Obata theorem on compact Finsler spaces

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Abstract. Let (M,g) be an n-dimensional $(n \geq 2)$ without boundary compact simply connected Finsler manifold. Then it admits a non-trivial solution for a certain second order differential equation, if and only if it is conformally homeomorphic to the standard n-sphere in the Euclidean space $\mathbb{R}^{n+1}$. This result generalizes the Obata theorem on compact Finsler spaces.

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1 Introduction

In 1960s Y. Tashiro studied a second order differential equation on Riemannian spaces, cf. [10]. Intuitively, the existence of non-trivial solutions for this differential equation describes the existence of certain coordinate system on the Riemannian manifold M, called adapted coordinates. Geometrically, the existence of a solution for this SODE, is equivalent to the existence of circle-preserving transformations on the Riemannian manifold M.

Recently, the circle-preserving transformations are studied in Finsler geometry by the present author and Z. Shen, cf. [5]. Previously, inspired by Tashiro's work, the present author in a joint paper, specialized adapted coordinates to the Finsler setting and proved (cf. [1, 3, 4]):

Theorem 1.1. *Let (M,g) be a complete connected Finsler manifold of dimension $n \geq 2$. If M admits a non-trivial solution of*

$$\nabla_i \nabla_j \rho = \phi g_{ij}, \tag{1.1}$$

where ∇ is the Cartan h-covariant derivative then, depending on the number of critical points of ρ - i.e. zero, one or two respectively - it is conformal to
(a) *A direct product $J \times \overline{M}$ of an open interval J of the real line and an $(n-1)$-dimensional complete Finsler manifold $\overline{M}$.*
(b) *An n-dimensional Euclidean space.*
(c) *An n-dimensional unit sphere in an Euclidean space.*

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In the present work we show that the converse is also true when (M, g) is compact. More precisely, we prove the following theorem.

Theorem 1.2. *Let (M, g) be an n-dimensional $(n \geq 2)$ without boundary compact simply connected Finsler manifold. Then it admits a non-trivial solution ρ of the equation (1.1), if and only if it is conformally homeomorphic to the standard n-sphere in the Euclidean space $\mathbb{R}^{n+1}$.*

This theorem is an extension of the Obata theorem to compact Finsler spaces, cf. [8]. The second order differential equation (1.1) is closely related to the concept of Hessian metrics, which has applications in the geometric approach to black hole thermodynamics, cf. [11].

2 Preliminaries

Let M be a real n-dimensional differentiable manifold and let (x, U) be a local chart on M. We denote by $TM \to M$ the tangent bundle and by $\pi : TM_0 \to M$ the slit tangent bundle. An element of TM is denoted by the pair (x, y), where $x \in M$ and $y \in T_xM$.

A *Finsler structure* on M is provided by a function $F : TM \to [0, \infty)$, with the following properties: F is differentiable (C^∞) on TM_0; F is positively homogeneous of degree one in y, i.e. $F(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0$; the Hessian matrix of F^2 is positive definite on TM_0, that is, $(g_{ij}) := \left(\frac{1}{2}\left[\frac{\partial^2}{\partial y^i \partial y^j} F^2\right]\right)$.

A *Finsler manifold* (M, g) is a pair consisting of a differentiable manifold M and the tensor field $g = (g_{ij})$. We denote here by G^i_j the coefficients of nonlinear connection on TM, where $G^i_j = \frac{\partial G^i}{\partial y^j}$ and $G^i = 1/4 g^{ih}(\frac{\partial^2 F^2}{\partial y^h \partial x^j} y^j - \frac{\partial F^2}{\partial x^h})$.

By means of this nonlinear connection, the tangent space of TM_0 can be split into the direct sum of the horizontal and vertical subspaces with the corresponding bases $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$. This basis is related to $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$, the typical basis of TM, by $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - G^j_i \frac{\partial}{\partial y^j}$. The dual basis is denoted by $\{dx^i, \delta y^i\}$, where $\delta y^i := dy^i + G^i_j dx^j$.

The coefficients of horizontal and vertical covariant derivatives with respect to the Cartan connection are denoted by $\Gamma^i_{jk} = 1/2 g^{ih}\Big(\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk}\Big)$ and $C^i_{jk} = 1/2 g^{ih} \dot{\partial}_h g_{jk}$, where $\delta_k = \frac{\delta}{\delta x^k}$ and $\dot{\partial}_k = \frac{\partial}{\partial y^k}$. The 1−form of the Cartan connection in this basis is given by $\omega^i_j = \Gamma^i_{jk} dx^k + C^i_{jk} \delta y^k$. Rewriting ω^i_j with respect to the basis $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ with dual basis $\{dx^i, dy^i\}$, we have $\omega^i_j = \overset{*}{\Gamma}{}^i_{jk} dx^k + C^i_{jk} dy^k$, where $\overset{*}{\Gamma}{}^i_{jk} = \Gamma^i_{jk} + G^a_k C^i_{aj}$.

