Discrete Morse theory, simplicial nonpositive curvature, and simplicial collapsibility

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Abstract. We find combinatorial conditions which ensure the collapsibility of a finite simplicial complex of dimension 3 or less. Our main result states that any finite systolic standard piecewise Euclidean simplicial complex of dimension 3 and 2, satisfying the property that any 2-simplex is a face of at most two 3-simplices in the complex, simplicially collapses to a point. A simplicial complex is systolic if it is simply connected, connected and locally 6-large. Our proof relies on the fact that any cycle in a systolic complex has a van Kampen diagram of minimal area whose disk is itself systolic. Our proof follows by applying discrete Morse theory.

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Introduction

In this paper we study necessary and sufficient conditions for the collapsibility of a finite simplicial complex of dimension 3 or less. The main tool we use in the proof is discrete Morse theory.

The combinatorial condition we have in mind is called local 6-largeness (see [10]). It was introduced by T. Januszkiewicz and J. Swiatkowski in [10] and independently by F. Haglund in [9]. In dimension 2, local 6-largeness is called the 6-property of the simplicial complex (see [4], [13]). A 2-dimensional simplicial complex has the 6-property if the link of each of its vertices is a graph of girth at least 6. The *girth* of a graph is defined as the minimum number of edges in a circuit.

In dimension 2, the 6-property is the natural combinatorial condition attached to the CAT(0) metric concept (see [1], [6], [8]). A geodesic metric space is a CAT(0) space if geodesic triangles are thinner than comparison triangles in Euclidean space. The standard piecewise Euclidean metric structure on a 2-dimensional simplicial complex is nonpositively curved if and only if the link of each of its vertices has girth at least

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6 (see [1], chapter II.5, page 207; [2], chapter 4.2, page 113). We emphasize that the equivalence no longer holds in higher dimensions (see [10], chapter 14, page 51). Our result in dimension 3 has therefore no implication with CAT(0).

It turns out that systolic spaces have many properties similar to CAT(0) spaces. In dimension 2, for instance, both CAT(0) and systolic simplicial complexes (not necessarily endowed with the standard piecewise Euclidean metric), collapse to a point (see [12], chapter 3.1, pages 36 – 48; [4]). Besides, the weaker condition of contractibility does characterize both nonpositively curved spaces (see [1], chapter $II.1$, page 161) and systolic ones (see [10], chapter 4, page 21). K. Crowley showed in [5] that CAT(0) standard piecewise Euclidean complexes of dimension 3 or less satisfying the property that any 2-simplex is a face of at most two 3-simplices in the complex, simplicially collapse to a point. The aim of this paper is to show that in dimension 2 and 3, under the same technical condition, systolic standard piecewise Euclidean complexes enjoy the same property. So the novelty of our approach is that, in dimension 3, our hypothesis is no longer of metric nature. It is important to note, however, that our result in dimension 2 is equivalent to K. Crowley's, although we call the 2-complex systolic, while she called it CAT(0). Namely, the curvature condition on the 2-complex in [5] seems metric, but it is in fact, just as in our paper, combinatorial. This is true because the CAT(0) 2-complex in K. Crowley's paper is constructed by endowing it with the standard piecewise Euclidian metric such that each of its interior vertices has degree at least 6. The standard piecewise Euclidian metric on the 2-complex becomes hence CAT(0).

It is interesting to note that, since 4−systolic cubical complexes are, according to M. Gromov's combinatorial description of nonpositively curved cubical complexes (see [6], Appendix I.6, page 516), CAT(0) spaces (see [11]), the collapsibility of CAT(0) cubical complexes of dimension three or less (see [12], chapters 5.5−5.8, pages 73−90) guarantees the collapsibility of 4-systolic cubical ones of the same dimension.

Our main result states that any finite systolic simplicial complex of dimension 2 and 3 endowed with the standard piecewise Euclidian metric and satisfying the property that any 2-simplex is a face of at most two 3-simplices in the complex, is collapsible. As in [5], the main tool used in the proof is discrete Morse theory (see [7]), a combinatorial analogue of the classical smooth Morse theory developed in the 1920s (see [14], [3]). The proof relies on the following result regarding van Kampen diagrams (see [13]) and proven in [10] (see chapter 1, page 11): there exists, for any cycle in a systolic simplicial complex, a simplicial nondegenerate van Kampen diagram of minimal area whose disk is itself systolic. Besides, when applying discrete Morse theory, we will make frequent use of well known results concerning systolic geometry. Namely, any two vertices in a systolic complex can be joined by a unique directed geodesic. Any sequence of vertices in a systolic complex, such that any two of its consecutive vertices belong to two consecutive simplices in a directed geodesic, is a geodesic in the 1-skeleton of the complex (see [10], chapter 9, page 40).

1 Preliminaries

We present in this section the notions we shall work with and the results we shall refer to. Let (X, d) be a metric space. Given a path $\gamma : [a, b] \to X$ in X, its *length* is defined by $L(\gamma) = \sup\{\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i))\}$, where the supremum is taken over all possible subdivisions of $[a, b]$, $a = t_0 < t_1 < ... < t_n = b$.

We call (X, d) a *geodesic space* if given two points p, q in X, there is a path from p to q whose length equals $d(p, q)$. Such a distance minimizing path is called a *geodesic segment*. We denote it by $[p, q]$.

