Generalized geodesics on almost Cliffordian geometries

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Abstract. In this paper we prove some facts about the geometric description of these structures, especially the generic rank of multiaffinor structures called almost Clifford geometries, i.e A–modules, where A is a Clifford algebra, particularly Clifford algebra $\mathcal{C}l(0,3), \mathcal{C}l(1,2), \mathcal{C}l(2,1)$ or $\mathcal{C}l(3,0)$. Finally, we focus on A–structures, where A is a Clifford algebra $\mathcal{C}l(0,3)$ and describe their generalized geodesics.

M.S.C. 2010: 53C15.
Key words: Clifford algebras; geometries over algebras; generic rank; A-structures; linear connection; geodesics.

1 Introduction

Recently, a generalized approach to differential geometric structures using the definition of an affinor action on the fibres of the tangent bundle has been developed. Indeed, in this sense an almost complex structure on a $2n$–dimensional manifold M is a $(1,1)$–tensor field $J : TM \to TM$ (i.e. an affinor) satisfying the identity $J^2 = -E$. An almost hypercomplex structure on a $4n$–dimensional manifold M is a triple (I, J, K) of almost complex structures I, J and K satisfying the conditions

$$I^2 = J^2 = K^2 = -E,\ K = IJ \text{ and } IJ + JI = 0,$$

see [1, 7]. In the Clifford algebra language, an almost complex structure on a $2n$ dimensional manifold M is a fibre preserving A–module $TM \to M$, where $A = \mathcal{C}l(0,1)$ and an almost hypercomplex structure on $4n$ dimensional manifold M is an fibre preserving A–module $TM \to M$, where $A = \mathcal{C}l(0,2)$. In these cases, elements of A act as affinors. A–modules based on the Clifford algebras $\mathcal{C}l(0,2)$, $\mathcal{C}l(1,1) \cong \mathcal{C}l(2,0)$ are examples of the so called triple structures and they are very well known too, see [4]. The key point of this approach to geometries as A–modules is that A is an unitary algebra, not necessary with inversions, which allows us to study their properties more effectively, see [7, 5, 6]. Finally, a geometry over Clifford algebra $\mathcal{O}$, for example

Balkan Journal of Geometry and Its Applications, Vol.17, No.1, 2012, pp. 41-48.

$\mathcal{O} := \mathcal{C}l(0,3)$, i.e. geometry based on the free algebra with generators I_1, I_2, I_3, i.e. the algebra of affinors: id, I_1, I_2, I_3, I_1I_2, I_1I_3, I_2I_3, $I_1I_2I_3$, such that

$$I_1^2 = I_2^2 = I_3^2 = -\mathrm{id}, \quad I_iI_j + I_jI_i = 0, \; i \neq j,$$

is called an almost Clifford if the structure group is $GL(m,\mathcal{O})$ and an almost Cliffordian if the structure group is $GL(m,\mathcal{O})\mathcal{O}(1)$, where $\mathcal{O}(1) = \{z \in \mathcal{O} | z^*z = 1\}$ provided that z^* is a conjugation (see [2, 3] or [10] for some physical motivations). Generally, geometries over Clifford algebras are called almost Cliffordian geometries if the structure group is $Gl(m,A)A(1)$ and almost Clifford geometries if the structure group is $Gl(m,A)$, where A is a corresponding Cliffford algebra.

2 The generic rank

Following the paper [2] we recall some basic facts about almost Clifford and almost Cliffordian manifolds, where $A = \mathcal{O}$ and we will prove similar facts about almost Clifford and Cliffordian manifolds, where $A = \mathcal{C}l(0,3)$, $\mathcal{C}l(1,2)$ and $\mathcal{C}l(2,1)$, respectively. First note that $\mathcal{C}l(3,0) \cong \mathcal{C}l(1,2) \cong \mathbb{C}(2)$, where $\mathbb{C}(2)$ is the set of 2×2 matrices over $\mathbb{C}$. This is why we focus only on Clifford algebras $\mathcal{C}l(0,3)$, $\mathcal{C}l(3,0)$ and $\mathcal{C}l(2,1)$. The ordered family of linearly independent complex structures $H = (I_1, \ldots, I_6)$ on vector space V is called a multicomplex structure on V (in [2, 3] author called these structures hypercomplex, but we would like to avoid any confusion with the cited papers [1, 4, 7, 5, 6]) if the algebra $\{Id, I_1, \ldots, I_6, I_7 = I_1 \circ I_6\}$ is isomorphic to the appropriate Clifford algebra $\mathcal{O}$. Clearly, the family of endomorphisms $\{Id, I_1, \ldots, I_6, I_7 = I_1 \circ I_6\}$ is linearly independent and V is an $\mathcal{O}$–module. Recall that if there exists a multicomplex structure on the vector space V then necessarily $\dim V = 8m$, $m \in \mathbb{N}$.

