Dirac operators over the flat 3-torus

J. Fabian Meier

Abstract. We determine spectrum and eigenspaces of some families of $\mathrm{Spin}^{\mathbb{C}}$ Dirac operators over the flat 3-torus. Our method relies on projections onto appropriate 2-tori on which we use complex geometry.

Furthermore we investigate those families by means of spectral sections (in the sense of Melrose/Piazza). Our aim is to give a hands-on approach to this concept. First we calculate the relevant indices with the help of spectral flows. Then we define the concept of a *system of infinitesimal spectral sections* which allows us to classify spectral sections for small parameters R up to equivalence in K-theory. We undertake these classifications for the families of operators mentioned above.

Our aim is therefore twofold: On the one hand we want to understand the behavior of $\mathrm{Spin}^{\mathbb{C}}$ Dirac operators over a 3-torus, especially for situations which are induced from a 4-manifold with boundary T^3. This has prospective applications in generalized Seiberg-Witten theory. On the other hand we want to make the term "spectral section", for which one normally only knows existence results, more concrete by giving a detailed description in a special situation.

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1 Introduction

In the study of smooth 4-manifolds, especially in the context of (generalized) Seiberg-Witten theory, it would be nice to understand $\mathrm{Spin}^{\mathbb{C}}$ Dirac operators which are induced on the boundary of a compact 4-manifold.

Manifolds with boundary T^3 where already studied in this context by [5]. But for generalized Seiberg-Witten theories, also families of operators in non-trivial $\mathrm{Spin}^{\mathbb{C}}$ structures become important. Therefore, we undertake a detailed study for some of these families. We now describe the object of investigation:

For every $\mathrm{Spin}^{\mathbb{C}}$ structure on $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ we analyse the family of Dirac operators given by connections $\nabla^K + \mathtt{i}\alpha$; here ∇^K is a fixed background connection (to be

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constructed below) for an appropriate line bundle K and α comes from the parameter space of closed one-forms.

Our first aim is to determine the spectrum and an orthogonal eigenbasis for these operators. Our strategy is as follows:

1. We write the 3-torus as S^1 bundle over a 2-torus (determined by the $\mathrm{Spin}^{\mathbb{C}}$ structure).

2. We equip the 2-torus with a complex structure and choose appropriate holomorphic line bundles.

3. We use complex geometry and methods from [1].

4. We combine the calculated terms with exponential functions to get the desired result.

The calculations above will help us to access our second aim: The construction of spectral sections.

For a lattice $\ell \subset H^1(T^3;\mathbb{Z}) \subset H^1(T^3;\mathbb{R})$ look at the family of operators parametrized by $B = \big(\ell \otimes \mathbb{R}\big)/\ell$. Since we know the concrete spectrum we can calculate all spectral flows in this torus which gives us direct access to the index in $K^1(B)$. By [4, section 2] the vanishing of this index corresponds to the existence of spectral sections.

For small parameters R we give a classification of all spectral sections up to equivalence in K-theory.

Remark 1.1. If $\iota : T^3 \hookrightarrow M$ is the boundary of a $\mathrm{Spin}^{\mathbb{C}}$ 4-manifold M and ℓ is chosen to be a subset of $\iota^*\big(H^1(M;\mathbb{Z})\big)$, then one can show that our family of operators is a boundary family in the sense of [4]; this guarantees the existence of spectral sections in this case but does not lead to concrete constructions of them.

2 Definitions

We take $T^3 := \mathbb{R}^3/\mathbb{Z}^3$ to be the flat 3-torus. We identify the first and second cohomology groups with each other by the Hodge star operation. Both of them will be identified with $\mathbb{Z}^3$ or $\mathbb{R}^3$ through the standard (positively oriented) basis dx_1, dx_2, dx_3 of $\mathrm{T}\mathbb{R}^3$.

The trivial Spin structure induces a $\mathrm{Spin}^{\mathbb{C}}$ structure with associated bundle $\underline{\mathbb{H}} = T^3 \times \mathbb{H}$. Here $\mathbb{H} = \mathrm{span}\{e_0, e_1, e_2, e_3\}$ denotes the space of quaternions. It is considered as a complex vector space by left multiplication with $\mathtt{i} = e_1$ and as a left-quaternionic vector space by inverse right multiplication.

