On the holonomy algebra of manifolds with pure curvature operator

M. Helena Noronha

Abstract. We study the holonomy algebra of Riemannian manifolds with pure curvature operator. We conclude that locally irreducible Kähler manifolds of dimension greater than four do not have pure curvature operator. A similar result is obtained for compact locally irreducible Kähler four-manifolds of nonnegative scalar curvature. We also study compact Riemannian manifolds with pure curvature operator and some special curvature conditions.

M.S.C. 2010: 53C21, 53C42.
Key words: pure curvature operator; holonomy algebra; Kähler manifolds; Weyl tensor.

1 Introduction

A Riemannian manifold is said to have *pure curvature tensor* if for every $p \in M$ there is an orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space such that the $R_{ijlk} = \langle R(e_i, e_j)e_l, e_k \rangle = 0$, whenever at least two of the indices $\{i, j, k, l\}$ are distinct. Let $\Lambda^2(T_pM)$ denote the exterior product of the tangent space T_pM endowed with its natural inner product, that is, $\{e_{ij}\}_{i<j}$ is an orthonormal basis of T_pM, where e_{ij} denotes the 2-form $e_i \wedge e_j$. It follows easily that pure curvature tensor implies that $\{e_{ij}\}_{i<j}$ is a basis of eigenvectors for the symmetric curvature operator $\mathcal{R} : \Lambda^2(T_pM) \to \Lambda^2(T_pM)$ given by

$$\mathcal{R}(e_{ij}) = \frac{1}{2} \sum_{k,l} R_{ijlk} e_{kl}.$$

We say in this case that M has *pure curvature operator* and call $\{e_1, \ldots, e_n\}$ an $\mathcal{R}$-*basis* . Conformally flat manifolds have pure curvature operator, since their Weyl tensor is zero. Other examples of Riemannian manifolds with pure curvature operator are all thee-manifolds and manifolds that admit an isometric immersion into a Space Form with zero normal curvature ($R^{\perp} = 0$) and, in particular, hypersurfaces of Space Forms.

Balkan Journal of Geometry and Its Applications, Vol.17, No.1, 2012, pp. 88-94.

In [4] we studied compact manifolds of pure curvature operator with nonnegative isotropic curvature. We concluded that the Betti numbers $b_p(M) = 0$, for $2 \leq p \leq n-2$. This result is proved using the Bochner technique. We show first that the condition

$$K_{ik} + K_{im} + K_{jk} + K_{jm} \geq 0, \tag{1.1}$$

for all sets of orthonormal vectors e_i, e_j, e_m, e_k in T_xM, where K_{ik} denotes the sectional curvature of the plane spanned by e_i, e_k, implies the nonnegativity of the p-Weintzenböck operator in the case of pure curvature tensor. Since nonnegative isotropic curvature implies Inequality (1.1), the result follows from the holonomy principle. Therefore, a crucial step to conclude the proof is the result that the holonomy algebra of manifolds of pure curvature operator and nonnegative isotropic curvature is the orthogonal algebra $o(n)$ (see Proposition 3.1 of [4]).

In this article we relax the condition on isotropic curvature and study the holonomy algebra of Riemanian of manifolds of pure operator, generalizing Proposition 3.1 of [4]. In fact, we prove:

Theorem 1.1. *Let M^n be a manifold with pure curvature operator. Then its universal cover $\tilde{M}$ splits into a Riemannian product $N_1^{n_1} \times \cdots \times N_k^{n_k} \times \mathbb{R}^m$, where $\mathbb{R}^m$ has its standard flat metric and the holonomy algebra h_i of N_i is one of the following:*
(i) The orthogonal algebra $o(n_i)$
(ii) The unitary algebra $u(2)$ and $n_i = 4$.