Definition 2.1. [7] Let A be a n–dimensional unitary $\mathbb{R}$–algebra and let V be a finite dimensional A–module. We say that A–module V has weak generic rank n if the subset of elements $X \in V$, such that the A–hull $A(X) := \{FX | F \in A\}$ generates a vector subspace of dimension n, is open and dense in V.

Let V be an $8m$–dimensional vector space and $H = (J_1, J_2, J_3, J_4, J_5, J_6)$ be a multicomplex structure. It is not hard to see the following facts. Clearly, we can choose the basis $H = (I_1, I_2, I_3, I_1I_2, I_1I_3, I_2I_3)$, without loss of generality. Now, for an arbitrary element $F \in \mathcal{O}$, i.e. the element of the form

$$F = a_0 + a_1I_1 + a_2I_2 + a_3I_3 + a_{12}I_1I_2 + a_{13}I_1I_3 + a_{23}I_2I_3 + a_{123}I_1I_2I_3,$$

and for $F^* = a_0 - a_1I_1 - a_2I_2 - a_3I_3 - a_{12}I_1I_2 - a_{13}I_1I_3 - a_{23}I_2I_3 + a_{123}I_1I_2I_3$, we get

$$\begin{aligned} F^*F &= a_0^2 + a_0a_1I_1 + a_0a_2I_2 + a_0a_3I_3 + a_0a_{12}I_1I_2 + a_0a_{13}I_1I_3 + a_0a_{23}I_2I_3 \\ &+ a_0a_{123}I_1I_2I_3 \\ &+ a_0a_1 - a_1^2 - a_1a_2I_1I_2 - a_1a_3I_1I_3 - a_1a_{12}I_2 - a_1a_{13}I_3 - a_1a_{23}I_1I_2I_3 \\ &- a_1a_{123}I_2I_3 + \cdots \\ &= (a_0^2 - a_1^2 - a_2^2 - a_3^2 + a_{12}^2 + a_{13}^2 + a_{23}^2 - a_{123}^2) \\ &+ 2(a_0a_{123} - a_1a_{23} + a_2a_{13} - a_3a_{12})I_1I_2I_3, \end{aligned}$$

Therefore $F^*F = b_0 + b_1 I_1 I_2 I_3$, where $b_i \in \mathbb{R}$ and $(F^*F)^*(F^*F) = b_0^2 - b_1^2 \in \mathbb{R}$. Finally, $(F^*F)^*(F^*F)X = 0$ implies $X = 0$ or $b_0^2 = b_1^2$, i.e. $b_0 = \pm b_1$.

In particular, if the element $F \in \mathcal{O}$ acts as a singular map, i.e. $\exists X \in V : FX = 0$, then also F^*F acts as a singular map and therefore $h_0 = \pm h_1$, where

$$h_0 = a_0^2 - a_1^2 - a_2^2 - a_3^2 + a_{12}^2 + a_{13}^2 + a_{23}^2 - a_{123}^2 \tag{2.1}$$

$$h_1 = a_0 a_{123} - a_1 a_{23} + a_2 a_{13} - a_3 a_{12} \tag{2.2}$$

and we have proved that the coefficients of those F which act as a singular map have to belong to the hyperspaces $h_0 = \pm h_1$. Now, we can see that the center of $\mathcal{O}$ is $\mathbb{R}I_0 + \mathbb{R}I_1 I_2 I_3$ and the only ideals in $\mathcal{O}$ are $\{0\}, \mathcal{O}, \mathcal{O}(\frac{1}{2}(I_0 + I_1 I_2 I_3))$ and $\mathcal{O}(\frac{1}{2}(I_0 - I_1 I_2 I_3))$. Also, the algebra $\mathcal{O}$ is not a real division algebra. Exactly, in case $h_0 = h_1 = 0$ we have

$$a_0^2 - a_1^2 - a_2^2 - a_3^2 + a_{12}^2 + a_{13}^2 + a_{23}^2 - a_{123}^2 = a_0 a_{123} - a_1 a_{23} + a_2 a_{13} - a_3 a_{12} = 0.$$