Let (X, d) be a geodesic space. A *geodesic triangle* in X consists of three points $p, q, r \in X$, called *vertices*, and a choice of three geodesic segments $[p, q], [q, r], [r, p]$ joining them, called *sides*. Such a geodesic triangle is denoted by $\triangle = \triangle(p, q, r)$. If a point $x \in X$ lies in the union of $[p, q], [q, r]$ and $[r, p]$, then we write $x \in \triangle$. A triangle $\overline{\triangle} = \triangle(\overline{p}, \overline{q}, \overline{r})$ in $\mathbf{R}^2$ is called a *comparison triangle* for $\triangle = \triangle(p, q, r)$ if $d(p, q) = d_{\mathbf{R}^2}(\overline{p}, \overline{q})$, $d(q, r) = d_{\mathbf{R}^2}(\overline{q}, \overline{r})$ and $d(r, p) = d_{\mathbf{R}^2}(\overline{r}, \overline{p})$. A point $\overline{x} \in [\overline{q}, \overline{r}]$ is called a *comparison point* for $x \in [q, r]$ if $d(q, x) = d_{\mathbf{R}^2}(\overline{q}, \overline{x})$.

We call (X, d) a *CAT(0) space* if it is a geodesic space all of whose geodesic triangles satisfy the so called CAT(0) inequality: for any $\triangle \subset X$ and for any $x, y \in \triangle$: $d(x, y) \leq d_{\mathbf{R}^2}(\overline{x}, \overline{y})$, where $\overline{x}, \overline{y} \in \overline{\triangle}$ are the corresponding comparison points in the comparison triangle $\overline{\triangle}$ of $\triangle$ in $\mathbf{R}^2$.

We call (X, d) *non-positively curved* if it is locally a CAT(0) space, i.e. for every $x \in X$, there exists $r_x > 0$ such that the ball $B(x, r_x)$, endowed with the induced metric, is a CAT(0) space.

Let X be a finite, connected standard piecewise Euclidean simplical complex. The *standard piecewise Euclidean metric* on $|X|$ is defined by declaring each simplex to be isometric to a regular simplex of edge length equal 1 in $\mathbf{R}^2$.

A finite sequence of vertices $v_1, v_2, ..., v_{k+1}$ in X such that any two consecutive vertices v_i, v_{i+1} span an edge, $1 \leq i \leq k$, is called a *combinatorial path* between the vertices v_1 and v_{k+1} in X. Note that the combinatorial path $[v_1, v_2, ..., v_{k+1}]$ from v_1 to v_{k+1} is the *edge-path* $[v_1, v_2][v_2, v_3]...[v_k, v_{k+1}]$ from v_1 to v_{k+1}. We call the edge-path $\alpha = [v_1, v_2][v_2, v_3]...[v_k, v_{k+1}]$ a *closed edge-path* or *cycle* if $v_1 = v_{k+1}$. We denote by $|\alpha|$ the number of 1-cells contained in α and we call it the *length* of α. If there exists a combinatorial path from v_1 to v_{k+1} of length k, but there does not exist a combinatorial path from v_1 to v_{k+1} of length less than k, then we call any combinatorial path of length k joining v_1 to v_{k+1}, a *combinatorial geodesic*. The *combinatorial distance* between two vertices v_1 and v_{k+1} in X, denoted by $d_c(v_1, v_{k+1})$, is the length of a combinatorial geodesic joining v_1 to v_{k+1}. We call the vertex v_2 a *neighbor* of v_1 if $d_c(v_1, v_2) = 1$.

Let σ be a simplex of X. The *link* of X at σ, denoted $\mathrm{Lk}(\sigma, \mathrm{X})$, is the subcomplex of X consisting of all simplices of X which are disjoint from σ and which, together with σ, span a simplex of X. The (closed) *star* of σ in X, denoted $\mathrm{St}(\sigma, \mathrm{X})$, is the union of all simplices of X that contain σ. A subcomplex L in X is called *full* (in X) if any simplex of X spanned by a set of vertices in L, is a simplex of L. A *full cycle* in X is a cycle that is full as subcomplex of X. We define the *systole* of X by $sys(X) = \min\{|\alpha| : \alpha \text{ is a full cycle in } X\}$.

We introduce further a combinatorial curvature condition on simplicial complexes. We call X 6-*large* if $sys(X) \geq 6$ and $sys(\mathrm{Lk}(\sigma, \mathrm{X})) \geq 6$ for each simplex σ of X. We call X *locally* 6-*large* if the star of every simplex of X is 6-large. We call X 6-*systolic* if it is connected, simply connected and locally 6-large. We abbreviate $6-$systolic to systolic. Systolic complexes are contractible (see [10, chapter 4, page 21]).

We define next a directed geodesic in a systolic simplicial complex and present a few basic results concerning this notion.

For a subcomplex L of X, we denote by $N_X(L)$ the subcomplex of X being the union of all (closed) simplices that intersect L. Given a simplex σ in a systolic complex X, we define a system $B_n = B_n(\sigma, X)$ of combinatorial balls in X of radii n centered at σ as $B_0 := \sigma$, $B_{n+1} = N_X(B_n)$ for $n \geq 0$.

A sequence (σ_n) of simplices in a systolic simplicial complex X is a *directed geodesic* if any two consecutive simplices in the sequence are disjoint and span a simplex of X, whereas any three consecutive simplices in the sequence satisfy $\mathrm{St}(\sigma_\mathrm{i}, \mathrm{K}) \bigcap \mathrm{B}_1(\sigma_{\mathrm{i}+2}, \mathrm{K}) = \sigma_{\mathrm{i}+1}$.