Now the $\mathrm{Spin}^{\mathbb{C}}$ structures can be canonically identified with elements $\hat{k} \in H^2\big(T^3;\mathbb{Z}\big)$ (for a general explanation of $\mathrm{Spin}^{\mathbb{C}}$ structures and their associated bundles see e.g. [6]). For every such element we choose a Hermitian line bundle K with $c_1(K) = \hat{k}$ and a unitary background connection ∇^K; possible choices and constructions will be detailed in the subsequent sections. Then the $\mathrm{Spin}^{\mathbb{C}}$ structure $\hat{k}$ has the associated bundle $\mathbb{H} \otimes K$.

For each K and closed one-form α we get a $\mathrm{Spin}^{\mathbb{C}}$ Dirac operator

$$\mathcal{D}^K_\alpha : \Gamma\big(\underline{\mathbb{H}} \otimes K\big) \to \big(\underline{\mathbb{H}} \otimes K\big)$$

for the connection $\nabla^K + \mathtt{i}\alpha$.

These operators will be analysed in the subsequent sections.

3 Spectrum and Eigenbasis

We distinguish two main cases.

3.1 Nontrivial Spin$^{\mathbb{C}}$ structure

We write $\hat{k} = h \cdot k$ with $k \in \mathbb{Z}^3$ and maximal $h \in \mathbb{Z}^+$. Let W be the plane in $\mathbb{R}^3$ orthogonal to k and π_k the orthogonal projection. By taking quotients we get a map $\pi_{\overline{k}} : T^3 \to T_\Lambda := W/\Lambda$ with $\Lambda = \pi_k(\mathbb{Z}^3)$.

Let w_1, w_2 be the basis of a fundamental parallelogram in Λ. We take $c^i \in [0,1)$, $i = 1,2$, with $w_i - c^i \cdot k \in \mathbb{Z}^3$.

Lemma 3.1. *The map $\pi_{\overline{k}} : T^3 \to T_\Lambda$ determines a trivial $\mathbb{R}/\mathbb{Z}$-bundle with trivialization:*

$$\begin{array}{ccc} T^3 & \overset{\pi_{\overline{k}}\times \mathrm{tri}}{\longrightarrow} & T_\Lambda \times \mathbb{R}/\mathbb{Z} \\ \Big[\chi_1 w_1 + \chi_2 w_2 + \chi k\Big] & \mapsto & \Big(\big[\chi_1 w_1 + \chi_2 w_2\big], \big[c^1\chi_1 + c^2\chi_2 + \chi\big]\Big). \end{array} \tag{3.1}$$

Proof. Direct calculation. □

We give T_Λ the induced metric and orientation and choose a Hermitian line bundle L over it with $c_1(L) = h$ (in the standard identification of $H^2\big(T_\Lambda; \mathbb{Z}\big)$ with $\mathbb{Z}$). Furthermore, we equip the bundle with an arbitrary unitary connection ∇^L.

Definition 3.1. We define $K := \pi_{\overline{k}}^{-1}(L)$ and $\nabla^K := \pi_{\overline{k}}^{-1}\big(\nabla^L\big)$. Then we have $c_1(K) = \hat{k}$.

3.1.1 Working on T_Λ

We now look at the corresponding problem on T_Λ. For each (positive) Chern class h, we have an associated bundle $\underline{\mathbb{H}} \otimes L$ over T_Λ. Then each closed one-form α_Λ over T_Λ defines a Dirac operator

$$\mathcal{D}^L_{\alpha_\Lambda} : \Gamma\big(\underline{\mathbb{H}} \otimes L\big) \to \big(\underline{\mathbb{H}} \otimes L\big).$$

We give W an arbitrary complex structure and scale everything so that we work on $\mathbb{C}/_{\{1,\tau\}}$ with $\operatorname{im}\tau > 0$. Now we can equip L with a *holomorphic* structure; we choose it so that $\nabla^L + \mathtt{i}\alpha_\Lambda$ becomes the Chern connection of the holomorphic bundle.

This specifies a problem for twisted Dirac operators on a Riemann surface. We use the results of [1, section 5.2], where the eigenspaces of $\mathcal{D}^L_{\alpha_\Lambda}$ are described in terms of holomorphic sections.