By homogeneity, we have $y^k \Gamma^i_{jk} = G^i_j$, and $y^j G^i_j = 2G^i$, cf. [2, 9, 12]. The Cartan connection is metric compatible, that is, $\nabla_l g_{jk} = 0$ and $\dot{\nabla}_l g_{jk} = 0$.

The components of the Cartan horizontal and vertical covariant derivatives of a tensor field S with the components $(S^i_{jk}(x, y))$ on TM are respectively given by

$$\nabla_l S^i_{jk} := \delta_l S^i_{jk} - S^i_{ak} \Gamma^a_{jl} - S^i_{ja} \Gamma^a_{kl} + S^a_{jk} \Gamma^i_{al}, \tag{2.1}$$

$$\dot{\nabla}_l S^i_{jk} := \dot{\partial}_l S^i_{jk} - S^i_{ak} C^a_{jl} - S^i_{ja} C^a_{kl} + S^a_{jk} C^i_{al}, \tag{2.2}$$

where $\nabla_l := \nabla_{\frac{\delta}{\delta x^l}}$ and $\dot{\nabla}_l = \nabla_{\frac{\partial}{\partial y^l}}$. Let c be a curve on TM given by $c : t \in I \subset R \longrightarrow (x^i(t), y^i(t)) \in TM$. We say that c is a *geodesic* of a Finsler connection ∇, if $\nabla_{\dot{c}}\dot{c} = 0$. Here,

$$\dot{c}(t) = \frac{dx^i}{dt}\frac{\partial}{\partial x^i} + \frac{dy^i}{dt}\frac{\partial}{\partial y^i} = \frac{dx^i}{dt}\frac{\delta}{\delta x^i} + \frac{\delta y^i}{dt}\frac{\partial}{\partial y^i},$$

is the tangent vector along c and $\frac{\delta y^i}{dt} := \frac{dy^i}{dt} + G^i_j\big(x(t), \frac{dx}{dt}\big)\frac{dx^j}{dt}$. From these equations we can see that a horizontal curve, that is, a curve for which we have $\frac{\delta y^i}{dt} = 0$, is a geodesic of the Finsler connection if and only if, cf. [6].

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0. \tag{2.3}$$

Consider a curve γ on (M, g) given by $\gamma : t \in I \subset R \longrightarrow \gamma(t) = (x^i(t)) \in M$. We say that γ is a *geodesic* of the Finsler space (M, g), if its natural lift $\tilde{\gamma}(t) = (x^i(t), dx^i(t)/dt)$ to TM, is a geodesic of ∇, or equivalently it is parallel (or horizontal) that is, $\nabla_{\frac{d\tilde{\gamma}}{dt}}\frac{d\tilde{\gamma}}{dt} = 0$. This implies the equation (2.3).

Two points p and q are said to be *conjugate points* along a geodesic γ if there exists a non-zero Jacobi field along γ that vanishes at p and q, cf. [2].

Throughout this paper, all manifolds are supposed to be connected.

Let $\rho : M \to [0, \infty)$ be a scalar function on M and let $\nabla_i\nabla_j\rho = \phi g_{ij}$, be a second order differential equation, where ∇_i is the Cartan horizontal covariant derivative and ϕ is a function of x alone; then we say that the equation (1.1) has a solution ρ. The solution ρ is said to be *trivial* if it is constant. The existence of a solution of the equation (1.1) is equivalent to the existence of some special conformal change of metric on M. We denote by $\mathtt{grad}\rho = \rho^i\partial/\partial x^i$ the gradient vector field of ρ, where $\rho^i = g^{ij}\rho_j$, $\rho_j = \partial\rho/\partial x^j$ and $i, j, ...$ run over the range $1, ..., n$.

We say that the point o of (M, g) is a *critical point* of ρ if the vector field $\mathtt{grad}\rho$ vanishes at o, or equivalently if $\rho'(o) = 0$, where $\rho' = d\rho/dt$. All the other points are called *ordinary points* of ρ on M.

It's noteworthy to recall that the partial derivatives ρ_j are defined on the manifold M, while ρ^i - the components of $\mathtt{grad}\rho$ - are defined on the slit tangent bundle TM_0. Hence, $\mathtt{grad}\rho$ can be considered as a section of $\pi^*TM \to TM_0$, the pulled-back tangent bundle over TM_0, and its trajectories lie in TM_0.