Theorem 1.1. *Let X be a systolic simplicial complex. Then:*

1. *given two vertices v, w in X, there is exactly one directed geodesic from v to w;*
2. *if v, w are two vertices in X such that $d_c(v, w) = n$, then the directed geodesic from v to w consists of $n + 1$ simplices.*

(For the proof see [10], chapter 9*, page* 40*.)*

Note that the sequence of simplices in a directed geodesic is, just as its name suggests, no longer a directed geodesic after reversing the order of its simplices. So there exists a unique directed geodesic δ_1 joining two vertices v and w in X and there exists another directed geodesic δ_2 from w to v in X which is also unique and differs therefore from δ_1. The above result implies that any sequence of vertices in X, such that any two consecutive vertices in the sequence belong to two consecutive simplices in a directed geodesic, is a geodesic in the 1-skeleton of the complex.

A *combinatorial map* $f : X_1 \to X_2$ between two simplicial complexes X_1 and X_2 is a continuous map such that each open simplex of X_1 is mapped homeomorphically onto an open simplex of X_2. We call a combinatorial map *nondegenerate* if it is injective on each simplex of the triangulation. A *combinatorial* 2*-complex* is a simplicial 2-complex such that the 2-cells are attached through continuous maps from S^1 to the 1-skeleton of the complex. S^1 denotes the unit circle in $\mathbf{R}^2$. A *combinatorial disk* is a combinatorial 2-complex homeomorphic to a disk.

We shall study the simplicial complex X by associating to each closed edge-path α in X a diagram in the Euclidean plane, called a van Kampen diagram, which contains all the essential information about α (see [13], [4]). Van Kampen diagrams turned out to be useful tools for showing collapsibility.

Let $\alpha = e_0 e_1 ... e_n$ be a closed edge-path in X. A *van Kampen diagram* for α is a pair (D, ϕ). D is a finite, simply connected combinatorial disk embedded in the plane, bounded by the closed edge-path $\beta = f_0 f_1 ... f_n$. $\phi : D \to X$ is a combinatorial map assigning to each edge f_i of β in ∂D an edge $\phi(f_i) = e_i$ of α in X such that $\phi(f_i^{-1}) = \phi(f_i)^{-1}$ for all $0 \leq i \leq n$. The *area* of the diagram is given by the number of 2-simplices of D. Let v be a vertex of X. The *degree* of v, denoted by deg v, is the number of edges having v as initial vertex.

The following theorems will be of crucial importance when showing the main result of the paper.

Theorem 1.2. *Let X be a simply connected simplicial complex and let α be a cycle in X. Then there exists a nondegenerate simplicial van Kampen diagram (D, ϕ) for α (i.e. ϕ is a nondegenerate combinatorial map) such that ϕ is an isomorphism from the boundary of D to α (for the proof see [10], chapter* 1*, page* 12*).*

Theorem 1.3. *Let X be a systolic simplicial complex and let α be a cycle in X. Let (D, ϕ) be a nondegenerate simplicial van Kampen diagram for α. If D has minimal area, then D is systolic. If moreover α is a full subcomplex of X, then D has at least one interior vertex (for the proof see [10], chapter 1, page 14).*

Theorem 1.4. *Let X be a simplicial complex and let α be a cycle in X. Let (D, ϕ) be a nondegenerate simplicial van Kampen diagram for α of minimal area. Then ϕ maps i-simplices to i-simplices, $0 \leq i \leq 2$ (for the proof see [5], Lemma 5).*

Let X be a simplicial complex and let α be an i-simplex of X. If β is a k-dimensional face of α but not of any other simplex in X, then we say there is an *elementary collapse* from X to $X \setminus \{\alpha, \beta\}$. If $X = X_0 \supseteq X_1 \supseteq ... \supseteq X_n = L$ are simplicial complexes such that there is an elementary collapse from X_{j-1} to X_j, $1 \leq j \leq n$, then we say that X *simplicially collapses* to L. $|X|$ denotes the underlying space of X and X^k denotes the $k-$ skeleton of X.

The main tool used in the proof of our main results is discrete Morse theory. In the remainder of the section we introduce the main notions in discrete Morse theory and we give a few basic results regarding these notions.

We denote by $\alpha^{(i)}$ an i-dimensional simplex of X and by $\alpha < \beta$ the fact that α is a face of β. A function $f : X \to \mathbf{R}$ is called a *discrete Morse function* on X if for every $\alpha^{(i)} \in X$, $\sharp\{\beta^{(i+1)} > \alpha \mid f(\beta) \leq f(\alpha)\} \leq 1$ and $\sharp\{\gamma^{(i-1)} < \alpha \mid f(\gamma) \geq f(\alpha)\} \leq 1$.

Let $f : X \to \mathbf{R}$ be a discrete Morse function on X. We call a simplex $\alpha^{(i)}$ of X *critical* if $\sharp\{\beta^{(i+1)} > \alpha \mid f(\beta) \leq f(\alpha)\} = 0$ and $\sharp\{\gamma^{(i-1)} < \alpha \mid f(\gamma) \geq f(\alpha)\} = 0$. For $c \in \mathbf{R}$, we define the *level subcomplex* $X(c) = \bigcup_{\alpha \in X, f(\alpha) \leq c} \bigcup_{\beta \leq \alpha} \beta$. If $a < b$ are real numbers such that $[a, b]$ contains no critical values of f, then $X(a)$ collapses to $X(b)$ (for the proof see [7], chapter 3, page 104). If there exists a critical 3-simplex β and a critical 2-simplex α with a unique gradient path from the boundary of β to α, then X admits a new discrete Morse function g with the same critical simplices as f, except that β and α are no longer critical (for the proof we refer to [7], chapter 11, page 140).