The eigenspaces can be made explicit using theta functions. A detailed discussion of all calculations and identifications can be found in [3, section 2.c]. The result is the following:

Lemma 3.2. *We can explicitly construct a basis of orthogonal eigensections σ_m, $m \in \mathbb{Z}$, for $\mathcal{D}^L_{\alpha_\Lambda}$ with respective eigenvalues*

$$\mu_m := \operatorname{sgn} m \sqrt{2\pi h \|k\| \left\lfloor \frac{|m|}{h} \right\rfloor}.$$

The eigenvalues are independent of α_Λ.

3.1.2 An eigenbasis for $(\mathcal{D}^K_\alpha)^2$

Remark 3.2. By a standard gauging argument, we can reduce the problem of finding spectrum and eigenspaces from closed one-forms to harmonic one-forms. So from now on we assume $\alpha \in H^1(T^3; \mathbb{R}) \cong \mathbb{R}^3$.

We now look at the map $s_l \circ \mathrm{tri}$, $l \in \mathbb{Z}$, where $s_l : \mathbb{R}/\mathbb{Z} \to S^1$ is defined to be $t \mapsto \exp(2\pi l t)$ and tri is the map from (3.1). Its exterior derivative is given by:

$$d\big(s_l \circ \mathrm{tri}\big) = 2\pi \mathtt{i} l \big(s_l \circ \mathrm{tri}\big)\,\big(c^1,\, c^2,\, 1\big).$$

We now want to separate this form into its parallel and orthogonal part with respect to W:

$$d\big(s_l \circ \mathrm{tri}\big) = 2\pi \mathtt{i} (s_l \circ \mathrm{tri}) \cdot \big(\omega^l_{\shortparallel} + \omega^l_{\perp}\big),$$

In the same way we split $\alpha = \alpha_{\shortparallel} + \alpha_\perp$.

We set $\alpha_\Lambda := \alpha_{\shortparallel} + 2\pi\omega^l_{\shortparallel}$ and use Lemma 3.2 to determine a basis of sections for $\Gamma(\underline{\mathbb{H}} \otimes L)$ which we call σ^l_m, $m \in \mathbb{Z}$.

The parameter $\omega^l_{\shortparallel}$ becomes necessary for our construction since the bundle $T^3 \to T_\Lambda$ is trivial but its metric differs from the orthogonal product $T_\Lambda \times S^1$.

We further denote

$$\hat\sigma_{l,m}(v) := (s_l \circ \mathrm{tri})(v) \cdot \pi^*_{\bar{k}}\big(\sigma^l_m\big)(v).$$

This can be interpreted as a combination of a basis of the Dirac operator over S^1 with bases over T_Λ. Let

$$\lambda_l := \big(2\pi l + \langle k,\, \alpha\rangle\big)/\|k\|,$$

where $\langle\,,\,\rangle$ means the standard scalar product of $\mathbb{R}^3$ (or, interpreted differently, the evaluation of $k \cup \alpha$ at the orientation class).

Theorem 3.3 (Eigenbasis for $(\mathcal{D}^K_\alpha)^2$). *The set $\big\{\hat\sigma_{l,m} \,\big|\, l, m \in \mathbb{Z}\big\}$ forms an orthogonal basis of eigensections for $(\mathcal{D}^K_\alpha)^2$ with the respective eigenvalues $\lambda_l^2 + \mu_m^2$.*

Proof. Applying $\mathcal{D}^K_\alpha$ twice and using the definition of ω^l, we see that these sections are indeed eigensections for the given eigenvalues. With a standard calculation (see [3, p.45]), we conclude that the set span $\big\{\hat\sigma_{l,m} \,\big|\, l, m \in \mathbb{Z}\big\}$ is dense in the space of L^2-sections. The orthogonality can be deduced from the orthogonality of the σ^l_m by using the fact that a change of $\alpha_\perp$ changes the spectrum but fixes σ^l_m. □