This result has some consequences. For instance, it implies that *locally irreducible manifolds of dimension greater than four and restricted holonomy group other than $SO(n)$ do not have pure curvature tensor*, and in particular, *a locally irreducible Kähler manifold of dimension greater than four does not have pure curvature operator.* This generalizes the well-known result that conformally flat Kähler manifolds of dimension greater than four are flat (see [9]). Another consequence is the following theorem:

Theorem 1.2. *Let $M^n, n \geq 4$, be a compact locally irreducible manifold with pure curvature operator. Suppose that the sectional curvatures of M satisfy one of the following conditions:*
(i) $K_{ik} + K_{im} \geq 0$ for all sets of orthonormal vectors e_i, e_m, e_k
(ii) $K_{ik} + K_{jm} \geq 0$ for all sets of orthonormal vectors e_i, e_j, e_m, e_k.
Then the Betti numbers $b_p(M) = 0$, for $2 \leq p \leq n-2$.

We now restrict ourselves to Kähler four-manifolds of pure curvature tensor. Our first result is:

Theorem 1.3. *Let M be a compact Kähler four-manifold with pure curvature operator and nonnegative scalar curvature. Then the universal covering of M is either $\mathbb{R}^4$ with its flat metric or a product of two surfaces.*

Recall that manifolds covered by the product of two surfaces of opposite constant curvature are conformally flat and kählerian. Matsushima in [6]) and Tanno in [8] proved that if $n \geq 4$ and the divergence of the Weyl tensor of a Kähler manifold is zero ($\delta W = 0$), then its Ricci tensor is parallel. It follows that product of two surfaces of opposite constant curvature are the only conformally flat Kähler four-manifolds. Assuming $\delta W = 0$, as Matsushima and Tanno, we prove the following:

Theorem 1.4. *Let M be a Kähler four-manifold with pure curvature operator. Suppose $\delta W = 0$. Then its universal cover $\tilde{M}$ splits in a Riemannian product of two surfaces of constant curvature.*

For the general case we prove:

Theorem 1.5. *Let M be a Kähler four-manifold with pure curvature operator. Then there exists an open and dense set U of M such that for every $p \in M$ there exists a neighborhood V of p in U that contains a totally geodesic surface S immersed in V with flat normal bundle. Moreover, if the scalar curvature is identically zero then M is conformally flat.*

2 The holonomy algebra

Before we prove Theorem 1.1, we recall some well known facts about the orthogonal algebra $o(U)$, where U is a vector space. First, $o(U)$ is an algebra with respect to the Lie bracket

$$[e_{ij}, e_{km}] = \delta_{im} e_{kj} + \delta_{jm} e_{ik} + \delta_{ik} e_{jm} + \delta_{jk} e_{mi}. \tag{2.1}$$

We also recall the Lemmas below and refer the reader to [3] for their proofs.

Lemma 2.1. *Let v be a non-zero element of U. Then*

$$vU = \{v \wedge u \mid u \in U\}$$

generates $o(U)$.

Lemma 2.2. *Let $U = V + W$ with $V = W^{\perp}$ and not both have dimension two. Then $o(V) + o(W)$ is a maximal proper subalgebra of $o(U)$.*

2.1 Proof of Theorem 1.1

Let $r(x)$ denote the Lie algebra generated by $Im\mathcal{R} \subset \Lambda_x(M)$, where $Im\mathcal{R}$ denote the image of $\mathcal{R}$ and h the holonomy algebra of M. It is well known that $r(x)$ is a subalgebra of h for all $x \in M$ (see [1] for instance).

If $h = o(n)$, then the restricted holonomy group of M is irreducible and so is the universal cover $\tilde{M}$. If $h \neq o(n)$, which implies that $r(x) \neq o(n)$, for all $x \in M$, let us consider an $\mathcal{R}$-*basis* $\{e_1, \ldots, e_n\}$ and let K_{ij} denote the eigenvalues of $\mathcal{R}$, that is, the sectional curvature of the plane spanned by e_i, e_j.