It is often easier to work with the gradient vector field associated to a discrete Morse function rather than the function itself. Associated to a discrete Morse function $f : X \to \mathbf{R}$ is a *gradient vector field* $W : X \to X \bigcup \{0\}$. We define $W(\alpha) = \beta$, if $\alpha^{(i)} < \beta^{(i+1)}$ such that $f(\alpha) \geq f(\beta)$. We define $W(\alpha) = 0$ for all simplices α for which there is no such β. A sequence $\alpha_0^{(i)}, \beta_0^{(i+1)}, \alpha_1^{(i)}, \beta_1^{(i+1)}, \alpha_2^{(i)}, \beta_2^{(i+1)}, ..., \beta_r^{(i+1)}$, $\alpha_{r+1}^{(i)}$ of simplices is a *W-path* if $W(\alpha_j) = \beta_j$ for $0 \leq j \leq r$ and $\beta_j > \alpha_{j+1} \neq \alpha_j$. Such a path is nontrivial if $r \geq 0$ and *closed* if $\alpha_0 = \alpha_{r+1}$. A *discrete vector field* on X is a map $W : X \to X \bigcup \{0\}$ such that for each $\alpha^{(i)}$:

1. there is at most one simplex γ in X with $W(\gamma) = \alpha$ (if $W(\gamma) = \alpha$ then γ must be in the boundary of α);
2. $W(\alpha) = 0$ or α is a codimension-one face of $W(\alpha)$;
3. if $\alpha \in ImageW$ then $W(\alpha) = 0$.

A discrete vector field W defined on X is the gradient vector field of the discrete Morse function f if and only if it has no nontrivial closed W-paths (for the proof see [7], chapter 9, page 131).

Let X be an n-dimensional simplicial complex containing exactly m_i simplices of dimension i, $0 \leq i \leq n$. Let $\mathbf{F}$ be any coefficient field. We denote by $C_i(X, \mathbf{F})$ the space $\mathbf{F}^{m_i}$, i.e. $C_i(X, \mathbf{F})$ denotes the free abelian group generated by the i-simplices of X. Then there are boundary maps $\partial_i : C_i(X, \mathbf{F}) \longrightarrow C_{i-1}(X, \mathbf{F})$, $0 \leq i \leq n$, such that $\partial_{i-1} \circ \partial_i = 0$ and such that the resulting differential complex $C : 0 \longrightarrow C_n(X, \mathbf{F}) \xrightarrow{\partial_n} C_{n-1}(X, \mathbf{F}) \xrightarrow{\partial_{n-1}} ... \xrightarrow{\partial_1} C_0(X, \mathbf{F}) \longrightarrow 0$ calculates the homology of X. That is, if we define $H_i(C, \partial) = \frac{Ker(\partial_i)}{Im(\partial_{i+1})}$ then for each i we have $H_i(C, \partial) \cong H_i(X, \mathbf{F})$, where $H_i(X, \mathbf{F})$ denotes the singular homology of X (for the proof we refer to [7], chapter 3, page 122). We call the differential complex C the *Morse complex* of X. Let $\mathfrak{M}_i$ denote the span of the critical i-simplices of X, i.e. $\mathfrak{M}_i = \{\sum_{\sigma \in X} a_\sigma \sigma \mid a_\sigma \in \mathbf{F}$, if $a_\sigma \neq 0$ then σ is a critical simplex of $X\}$. The Morse complex of X is isomorphic to $\mathfrak{M} : 0 \to \mathfrak{M}_n \xrightarrow{\overline{\partial}} \mathfrak{M}_{n-1} \xrightarrow{\overline{\partial}} ... \xrightarrow{\overline{\partial}} \mathfrak{M}_0 \to 0$ (for the proof see [7], chapter 8, page 124). Let $\chi(X)$ denote the Euler characteristic of X. Then the so called weak Morse inequalities hold: $m_0 - m_1 + m_2 - ... + (-1)^n m_n = \chi(X)$ (for the proof see [7], chapter 8, page 124).

Lemma 1.5. *Let X be an n-dimensional simplicial complex satisfying the property that every $(n-1)$-simplex is a face of at most two n-simplices in the complex. Then there exist at most two gradient paths from any critical n-simplex in X to any critical $(n-1)$-simplex in X. (For the proof we refer to [5], section 6.)*

2 The geometry of systolic triangulated disks

The proof of the paper's main result relies on a certain result shown on a systolic triangulated disk whose boundary is mapped by an isomorphism into a cycle of a systolic 3-complex. It is therefore necessary to understand the geometry of systolic triangulated disks first. We will show that any systolic triangulated disk possesses a good direction of flow along the edges of its triangulation. Because systolic triangulated disks are endowed with the standard piecewise Euclidian metric, such disks are CAT(0) spaces. The results of this section are therefore similar to the ones obtained by K. Crowley on CAT(0) triangulated disks (see [5], section 3).

The following lemma presents an important inequality that holds in any triangulated disk whose interior vertices have degree at least 6.