3.1.3 An eigenbasis for $\mathcal{D}_\alpha^K$

Theorem 3.3 gives a quadratic equation for $\mathcal{D}_\alpha^K$. Furthermore, we know that the Dirac operator on T_Λ is graded, so the bases σ_m^l split into $\sigma_m^{l+} + \sigma_m^{l-}$. Together this leads us to the following denominations

$$\begin{aligned}\sigma_{l,m}^{\pm} &:= (s_l \circ \mathrm{tri}) \cdot \Big(\Big(\lambda_l + \mu_m \pm \sqrt{\lambda_l^2 + \mu_m^2} \Big) \pi_{\bar{k}}^*(\sigma_m^{l+}) \\ &\qquad + \Big(-\lambda_l + \mu_m \pm \sqrt{\lambda_l^2 + \mu_m^2} \Big) \pi_{\bar{k}}^*(\sigma_m^{l-}) \Big) \\ \sigma_{l,m}^0 &:= \hat{\sigma}_{l,m}\end{aligned}$$

and

$$\nu_{l,m}^{\pm} := \pm\sqrt{\lambda_l^2 + \mu_m^2}, \qquad \nu_{l,m}^0 := \begin{cases} \lambda_l & \text{for } 0 \le m \le h-1 \\ \mu_m & \text{otherwise.} \end{cases}$$

From this set of vectors we have to choose a subset of nonzero vectors whose span is dense.

Theorem 3.4. *We get an orthogonal eigenbasis of $\mathcal{D}_\alpha^K$ by*

$$\Big\{ \sigma_{l,m}^{\pm} \,\Big|\, (l,m) \in \mathbb{Z}^2 \quad \textit{with } \lambda_l \neq 0 \textit{ and } m \ge h \Big\} \cup \Big\{ \sigma_{l,m}^0 \,\Big|\, (l,m) \in \mathbb{Z}^2 \quad \textit{with } \lambda_l = 0 \textit{ or } 0 \le m \le h-1 \Big\},$$

which will be written as $M_\alpha^{\pm} \cup M_\alpha^0$. The respective eigenvalues are $\nu_{l,m}^{+/0/-}$.

Proof. We check that all these vectors are nonzero and belong to the defined eigenspaces.

From the construction in [1] we know that $\sigma_m^l = \sigma_m^{l+} + \sigma_m^{l-}$ implies

$$\sigma_{h-m-1}^l = \sigma_m^{l+} - \sigma_m^{l-}.$$

Therefore, we have the $\mathcal{D}_\alpha^K$-invariant subspaces

$$\operatorname{span}\big\{ \hat{\sigma}_{l,m},\, \mathcal{D}_\alpha^K\, \hat{\sigma}_{l,m} \big\} = \operatorname{span}\big\{ \hat{\sigma}_{l,m},\, \hat{\sigma}_{l,h-m-1} \big\}.$$

They can be used to prove the orthogonality and density of the constructed sections. □

3.2 Trivial Spin$^{\mathbb{C}}$ structure

We look at $\mathcal{D}_\alpha$ on $\Gamma(\underline{\mathbb{H}}) = \Gamma(\underline{\mathbb{C}}^2)$ for the standard connection ∇^K. Let

$$\sigma_b(x_1,x_2,x_3) := \exp\big(2\pi \mathtt{i}(b_1x_1 + b_2x_2 + b_3x_3)\big).$$

Then we get the basis of sections:

$$\operatorname{span}\big\{ \sigma_b^+ = (\sigma_b, 0) \,\big|\, b \in \mathbb{Z}^3 \big\} \cup \big\{ \sigma_b^- = (0, \sigma_b) \,\big|\, b \in \mathbb{Z}^3 \big\}.$$

Define $\beta = \alpha + 2\pi b$. We use the classical methods of [2] to determine:

Theorem 3.5. *We get an orthogonal eigenbasis for $\mathcal{D}_\alpha$ as*

$$\left\{ \|\beta\|\sigma_b^+ - \mathcal{D}_\alpha\sigma_b^+ \,\middle|\, b \in \mathbb{Z}^3 \text{ with } \beta_2 \neq 0 \text{ or } \beta_3 \neq 0 \right\}$$
$$\cup \left\{ \|\beta\|\sigma_b^+ + \mathcal{D}_\alpha\sigma_b^+ \,\middle|\, b \in \mathbb{Z}^3 \text{ with } \beta_2 \neq 0 \text{ or } \beta_3 \neq 0 \right\}$$
$$\cup \left\{ \sigma_b^\pm \,\middle|\, \beta_2 = \beta_3 = 0 \right\}.$$