For x such that $r(x) \neq 0$, we reorder the indices and suppose $K_{1i} \neq 0, i = 2, ..., k_1$ and $K_{1i} = 0, i > k_1$. Let $V_1 = span\{e_1, \ldots, e_{k_1}\}$. Since $r(x) \neq 0$, $k_1 < n$ and after reordering the indices, we define

$$V_2 = span\{\{e_{k_1+1}\} \cup \{e_i \in V_1^{\perp} \mid K_{k_1+1,i} \neq 0\}\}.$$

Let $W_2 = V_1 + V_2$ and let $k_2 = dimW_2$. Since

$$e_{1i}, e_{k_1+1,j} \in Im\mathcal{R}, \quad 1 < i \leq k_1, \quad k_1 + 1 \leq j \leq k_2,$$

it follows from (2.1) that for all r and s such that $r < s \leq k_1$, $\forall k_1 + 1 \leq r < s \leq k_2$, it holds that

$$e_{rs} \in r(x).$$

If for some $e_i \in V_1$ and $e_j \in V_2$, $K_{ij} \neq 0$, that is, $e_{ij} \in Im\mathcal{R}$, we conclude that $e_{rs} \in r(x)$ for all $1 \leq r < s \leq k_2$ and $o(W_2) \subset r(x)$. By continuing this procedure, our assumption that $r(x) \neq o(n)$ implies that there exists a subspace $V_m = span\{e_{k_{m-1}+1}, \dots, e_{k_m} = e_n\}$ such that $K_{ij} = 0$, for all $e_i \in W_{m-1} = V_1 + \cdots + V_{m-1}$ and all $e_j \in V_m$ and $o(W_{m-1}) \subset r(x)$.

Now we have $T_xM = W_{m-1} + V_m$ and $W_{m-1}^{\perp} = V_m$. If both W_{m-1} and V_m have dimension two, then the previous paragraph shows that $r(x) = o(2) + o(2)$ and we conclude that either $h = o(4)$, $h = o(2) + o(2)$ or $h = u(2)$. If not both have dimension two, since $r(x) \neq o(n)$, Lemma 2.2 implies that $r(x) = o(W_{m-1}) + r_1(x)$, where $r_1(x)$ is a subalgebra of $o(V_m)$.

We then repeat the procedure above for the space V_m and obtain that $r_1(x)$ is either $o(dimV_m)$, or $o(l) + r_2(x)$, where $r_2(x)$ is a subalgebra of $o(dimV_m - l)$, or $o(2) + o(2)$ and $dimV_m = 4$.

Now it is easy to conclude that $r(x) = o(n_1) + \cdots + o(n_i) + o(2) + \cdots + o(2)$ and the de Rham decomposition theorem implies that the holonomy algebra of each non-flat factor N_i of the universal cover of M is $o(n_i)$ or $u(2)$ and $dimN_i = 4$. □

2.2 Proof of Theorem 1.2

Let a ω be a p-form ω. The Weintzenböck formula is given by

$$(\Delta\omega, \omega) = \sum_{i=1}^{n} (\nabla_{X_i}\omega, \nabla_{X_i}\omega) + (Q_p\omega, \omega),$$

where

$$(Q_p\omega, \omega) = \int_M \langle Q_p\omega(x), \omega(x)\rangle dM.$$

If M has pure curvature operator and $\{e_i\}, i = 1, \dots, n$ is an $\mathcal{R}$-*basis*, the formula above simplifies to (see [4], Section 4).

$$\begin{aligned} Q_p(e_{i_1} \wedge \dots \wedge e_{i_p}) &= -\textstyle\sum_{s<t, s\in\{i_1,\dots,i_p\}, t\notin\{i_1,\dots,i_p\}} K_{st}(-e_{i_1} \wedge \dots \wedge e_{i_p}) \\ &= (\textstyle\sum_{h=1,k=p+1}^{p,n} K_{i_h i_k}) e_{i_1} \wedge \dots \wedge e_{i_p}. \end{aligned}$$

Note that the hypotheses of the theorem implies Inequality (1.1), which in turn implies that Q_p is nonnegative for all $2 \leq p \leq n-2$ (see Lemma 2.2 of [4]). Since, for $n \geq 5$, Theorem 1.1 implies that the only possibility for the holonomy group G of M is $SO(n)$. Therefore, if $\beta_p(M) > 0$ for $2 \leq p \leq n-2$, there would exist a parallel p-form ω that would be left invariant by $SO(n)$. But, by the holonomy principle, the existence of such ω would give rise to a parallel and hence harmonic p-form on the sphere S^n, which is a contradiction.