Lemma 2.1. *Let D be a triangulated disk whose interior vertices have degree at least 6. Then:* $\sum\limits_{v \in \partial D} (4 - \deg\, v) \geq 6$*, summing over the boundary vertices of D.*

Proof. We denote by V, V_{int}, V_{ext}, E, E_{ext} and F the number of vertices, interior vertices, exterior vertices, edges, exterior edges and 2-simplices of D, respectively. The following relations hold in any triangulated disk: $1 = V - E + F$ (Euler's characteristic), $2E - E_{ext} = 3F$, $V_{ext} = E_{ext}$, $\sum_v \deg\, v = 2E$. Using these relations, we obtain: $6 = 6(V - E + F) = 6V - 6E + 6(\frac{2}{3}E - \frac{1}{3}E_{ext}) = 6V - 2E_{ext} - 2E = = 6V_{int} + 4V_{ext} - (\sum\limits_{v \in int(D)} \deg\, v + \sum\limits_{v \in \partial D} \deg\, v) = (6V_{int} - \sum\limits_{v \in int(D)} \deg v) + (4V_{ext} - \sum\limits_{v \in \partial D} \deg v)$. Thus $6 = \sum\limits_{v \in int(D)} (6 - \deg\, v) + \sum\limits_{v \in \partial D} (4 - \deg\, v)$. Because D is a disk whose interior vertices have degree at least 6, the above relation implies $\sum\limits_{v \in \partial D} (4 - \deg\, v) \geq 6$. □ □

A *geodesic disk* is a triangulated disk D whose interior vertices have degree at least 6, and whose exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$

and $[\overline{v}_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$. If $v_n = \overline{v}_n$ then D is called a *geodesic disk of type I.* If $d_c(v_n, \overline{v}_n) = 1$ then D is called a *geodesic disk of type II.*

We note that a geodesic disk is in fact a finite locally 6-large triangulated disk with given exterior vertices.

Lemma 2.2. *Let D be a geodesic disk whose exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[\overline{v}_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$. Then D has an exterior vertex $v \in \{v_1, \overline{v}_1, ..., v_{n-1}, \overline{v}_{n-1}, v_n\}$ such that* $\deg v = 3$.

Proof. For $1 \leq k \leq n-1$, the degree of v_k must be at least 3. Otherwise the vertices v_{k-1}, v_k, v_{k+1} would span a 2-simplex in D, contradicting the fact that $[v_n, ..., v_{k-1}, v_k, v_{k+1}, ..., v_0]$ is a combinatorial geodesic in D. Similarly, $\deg \overline{v}_k \geq 3$, $1 \leq k \leq n-1$.

In a geodesic disk of type I, the vertices v_0 and v_n have each at least two distinct neighbors. Thus $(4 - \deg v_0) + (4 - \deg v_n) \leq 4$. In a geodesic disk of type II, at least one of of the vertices v_n and $\overline{v}_n$ has degree at least 3. In a geodesic disk of type I or II, we have: $(4 - \deg v_0) + (4 - \deg v_n) + (4 - \deg \overline{v}_n) \leq 5$. Lemma 2.1 therefore implies: $6 \leq (4 - \deg v_0) + (4 - \deg v_n) + (4 - \deg \overline{v}_n) + \sum_{v \in \partial D, v \notin \{v_0, v_n, \overline{v}_n\}} (4 - \deg v) \leq 5 + \sum_{v \in \partial D, v \notin \{v_0, v_n, \overline{v}_n\}} (4 - \deg v)$. So there exists an exterior vertex $v \in \{v_1, \overline{v}_1, ..., v_{n-1}, \overline{v}_{n-1}\}$ such that $\deg v \leq 3$. This guarantees that D has an exterior vertex $v \in \{v_1, \overline{v}_1, ..., v_{n-1}, \overline{v}_{n-1}\}$ with $\deg v = 3$. □ □

We introduce further a notion which generalizes the one of geodesic disk. Let J be a simplicial complex whose underlying space is homeomorphic to $\mathbf{R}^2$ and whose interior vertices have degree at least 6. A finite connected, simply connected subcomplex S of J is called a *string of pearls* if its exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[\overline{v}_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$ in J. If $v_n = \overline{v}_n$ then S is called a *string of pearls of type I.* If $d_c(v_n, \overline{v}_n) = 1$ then S is called a *string of pearls of type II.*

So every exterior vertex of S lies on a combinatorial geodesic either from v_n to v_0 or from $\overline{v}_n$ to v_0. We show further that every vertex of S lies on a combinatorial geodesic either from v_n to v_0 or from $\overline{v}_n$ to v_0, not only its boundary vertices. A string of pearls has therefore a good direction of flow along the edges of its triangulation.

We note that a string of pearls is in fact a finite systolic triangulated disk with given exterior vertices.

Theorem 2.3. *Let S be a string of pearls whose exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[\overline{v}_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$. Then every vertex of S lies on a combinatorial geodesic either from v_n to v_0 or from $\overline{v}_n$ to v_0.*

Proof. According to Lemma 2.2, S has an exterior vertex v_{k_1}, $1 \leq k_1 \leq n-1$, such that $\mathrm{degv_{k_1}} = 3$. So the closed star of v_{k_1} in S contains two 2-simplices and their faces. We consider the subcomplex $S_1 = S - \overline{\mathrm{St}} v_{k_1}$ of S. Because S is a string of pearls, it is simply connected. Since S deformation retracts to S_1, S_1 remains simply connected. Because every interior vertex of S_1 is also an interior vertex of S, its degree is at least 6. So S_1 is also a string of pearls. Every exterior vertex of S_1 lies therefore on a combinatorial geodesic either from v_n to v_0 or from $\overline{v}_n$ to v_0.