Furthermore, we have for $\beta_2 \neq 0$ or $\beta_3 \neq 0$:

$$\operatorname{span}\left\{\sigma_b^+,\, \sigma_b^-\right\} = \operatorname{span}\left\{\|\beta\|\sigma_b^+ - \mathcal{D}_\alpha\sigma_b^+,\, \|\beta\|\sigma_b^+ + \mathcal{D}_\alpha\sigma_b^+\right\}.$$

The spectrum consists of all numbers $\pm\|\beta(b,\alpha)\|$ for $b \in \mathbb{Z}^3$.

Remark 3.3. In the case $\hat{k} \neq 0$ the spectrum is determined by $\alpha_\perp$ while the eigenbasis is determined by $\alpha_{\shortparallel}$. Here every change of α has influence on both eigenbasis and spectrum.

4 Spectral sections

We look at families of Dirac operators over a compact base space B. [4] defined the concept of a *spectral section* for a constant $R > 0$. The most interesting spectral sections are those for small R; they should be classified in the sense of the following definition.

Definition 4.1. Let R_{inf} be defined as the infimum of the set

$$\{R > 0 \,|\, \text{for } R \text{ exists at least one spectral section}\}.$$

Furthermore, choose a (small) positive number ε_P. Then a *system of infinitesimal spectral sections* is a map

$$\begin{aligned} \left]R_{\text{inf}}, R_{\text{inf}} + \varepsilon_P\right] \times I \quad &\to \quad \left\{\text{spectral sections for a fixed operator } D\right\} \\ (R, i) \quad &\mapsto \quad P_R^i, \end{aligned}$$

where

1. I is an arbitrary index set,
2. P_R^i is a spectral section for the constant map R,
3. every $\left(P_R^i\right)_\alpha$, $\alpha \in B$, depends continuously on R (where we consider $\left(P_R^i\right)_\alpha$ as operator between L^2 spaces), and
4. $\cup_{i\in I}\{P_R^i\}$ is a representation system for all spectral sections for R, i.e. for all possible spectral sections P_R there is a P_R^i with $i \in I$, so that $\mathtt{Im}\, P_R - \mathtt{Im}\, P_R^i$ is zero in K-theory.

A *minimal system of infinitesimal spectral sections* is one in which I is chosen minimal (under the inclusion relation).

4.1 Definition of the family

Let $\ell \subset H^1\big(T^3;\mathbb{Z}\big)$ be a lattice (of non-maximal dimension) and let $B := (\ell \otimes \mathbb{R})/\ell$.

We need the following ingredients for our definition:

- $\ker(d)_{l\otimes\mathbb{R}}$: The subset of $\ker(d)$ representing elements in $\ell \otimes \mathbb{R}$.
- $\mathcal{G}_\ell$: The subgroup of the gauge group $\mathrm{Map}\big(T^3, S^1\big)$ determined by ℓ.
- The projection
$$\mathrm{pr}_{T^3} : T^3 \times \big(\nabla^K + \mathtt{i}\ker(d)_{l\otimes\mathbb{R}}\big) \to T^3$$
together with the induced vector bundle $\mathrm{pr}^*_{T^3}\big(\underline{\mathbb{H}} \otimes K\big)$.

If v is an element of the fibre of $\mathrm{pr}^*_{T^3}\big(\underline{\mathbb{H}} \otimes K\big)$ over
$$(y, \nabla^K + \mathtt{i}\alpha^c) \in T^3 \times \big(\nabla^K + \mathtt{i}\ker(d)_{\ell\otimes\mathbb{R}}\big),$$
we can define the following action of $\mathcal{G}_\ell$:
$$\begin{aligned} &\mathcal{G}_\ell \times \mathrm{pr}^*_{T^3}\big(\underline{\mathbb{H}} \otimes K\big) \to \mathrm{pr}^*_{T^3}\big(\underline{\mathbb{H}} \otimes K\big),\\ &\Big(u\,,\ \big(v, y, \nabla^K + \mathtt{i}\alpha\big)\Big) \mapsto \big(u(y)\cdot v, y, \nabla^K + \mathtt{i}\alpha + udu^{-1}\big). \end{aligned} \tag{4.1}$$

The quotient is a bundle over $T^3 \times B$. The connection from the parameter space determines a family of Dirac operators called $\mathcal{D}$.