Now, for the case of $n = 4$, if the holonomy group G of M is $SO(4)$ we conclude again that $b_2(M) = 0$. If $b_2(M) > 0$, then the holonomy group G of M is the unitary group $U(2)$. The Weintzenböck formula implies that an harmonic 2-form ω is parallel. Since the Complex Projective Plane $\mathbb{C}P^2$ also has holonomy $U(2)$, if $b_2(M) > 1$, each of these parallel 2-forms would give rise to a parallel 2-form in $\mathbb{C}P^2$, implying that $b_2(\mathbb{C}P^2) > 1$, which is a contradiction. Therefore we get that $b_2(M) = 1$. It follows that the signature of M, $\sigma(M) = \pm 1$. But the Signature Theorem of Hirzebruch states that $\sigma(M)$ is a linear function of the Pontrjagin numbers of M (see [7], p. 224) and manifolds of pure curvature tensor have zero Pontrjagin forms (see [2] p. 439 or [5])). We then have a contradiction. Therefore $b_2(M) = 0$. □

3 Four dimensional Kähler manifolds with pure curvature operator

The proof of Theorem 1.1 shows that if M^4 is a Kähler manifold with pure curvature operator then the algebra $r(x)$ generated by $Im\mathcal{R}$ is contained in $o(2) + o(2)$. Thus, henceforth, $\{e_1, e_2, e_3, e_4\}$ denotes the $\mathcal{R}$- basis and we will assume that $K_{13} = K_{23} = K_{14} = K_{24} = 0$.

Proposition 3.1. *Let us suppose $r(x) = o(2) + o(2)$. Then there exists an open set V containing x that splits as a Riemannian product of two surfaces.*

Proof. With the convention above, if $r(x) = o(2) + o(2)$ we have that $K_{12} \neq 0$ and $K_{34} \neq 0$. Let us consider an open set V containing x such that K_{12} and K_{34} do not vanish on V. This defines two orthogonal distributions $D_1 = span\{e_1, e_2\}$ and $D_2 = span\{e_3, e_4\}$ on the tangent bundle of V. We will show that they are both parallel and involutive and the proposition will follow from Frobenius theorem. For that we will consider the Second Bianchi Identity:

$$\left[\nabla_{e_k} R(e_1, e_2) + \nabla_{e_2} R(e_k, e_1) + \nabla_{e_1} R(e_2, e_k)\right](e_1, e_l) = 0.$$

Expanding this expression and taking into account that $\langle R(e_i, e_j)e_l, e_k\rangle = 0$, if $\{i, j, k, l\}$ contains more than two elements, we are left with

$$\langle R(e_1, e_2)e_1, \nabla_{e_k} e_l\rangle = 0.$$

Therefore, if $K_{12} \neq 0$, we get that

$$\langle e_2, \nabla_{e_k} e_l\rangle = 0, \qquad \forall k, l > 2.$$

Similarly we obtain

$$\begin{aligned} \langle e_1, \nabla_{e_k} e_l\rangle &= 0, \quad \forall k, l > 2 \\ \langle e_i, \nabla_{e_k} e_l\rangle &= 0, \quad \forall k, l < i, i = 3, 4, \end{aligned}$$

which completes the proof. □

3.1 Proof of Theorem 1.3

With the convention above, the Scalar Curvature S is given by $S = K_{12} + K_{34}$. Therefore $S \geq 0$ implies that the Weintzenböck operator Q_2 is nonnegative. In this case, the arguments used in the last part of the proof of Theorem 1.2 shows that M

cannot have holonomy group $U(2)$ and hence M is locally reducible. If M is not flat and there is a point x such that $r(x) = o(2) + o(2)$, then the universal covering of M splits as a a Riemannian product of two surfaces. If $r(x) = o(2)$ for all $x \in M$, then the universal covering of M also splits as a a Riemannian product of two surfaces and one factor is $\mathbb{R}^2$ with its flat metric. □