Because S_1 is a string of pearls, Lemma 2.2 guarantees that there exists an exterior vertex v_{k_2}, $1 \leq k_2 \leq n-1$, such that $\mathrm{degv_{k_2}} = 3$. So the closed star of v_{k_2} in S_1

contains two 2-simplices and their faces. Because the subcomplex $S_2 = S_1 \setminus \mathrm{Stv}_{\mathrm{k}_2}$ remains a string of pearls, every exterior vertex of S_2 lies on a combinatorial geodesic either from v_n to v_0 or from $\overline{v}_n$ to v_0.

We retract further and obtain each time a string of pearls. Because S is finite, we reach, after a finite number of steps, a string of pearls S' with no interior vertices. Its exterior vertices lie therefore on a combinatorial geodesic either from v_n to v_0 or from $\overline{v}_n$ to v_0.

So every vertex of S lies on a combinatorial geodesic either from v_n to v_0 or from $\overline{v}_n$ to v_0. □ □

Corollary 2.4. *Let S be a string of pearls whose exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[\overline{v}_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$. Then $d_c(v, v_0) < n$ for every interior vertex v of S.*

Proof. By Theorem 2.3, every vertex of S lies on a combinatorial geodesic either from v_n to v_0 or from $\overline{v}_n$ to v_0. Hence $d_c(v, v_0) < d_c(v_n, v_0) = n$. □ □

3 Collapsing a systolic simplicial complex of dimension 2 and 3

In this section we prove that finite systolic standard piecewise Euclidean simplicial complexes of dimension 3 or less are collapsible. Our main tool is discrete Morse theory. Similarly, K. Crowley showed in [5] (section 6), also by applying discrete Morse theory, that finite CAT(0) simplicial complexes of dimension 3 or less, collapse to a point, when endowed with the standard piecewise Euclidean metric. Our approach is based on her considerations. Applying discrete Morse theory on systolic complexes, however, turns out to be easier due to certain results regarding systolic geometry (see [10], chapter 9).

We start by investigating the geometry of systolic complexes. Our investigation relies on T. Januszkiewicz and J. Swiatkowski's results (see [10], Lemma 1.6, Lemma 1.7). Namely, Theorem 1.2 and Theorem 1.3 ensure that there exists, for each cycle in a systolic simplicial complex, a simplicial nondegenerate van Kampen diagram of minimal area whose disk is itself systolic. This fact will allow us to use a result obtained in section 3 on systolic triangulated disks in order to obtain a similar one on systolic 3-complexes.

Lemma 3.1. *Let X be a systolic simplicial complex of dimension three or less endowed with the standard piecewise Euclidian metric. Let $\alpha = [w_0, w_1][w_1, w_2] ...[w_{n-1}, w_n][w_n, \overline{w}_m][\overline{w}_m, \overline{w}_{m-1}]...[\overline{w}_2, \overline{w}_1][\overline{w}_1, w_0]$ be a cycle in X, $m \in \{n, n-1\}$ such that the distinct vertices $w_n, ..., w_1, w_0, \overline{w}_1, ..., \overline{w}_{m-1}, \overline{w}_m$ lie on the combinatorial geodesics $[w_n, w_{n-1}, ..., w_1, w_0]$ and $[\overline{w}_m, \overline{w}_{m-1}, ..., \overline{w}_1, w_0]$ in X. Let (D, ϕ) be a nondegenerate simplicial van Kampen diagram for α of minimal area such that the boundary vertices of D lie on the cycle $\beta = [v_0, v_1][v_1, v_2] ...[v_{n-1}, v_n][v_n, \overline{v}_m] [\overline{v}_m, \overline{v}_{m-1}]...[\overline{v}_2, \overline{v}_1][\overline{v}_1, v_0]$ in D. Let β denote the unique 2-simplex of D containing the edge $[v_n, \overline{v}_m]$. Let w denote the third vertex of $\phi(\beta)$ opposite $[w_n, \overline{w}_m]$. Then $d_c(w, w_0) = n - 1$.*

Proof. Let v denote the vertex of β opposite the edge $[v_n, \overline{v}_m]$. Because X is systolic and D has minimal area, Theorem 1.3 guarantees that D is a systolic triangulated disk. Since the exterior vertices of D lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[\overline{v}_m, \overline{v}_{m-1}, ..., \overline{v}_1, v_0]$, D is a string of pearls. So, by Theorem 2.3, each vertex of D lies on a combinatorial geodesic either from v_n to v_0 or from $\overline{v}_m$ to v_0. So $d_c(v, v_0) = n-1$. Let $[v, v'_{n-2}, ..., v'_1, v_0]$ be a combinatorial geodesic in D from v to v_0. Because D has minimal area, according to Theorem 1.4, the combinatorial map ϕ preserves the dimension of simplices. So $[\phi(v), \phi(v'_{n-2}), ..., \phi(v'_1), \phi(v_0)]$ is a combinatorial path in X from w to w_0. Since this combinatorial path has length $n-1$, the combinatorial distance between w and w_0 is at most $n-1$. So $d_c(w, w_0) \leq n-1$. Because w is a neighbor of w_n and $[w_n, w_{n-1}, ..., w_1, w_0]$ is a combinatorial geodesic, $d_c(w, w_0) \geq n-1$. So $d_c(w, w_0) = n-1$. □ □

We prove further that any systolic standard piecewise Euclidean simplicial complex of dimension 3 or less, admits a discrete Morse function with no critical edges and a single critical vertex.