Depending on $\hat{k}$ and ℓ we want to know:

1. Do spectral sections exist?
2. If they exist: What do they look like?

4.2 Existence of spectral sections

Following [4] we know that spectral sections for $\mathcal{D}$ exist if and only if the index of $\mathcal{D}$ in $K^1(B)$ vanishes. Let $\mathcal{I}$ be the following composition of isomorphisms (remember that B is a torus of maximal dimension 2):
$$K^1(B) \overset{\text{Chern}}{\longrightarrow} H^1(B;\mathbb{Z}) \longrightarrow \big(H_1(B;\mathbb{Z})\big)^* \longrightarrow \ell^*.$$

Lemma 4.1. *Let $a \in H^1\big(T^3;\mathbb{Z}\big)$ and let $f : (\mathbb{R}\cdot a)/a \to B$ be the map induced by the inclusion. In this way we get a pullback family $\mathcal{D}^a$ over $(\mathbb{R}\cdot a)/a$. Then the spectral flow of $\mathcal{D}^a$ in positive direction is given by*
$$\langle \hat{k},\, a\rangle = \langle \hat{k} \cup a,\, \big[T^3\big]\rangle.$$

Proof. We use our explicit knowledge of the spectrum. First we assume $\hat{k} \neq 0$: From all eigenvalues $\nu^{+/0/-}_{l,m}$ only those of the form $\nu^0_{l,m}$ for $0 \le m \le h-1$ have a chance to cross zero. From the definition we know that
$$\nu^0_{l,m} = \lambda_l = \big(2\pi l + \langle k,\, \alpha\rangle\big)/\|k\|,$$
for which we can count the crossings while running around the circle. For $\hat{k} = 0$ the spectrum is always symmetric with respect to zero. We see that every spectral flow has to vanish. □

Remark 4.2. The spectral flow of $\mathcal{D}^a$ for $\hat{k}$ is, by a folklore result of Atiyah, the same as the index of the positive Dirac operator over $T^3 \times S^1$ equipped with the $\mathrm{Spin}^{\mathbb{C}}$ structure belonging to $\hat{k} + a \cup e_{S^1}$, where e_{S^1} is the positive generator of $H^1(S^1;\mathbb{Z})$. Since every two-form over $T^3 \times S^1 \cong T^4$ can be written in this form, this allows us to calculate the index of $\mathcal{D}_b^+$ for every $b \in H^2\big(T^4;\mathbb{Z}\big)$. A direct computation yields $\Big\langle b \cup b,\ \big[T^4\big]\Big\rangle$.

With this Lemma we get a direct access to the following statement:

Theorem 4.2. *The isomorphism $\mathcal{I}$ maps the index of $\mathcal{D}$ to the map $x \mapsto \langle \hat{k} \cup x,\ \big[T^3\big]\rangle$ in ℓ^*.*

Proof. Take a fundamental basis a_1, a_2 of the torus B; then an element in $K^1(B)$ is determined by its images in $K^1\big((\mathbb{R}\cdot a_i)/a_i\big)$, which we calculate with the formula from the preceding lemma. Since the maps are linear, it is enough to check the theorem for a_1, a_2 which is an easy exercise. □

Corollary 4.3. *Spectral sections for $\mathcal{D}$ exist if and only if $k \cup \ell = 0$.*

4.3 Construction of spectral sections for $\hat{k} \neq 0$

Theorem 4.4. *If spectral sections exist, the spectrum is constant.*

Proof. From $k \cup \ell = 0$ we know that for every $\alpha \in (\ell \otimes \mathbb{R})$ we have $\alpha_\perp = 0$. From section 3.1.3 we know that this implies a constant spectrum. □

Therefore, we have $R_{\mathrm{inf}} = 0$. For ε_P smaller than the smallest eigenvalue of $\mathcal{D}$, the spectral sections are fixed everywhere except for the h-dimensional kernel of $\mathcal{D}$. Let

$$I := \big\{F \,\big|\, F\,\text{subbundle of } B \times \mathbb{C}^h\big\}/_{\cong}\ \cong \mathbb{Z}^{h-1} \cup \{0\} \cup \{\mathbb{C}^k\}$$

and define $P_F|_{\ker \mathcal{D}}$ for $R < \varepsilon_P$ as the orthogonal projection onto F. This defines a system of infinitesimal spectral sections which is obviously also minimal.