Let the Finsler manifold (M, g) admit a non-trivial solution ρ of (1.1); then for any ordinary point $p \in M$ there exists a coordinate neighborhood $\mathcal{U}$ of p which contains no critical point, and where we can choose a system of coordinates $(u^1 = t, u^2, ..., u^n)$ having the following properties, cf. [1]:

- the function ρ depends only on the first variable $u^1 = t$ on $\mathcal{U}$;

- the integral curve of $\mathtt{grad}\rho$ is a geodesic; any geodesic containing such a curve is called a *ρ-curve* or a *t-geodesic* of ρ;

- the connected component of a regular hyper-surface defined by $\rho = constant$, is called a *level set of* ρ or simply a t-level. Given a solution ρ and a point $q \in \mathcal{U}$, there exists one and only one t-level set of ρ passing through q. The t-geodesics form the normal congruence to the family of t-level sets of ρ;

- the curves defined by u^α =const are t-geodesics of ρ, and the parameter $u^1 = t$ may be regarded as the arc-length parameter of t-geodesics;

- the components g_{ij} of the Finsler metric tensor g satisfy $g_{\alpha 1} = g_{1\alpha} = 0$, where the Greek indices α, run over the range $2, 3, ..., n$ and the Latin indices i, j, run over the range $1, 2, ..., n$;

- in adapted coordinates the first fundamental form of (M, g) is given by

$$ds^2 = (dt)^2 + \rho'^2 f_{\gamma\beta} du^\gamma du^\beta, \tag{2.4}$$

where $f_{\gamma\beta}$ given by $g_{\gamma\beta} = \rho'^2 f_{\gamma\beta}$ are the components of the metric tensor on a t-level of ρ and $g_{\gamma\beta}$ is the induced metric tensor of this t-level.

For more details about our purpose in adapted coordinates, we refer to [1, 3, 10].

3 Proof of Theorem 1

Let (M, g) be a an n-dimensional $n \geq 2$ Finsler manifold which admits a non trivial C^∞ solution ρ of the equation(1.1). Consider the so called t-geodesic which is integral curve of the gradient vector field $\mathtt{grad}\rho$ on M. It is well known that every t-geodesic is a geodesic on M.

Since M is compact, by the extension of Extreme Value Theorem to differentiable manifolds, every solution ρ of the equation (1.1) is bounded and attains its extremum values on M. Once the assumption is made that M is without boundary, the differentiability of ρ requires that these extremal values are critical points.

Let O be a critical point for a t-geodesic on M. By compactness, M must have finite diameter D and no t-geodesic longer than D may remain minimizing. Thus every t-geodesic longer than D emerging from O contains at least two critical points.

Before proceeding further, we shall recall that on a Finsler manifold there exist no more than two critical points of ρ on every t-geodesic emanating from O, cf. [1]. Therefore, every t-geodesic on (M, g) contains exactly two critical points.

Thus, by means of Theorem A, (M, g) is conformal to an n-dimensional sphere in the Euclidean space $\mathbb{R}^{n+1}$, with the first fundamental form (2.4). Moreover, M is assumed to be simply connected and an extension of the Milnor theorem to Finslerian category, cf. [7], implies that M is topologically homeomorphic to the sphere S^n.

Conversely, let (M, g) be compact and conformally homeomorphic to the n-sphere $S^n \subset \mathbb{R}^{n+1}$. The first fundamental form of S^n is given by

$$g_{_{S^n}} = dt^2 + \sin^2 t g_{_{S^{n-1}}}, \tag{3.1}$$

where $g_{_{S^{n-1}}}$ is the first fundamental form of the hypersphere S^{n-1}, cf. [9]. Let $\gamma := x^i(t)$ be a geodesic on (M, g), by definition its differential equation is given by

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = \varphi\frac{dx^i}{dt}, \tag{3.2}$$

where t is an arbitrary parameter and φ is a function of t. If we consider the vector fields on TM for which the projection of their integral curves on M is γ, then we can put $\frac{dx^j}{dt} = \gamma^j$, and by virtue of the equation (3.2) we have

$$\gamma^k\frac{d\gamma^l}{dx^k} + \Gamma^l_{jk}\gamma^j\gamma^k = \varphi\gamma^l. \tag{3.3}$$

This is equivalent to $\gamma^k(\nabla_k\gamma^l) = \varphi\gamma^l$, where ∇_k is the Cartan h-covariant derivative. Denoting $\gamma_i := g_{il}\gamma^l$ and contracting with g_{il}, we get $\gamma^k(\nabla_k\gamma_i) = \varphi\gamma_i$, which leads to

$$\gamma^k(\nabla_k\gamma_i - \varphi g_{ik}) = 0. \tag{3.4}$$

The conformality assumption of (M, g) to the standard sphere (S^n, g_{S^n}) implies that the Finsler metric g is positively proportional to g_{S^n}, that is $g = e^{2\psi}g_{S^n}$ where, by the Knebelman theorem, ψ is a function on M. Therefore, g is also a function on M and hence a Riemannian metric. By compactness of M, the vector field γ^k is complete and the equation (3.4) leads to $\nabla_k\gamma_i = \varphi g_{ik}$, which is equivalent to the equation (1.1). This completes the proof of Theorem. □

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