Theorem 3.2. *Let X be a finite systolic simplicial complex of dimension 3 or less endowed with the standard piecewise Euclidian metric. Then X admits a discrete Morse function with no critical edges and a single critical vertex.*

Proof. Let $W : X \to X \cup \{0\}$ be a vector field defined on X. We fix a vertex w_0 of X and we define $W(w_0) = 0$. For each vertex w different from w_0, there exists, according to Theorem 1.1, a unique directed geodesic $\delta = \sigma_n \sigma_{n-1} ... \sigma_0$ from w to w_0. Recall that there exists another directed geodesic from w_0 to w which is also unique and differs from δ. Any sequence of vertices $w = w_n, ..., w_0$ with $w_i \in \sigma_i$, $0 \leq i \leq n$ is, by Theorem 1.1, a combinatorial geodesic in X. Let $e_i = [w_i, w_{i-1}]$, $1 \leq i \leq n$. We define $W(w_i) = e_i$, $1 \leq i \leq n$. We note that for each edge $e_i = [w_i, w_{i-1}]$ such that $W(w_i) = e_i$, the vertex w_i is unique. For such edges e_i, we define $W(e_i) = 0$.

Let e be an edge of X for which there does not exist a vertex w such that $W(w) = e$. We denote the endpoints of e by w_n and $\overline{w}_m$, $m \in \{n, n-1\}$. There exists a unique directed geodesic $\sigma_n \sigma_{n-1} ... \sigma_0$ ($\overline{\sigma}_m \overline{\sigma}_{m-1} ... \overline{\sigma}_0$) from w_n ($\overline{w}_m$) to w_0. Any sequence of vertices $w_n, w_{n-1}, ..., w_0$ ($\overline{w}_m, \overline{w}_{m-1}, ..., w_0$) with $w_i \in \sigma_i$ ($\overline{w}_i \in \overline{\sigma}_i$), $0 \leq i \leq n$, ($0 \leq i \leq m$) is a combinatorial geodesic in X. We consider the cycle $\alpha = [w_n, w_{n-1}]$ $[w_{n-1}, w_{n-2}] ... [w_1, w_0][w_0, \overline{w}_1][\overline{w}_1, \overline{w}_2] ... [\overline{w}_{m-1}, \overline{w}_m][\overline{w}_m, w_n]$ in X. We note that α is a full subcomplex of X. Because X is simply connected, there exists, according to Theorem 1.2, a simplicial nondegenerate van Kampen diagram (D, ϕ) for α such that ϕ maps the boundary of D isomorphically on α. So there exists an edge $f \in \partial D$ such that $\phi(f) = e$. Choose D to be of minimal area. X being systolic, Theorem 1.3 implies that D is itself systolic and it has at least one interior vertex. So there exists a 2-simplex β of D such that $f < \beta$. Since f belongs to the boundary of D, β is unique. Theorem 1.4 ensures that the combinatorial map ϕ preserves the dimension of simplices. So $\phi(\beta) = \tau$ is a 2-simplex of X. We define $W(e) = \tau$. There exists a unique directed geodesic θ joining one endpoint of e, say w_n, with w_0. The 2-simplex τ is therefore either the first simplex of θ or it belongs to the star of the first simplex of θ. We note that the directed geodesic from w_n to w_0 is unique and differs from the one from w_0 to w_n. Lemma 3.1 guarantees that the third vertex of τ, the one that differs from w_n and $\overline{w}_m$, say w, satisfies $d_c(w, w_0) = n-1$. So for any 2-simplex τ of X for which there exists an edge e such that $W(e) = \tau$, such edge is unique.

For all simplices γ of X of dimension at least 2, we define $W(\gamma) = 0$. So the vector field $W : X \to X \cup \{0\}$ is defined on X such that it has a single critical vertex and no critical edges. We show further that W is the gradient vector field of a discrete Morse function defined on X with no critical edges and a single critical vertex.

As shown above, for each simplex $\beta \in ImW$ there exists a unique simplex γ in X such that $W(\gamma) = \beta$. According to the definition of W, if $\beta \in Im(W)$, then $W(\beta) = 0$ for all simplices $\beta \in X$. The definition of W also ensures that either $W(\beta) = 0$ or $W(\beta)$ is a codimension-one face of β for all $\beta \in X$. So W is a discrete vector field defined on X.

We show next that W contains no non-trivial closed W-paths, neither of vertices and edges nor of edges and 2-simplices.

Suppose, on the contrary, that there exists a nontrivial closed W-path of vertices and edges in X: $u_0^{(0)}, e_0^{(1)}, u_1^{(0)}, e_1^{(1)}, ..., u_r^{(0)}, e_r^{(1)}, u_{r+1}^{(0)} = u_0^{(0)}$. Because this W-path is non-trivial, $r \geq 0$. Since W-paths of vertices and edges in X point along geodesic paths, $d_c(u_i, w_0) = d_c(u_{i+1}, w_0) + 1$ for $0 \leq i \leq r$. Hence $d_c(u_0, w_0) = d_c(u_{r+1}, w_0) + (r + 1)$. Thus $d_c(u_{r+1}, w_0) = d_c(u_0, w_0) - (r + 1) < d_c(u_0, w_0) = d_c(u_{r+1}, w_0)$ which is a contradiction. So W contains no nontrivial closed W-paths of vertices and edges.