4.4 Construction of spectral sections for $\hat{k} = 0$

We split $\Gamma_{L^2}(\mathbb{H})$ into the 2-dimensional $\mathcal{D}_\alpha$-invariant subspaces

$$\Sigma_b = \mathrm{span}\{\sigma_b^+,\, \sigma_b^-\}.$$

On each of them, we have the two eigenvalues $\pm\|\beta\| = \pm\|\alpha + 2\pi b\|$. For small R we know that for each α there is at most one b with $\|\beta\| \leq R$. So for any spectral section P for $\mathcal{D}$ with small R we know that it fixes all Σ_b. Since $P_\alpha|\Sigma_b : \Sigma_b \to \Sigma_b$ is a one-dimensional orthogonal projection for $\|\beta\| > R$, it has to be a one-dimensional orthogonal projection for all β (and, therefore, for all α, since α and β are in bijective correspondence).

We now assume that ℓ is a plane since $\dim \ell \leq 1$ does not lead to interesting conclusions. In addition to the assumptions about R above we assume that ε_P is smaller than the minimal distance between $\ell \otimes \mathbb{R}$ and any point $b \in \mathbb{Z}^3 \backslash \ell$. This implies that for such b there are no eigenvalues with $\|\beta\| < R$ on Σ_b.

The space of one-dimensional orthogonal projections on $\mathbb{C}^2$ equals $\mathbb{CP}^1 \cong S^2$. Fix an element $b \in \ell_{\mathbb{Z}} = (\ell \otimes \mathbb{R}) \cap \mathbb{Z}^3$ and look at the corresponding map $P_\beta|_{\Sigma_b} : \ell \otimes \mathbb{R} \to \mathbb{CP}^1$ (written as function of β). For $\|\beta\| \geq R$ every ray coming from zero will be mapped to one point, producing a circle in $\mathbb{CP}^1$ (this follows from the construction of the eigenbasis). For $\|\beta\| < R$ we have to continue this map in some way; topologically, the problem is as follows: We have to construct a map from the 2-disc to the 2-sphere which maps the boundary pointwise to the equator. Up to homotopy, there are $\pi_2(S^2) \cong \mathbb{Z}$ many choices for that.

4.4.1 A system of infinitesimal spectral sections

The preceding discussion leads to the following:

Since we had imposed no lower bounds for R, we have $R_{\text{inf}} = 0$. Let ε_P be so small that if fulfills all conditions mentioned above. We take

$$I = \left\{ g : \ell_{\mathbb{Z}}/\ell \to \pi_2\big(\mathbb{CP}^1\big) \right\}$$

and define for each $R < \varepsilon_P$ spectral projections P^g. For $b \notin \ell_{\mathbb{Z}}$ these maps are already defined on Σ_b. For $b \in \ell_Z$, we define P^g_α on Σ_b to be a continuation specified by $g(b) \in \pi_2\big(\mathbb{CP}^1\big)$ as discussed in the preceding subsection (These continuations can be chosen to depend continuously on the parameters).

Conditions 1 and 2 (from the definition of infinitesimal spectral sections) are clear, 3 can be checked directly (if we specify the continuations explicitly), and 4 follows from the discussion above.

In general this system is not minimal. We can choose a minimal system J by fixing an element $g_0 \in I$ and a point $l_0 \in \ell_{\mathbb{Z}}/\ell$ and defining

$$J = \big\{ g \in I \,\big|\, g(l) = g_0(l) \quad \text{for } l \neq l_0 \big\}.$$

This is true because J represents all element of the form $(0, z)$ from

$$K(B) \cong H^0(B;\mathbb{Z}) \oplus H^2(B;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

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Author's address:

Johannes Fabian Meier
Institute of Mathematics, Endenicher Allee 60,
53115 Bonn, Germany.
E-mail: brief@fabianmeier.de