νF gives canonically a positively admissible Lagrangian $L^{(r)}$ of order r and a canonical vertically exact invariant Riemannian metric $g^{(r)}$ on $\nu^r\mathcal{F}$, for any $r\geq 1$.*

We can state the following results.

Theorem 4.3. *The lifted foliation $\mathcal{F}^r$ is Riemannian for some $r \geq 1$ iff $\mathcal{F}$ is Riemannian.*

We say that a foliation $\mathcal{F}$ is transversely almost parallelizable if there is a $\mathcal{F}$-transverse vector bundle ξ over M, such that $\xi \oplus \nu\mathcal{F}$ is transversely parallelizable. Obviously, if a foliation $\mathcal{F}$ is transversely parallelizable, then it is a Riemannian one.

Corollary 4.4. *If the lifted foliation $\mathcal{F}^r$ is transversely parallelizable of almost parallelizable, then $\mathcal{F}$ is a Riemannian foliation.*

The proof of Theorem 4.3 can not give any answer to the following question: *when is $\mathcal{F}$ Riemannian if the foliation induced on $\nu^r\mathcal{F}\backslash I^r_{r-1}(\nu^{r-1}\mathcal{F})$ is Riemannian for some $r \geq 1$?* We are going to relate this question to the existence of a certain transverse slashed Lagrangian L^r of order r, asking that the open subset $\nu^r_*\mathcal{F} \subset \nu^r\mathcal{F}$ does not contains $I^r_{r-1}(\nu^{r-1}\mathcal{F})$. We say that a such Lagrangian L^r is r-regular if its vertical Hessian, according to the induced affine bundle structure $\pi^r_{r-1} : \nu^r\mathcal{F} \to \nu^{r-1}\mathcal{F}$, is non-degenerate. In order to give an answer to the above question, we are going to consider below some other regularity conditions for these slashed Lagrangians of order r, as it follows.

A transverse bundle of order r, $\nu^r\mathcal{F}$ can be regarded as a fibered manifold $\pi^r_{r'}$: $\nu^r\mathcal{F} \to \nu^{r'}\mathcal{F}$, $(\forall)0 \leq r' < r$. We denote $\nu^{r',(r-r')*}\mathcal{F} = \nu^{r'}\mathcal{F} \times_M (\nu^*\mathcal{F})^{r-r'}$ (where $(\nu^*\mathcal{F})^{r-r'} = \nu^*\mathcal{F} \times_M \cdots \times_M \nu^*\mathcal{F}$, with the fibered product of $(r-r')$-times and $\nu^*\mathcal{F}$ is the transverse bundle dual to $\nu\mathcal{F}$).

In particular, according to the case of trivial foliation of M by points in [9], $\nu^{1.(r-1)*}\mathcal{F} = \nu^{r-1}\mathcal{F} \times_M \nu^*\mathcal{F}$ is denoted by $\nu^{r*}M$ and play the role of the vectorial dual of the affine bundle $\nu^r\mathcal{F} \to \nu^{r-1}\mathcal{F}$.

A transverse slashed Lagrangian of order r is a map $L^r : \nu^r\mathcal{F} \to I\!R$ that is differentiable on an open subset $\nu^r_*\mathcal{F} \subset \nu^r\mathcal{F}$, where $\nu^r\mathcal{F} \backslash \nu^r_*\mathcal{F}$ contains $I^r_{r-1}(\nu^{r-1}\mathcal{F})$.

We can now state and prove the following Theorems, where the main technical tool to prove the necessity is Proposition 2.4.

Theorem 4.5. *Let $\mathcal{F}$ be a foliation on a manifold M and $\mathcal{F}^r_0$ be the lifted foliation in a suitable slashed bundle $\nu^r_*\mathcal{F}$ of the r-normal bundle $\nu^r\mathcal{F}$. Then $\mathcal{F}^r_0$ is Riemannian and vertically exact for some $r \geq 1$ iff $\mathcal{F}$ is Riemannian.*

In particular, it follows that any transverse metric g on νF gives rise to a canonical Lagrangian on $\nu^r\mathcal{F}$, coming from the vertical part of the vertically exact invariant Riemannian metric on $\nu\mathcal{F}^r$. So, it is natural to ask that only the existence of a Lagrangian on $\nu^r\mathcal{F}$ guaranties that $\mathcal{F}$ is Riemannian. One have a positive answer, as it follows.

Theorem 4.6. *If $(M, \mathcal{F})$ is a foliated manifold, then there is a positively admissible Lagrangian on $\nu^r\mathcal{F}$ for some $r \geq 1$ iff the foliation $\mathcal{F}$ is Riemannian.*

Finally, as in the jet bundle case, the following question arises: *can we drop in Theorem 4.5 the condition that $\mathcal{F}^r_0$ is vertically exact?*

As a conclusion, the results in both cases (jets and accelerations), confirm that imposing some minimal conditions in each case on some higher order Lagrangians, the given foliation must be Riemannian; thus: *Riemannian foliations are necessary setting to study certain transverse Lagrangians, subject to some natural conditions, considered on jet transverse bundles or on higher order transverse bundles of a foliation.*

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