3.2 Proof of Theorem 1.4

The fact that $K_{13} = K_{23} = K_{14} = K_{23} = 0$ implies that $Ric(e_1, e_1) = Ric(e_2, e_2) = K_{12}$ and $Ric(e_3, e_3) = Ric(e_4, e_4) = K_{34}$. Since $\delta W = 0$ for Kähler manifolds imply that the Ricci tensor is parallel, we have that either M is Einstein or (locally) a product of Einstein manifolds. The latter case implies that the universal covering $\tilde{M}$ is the Riemannian product of two surfaces of constant curvature. If M is Einstein and the scalar curvature $S \neq 0$ then Proposition 3.1 implies that $\tilde{M}$ splits in the Riemannian product of two surfaces with same constant curvature. If $S = 0$, then M is flat. □

3.3 Proof of Theorem 1.5

Recall that the Weyl tensor W is given by

$$W(X,Y)Z = R(X,Y)Z - \langle Y, Z\rangle B(X) + \langle B(X), Z\rangle Y + \langle X, Z\rangle B(Y) - \langle B(Y), Z\rangle X,$$

where

$$B(X) = \frac{1}{2}\big[Ric(X) - \frac{S}{6}X\big].$$

Note that we have $\langle W(e_i, e_j)e_l, e_k\rangle = 0$, if $\{i, j, k, l\}$ contains more than two elements and denoting by $W_{ij} = \langle W(e_i, e_j)e_i, e_j\rangle$, we have

$$W_{12} = W_{34} = \tfrac{1}{3}(K_{12} + K_{34})$$
$$W_{13} = W_{14} = W_{23} = W_{24} = -\tfrac{1}{6}(K_{12} + K_{34}).$$

Therefore $S \equiv 0$ implies $W \equiv 0$. Now we consider the open dense subset U of M so that each point $p \in U$ has a neighborhood V with the property that $dim\ Im\mathcal{R}$ is constant on V. The case $dim\ Im\mathcal{R} = 2$ implies that V is the product of two surfaces by Proposition 3.1. If $dim\ Im\mathcal{R} = 1$, say, $K_{12} \neq 0$, then Proposition 3.1 implies that leaves S of the integrable distribution $span\{e_3, e_4\}$ are totally geodesic. Since $\langle R(e_3, e_4)e_1, e_2\rangle = 0$, the Ricci equation implies that $i : S \to V$ has flat normal bundle. □

References

[1] W. Ambrose, I. Singer *A theorem on holonomy*, Trans. Amer. Math Soc. 75 (1953), 428-443.

[2] A. Besse, *Einstein Manifolds*, Ergeb. Math. Grenzgeb (3) 10, Springer-Verlag, Berlin, 1987.

[3] R. L. Bishop, *The holonomy algebra of immersed manifolds of codimension two*, J. Diff. Geom. 6 (1971), 119-128.

[4] M. Dussan, H. Noronha,*Compact manifolds of nonnegative isotropic curvature and pure curvature tensor* Balkan J. Geom. Appl. (BJGA) 10, 2 (2005), 58-66.

[5] H. Maillot, *Sur les varietes riemannienness a operateur de courbure pur*, C.R. Acad. Sci. Paris A 278 (1974), 1127-1130.

[6] Y. Matsushima, *Remarks on Kähler-Einstein manifolds*, Nagoya Math. J. 46 (1972), 161-173.

[7] J. Milnor, J. Stasheff, *Characteristic Classes*, Ann. of Math. Studies 76, Princeton, 1974.

[8] S. Tanno, *Curvature tensors and covariant derivatives*, Ann. Mat. Pura Appl. 96 (1973), 233-241.

[9] K. Yano, I. Mogi, *On real representations of Kählerian manifolds*, Ann. of Math 61 (1995), 170-189.

Maria Helena Noronha
Department of Mathematics,
California State University Northridge,
Northridge, CA 91330-8313, USA.
E-mail: maria.noronha@csun.edu