Let $e_0^{(1)}, \sigma_0^{(2)}, e_1^{(1)}, \sigma_1^{(2)}, ..., e_r^{(1)}, \sigma_r^{(2)}, e_{r+1}^{(1)}$ be a W-path of edges and 2-simplices in X. We denote by a_i and b_i the endpoints of the edge e_i and by c_i the opposite vertex of e_i in σ_i, $0 \leq i \leq r + 1$.

In case $d_c(a_i, w_0) = k$ and $d_c(b_i, w_0) = k - 1$, Lemma 3.1 implies that $d_c(c_i, w_0) = k-1$. According to the definition of W, $W(a_i) = [a_i, c_i]$ and $W(e_i) = \sigma_i$. Hence $[b_i, c_i]$ is the only edge of σ_i that is neither in $Im(W)$ nor is it mapped by W to a 2-simplex. So $e_{i+1} = [b_i, c_i]$ and $W(e_{i+1}) = \sigma_{i+1}$. According to Lemma 3.1, $d_c(c_{i+1}, w_0) = k - 2$ and $d_c(c_{i+2}, w_0) = k - 2$.

In case $d_c(a_i, w_0) = d_c(b_i, w_0) = k$, Lemma 3.1 implies that $d_c(c_i, w_0) = k - 1$. As shown above, due to the definition of W, we have $e_{i+1} = [b_i, c_i]$ and $W(e_{i+1}) = \sigma_{i+1}$. Lemma 3.1 further implies that $d_c(c_{i+1}, w_0) = k - 2$ and $d_c(c_{i+2}, w_0) = k - 2$.

We note that in both cases $\{d_c(c_i, w_0)\}_{i=0}^{r}$ is a non-increasing sequence and that $d_c(c_i, w_0) = d_c(c_{i+2}, w_0) + 1$.

Suppose that there exists a nontrivial closed W-path of edges and 2-simplices in X: $e_0^{(1)}, \sigma_0^{(2)}, e_1^{(1)}, \sigma_1^{(2)}, ..., e_r^{(1)}, \sigma_r^{(2)}, e_{r+1}^{(1)} = e_0^{(1)}$. Because this W-path is nontrivial, the intersection of any two of its 2-simplices is a face of each of them. Thus $r \geq 2$. Because $d_c(a_o, w_0) = k - 1$ and $d_c(b_o, w_0) = k$, Lemma 3.1 ensures that $d_c(c_o, w_0) = k - 1$. Because $e_{r+1} = e_0$, c_r coincides either with a_0 or with b_0. So $d_c(c_r, w_0) \geq k - 1$. Hence

$$k - 1 = d_c(c_0, w_0) \geq d_c(c_{r-2}, w_0) = d_c(c_r, w_0) + 1 > d_c(c_r, w_0) \geq k - 1$$

which is a contradiction. So there exist no nontrivial closed W-paths of edges and 2-simplices in X.

In conclusion W is the gradient vector field of a discrete Morse function defined on X with no critical edges and a single critical vertex. □ □

We present the main result of the paper.

Corollary 3.3. *Let X be a finite systolic simplicial complex of dimension 3 or less endowed with the standard piecewise Euclidian metric. If X satisfies the property that every 2-simplex is a face of at most two 3-simplices in the complex, then X simplicially collapses to a point.*

Proof. The previous theorem implies that X admits a discrete Morse function with no critical edges and a single critical vertex w_0. If X is 2-dimensional, because it is contractible, the weak Morse inequalities imply that the number of critical simplices of dimension 2 equals zero. So X simplicially collapses to the critical vertex w_0.

If X is 3-dimensional, by the weak Morse inequalities we have $\chi(X) = m_0 - m_1 + m_2 - m_3 = 1 + m_2 - m_3$, where m_i denotes the number of critical simplices of dimension i. Because X is contractible, the above relation implies that the number of critical simplices of dimension 2 equals the number of critical simplices of dimension 3. So, once we have shown that there exists a unique W-path from each critical 2-simplex to each critical 3-simplex in X, these critical simplices can be canceled out in pairs.

We consider the Morse complex of the function f with coefficients in any field $\mathbf{F}$: $\ldots \to \mathfrak{M}_3 \stackrel{\partial_3}{\to} \mathfrak{M}_2 \stackrel{\partial_2}{\to} 0 \to < w_0 > \to 0$. Because X is contractible, $0 = H_2(K, \mathbf{F}) = \frac{Ker\partial_2}{Im\partial_3} = \frac{\mathfrak{M}_2}{Im\partial_3}$. The boundary map ∂_3 is therefore surjective. So there exists a gradient path from any critical 2-simplex to any critical 3-simplex in X.

Let $\mathbf{F} = \mathbf{Z}_2$. Because the map ∂_3 is surjective, there exists, for any critical 2-simplex α, a critical 3-simplex β such that $< \partial_3\beta, \alpha >= 1 \text{mod } 2$. Hence, mod2, there exists a unique gradient path from β to α. Computing with coefficients in $\mathbf{Z}$, we notice that there exists an odd number of gradient paths from β to α. Because X satisfies the property that every 2-simplex in X is the face of at most two 3-simplices in X, Lemma 1.5 guarantees that there exists a unique gradient path from β to α. X admits therefore a new discrete Morse function with no critical simplices of dimension $1, 2$ or 3 and a single critical vertex w_0. So X simplicially collapses to w_0. □

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