Then we can choose a vector $(a_0, a_{12}, a_{13}, a_{23})$ and compute $F \in A$ by

$$\| a_0, a_{12}, a_{13}, a_{23} \| = \| a_1, a_2, a_3, a_{123} \| \text{ and}$$

$$(a_0, a_{23}, a_{13}, a_{12}) \cdot (a_{123}, -a_1, a_2, -a_3) = 0.$$

In fact, for any $a = (a_0, a_{12}, a_{13}, a_{23})$ there are eight possibilities:

$$\begin{aligned}
&\pm(a_{123}, -a_1, a_2, -a_3), && \\
&\pm(a_{123}, v_1, v_2, v_3), && \text{where } (v_1, v_2, v_3) = (a_{23}, a_{13}, a_{12}) \times (-a_1, a_2, -a_3) \\
&\pm(v_1, -a_1, v_2, v_3), && \text{where } (v_1, v_2, v_3) = (a_0, a_{13}, a_{12}) \times (a_{123}, a_2, -a_3) \\
&\pm(v_1, v_2, a_2, v_3), && \text{where } (v_1, v_2, v_3) = (a_0, a_{23}, a_{12}) \times (a_{123}, -a_1, -a_3)
\end{aligned}$$

Thus in this case the subset of the elements which act as a singular map is isomorphic to $\mathcal{C}l^+(0,3)$.

Concerning the almost hypercomplex structure, the polynomials corresponding to h_0 and h_1 vanish by default. In particular, if V is an $4n$–dimensional vector space then A–module with A being the algebra of quaternions has generic rank 4 independent on the representation. In the following text this is not the case and we have to work with explicit representations.

Let us mention that the faithful representation is a representation $\rho : A \to \mathrm{Hom}(V)$ such that ρ is injective. In the sequel we will use the symbol $\bar{\mathcal{O}}$ for the Clifford algebras $\mathcal{C}l(0,3), \mathcal{C}l(2,1)$ or $\mathcal{C}l(3,0)$. If the $\bar{\mathcal{O}}$–module is based on faithful representation, we may choose a local basis of $\bar{\mathcal{O}}$. For example,

$$I_1 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}, I_2 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix}$$

$$I_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

for $\bar{\mathcal{O}} = \mathcal{C}l(3,0)$ and

$$I_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, I_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

for $\bar{\mathcal{O}} = \mathcal{C}l(2,1)$. For the explicit matrices of faithful representation of $\mathcal{O}$ see [2]. We outlined the blocks in the above matrices which represent identity, complex structure and product structure (i.e. such affinor P that $P^2 = E$).

Theorem 2.1. *[6] Let $X_1, \dots, X_m$ be a basis of an A–module V and let A be an n–dimensional unitary associative $\mathbb{R}$–algebra, where $n < m$. If there exists $X \in V$ such that $\dim(A(X)) = n$ then the A–module V has weak generic rank n.*

Theorem 2.2. *Let V be an A–module, where $A = \mathcal{C}l(0,3), \mathcal{C}l(2,1)$ or $\mathcal{C}l(3,0)$, respectively. If $\dim(V) = 8m$ then A–module V has weak generic rank 8.*

Proof. For $A = \mathcal{C}l(3,0)$, by means of real representation of $GL(m, A)$, for vector $X \in V$, $X = (1, 0, \dots, 0)^T$ we obtain:

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & -a_{12} & -a_{13} & -a_{23} & -a_{123} \\ a_1 & a_0 & a_{12} & a_{13} & -a_2 & -a_3 & a_{123} & -a_{23} \\ a_2 & -a_{12} & a_0 & a_{23} & a_1 & a_{123} & -a_3 & a_{13} \\ a_3 & -a_{13} & -a_{23} & a_0 & -a_{123} & a_1 & a_2 & -a_{12} \\ a_{12} & -a_2 & a_1 & a_{123} & a_0 & a_{23} & -a_{13} & a_3 \\ a_{13} & -a_3 & -a_{123} & a_1 & -a_{23} & a_0 & a_{12} & -a_2 \\ a_{23} & -a_{123} & -a_3 & a_2 & a_{13} & a_{12} & a_0 & a_1 \\ a_{123} & a_{23} & -a_{13} & a_{12} & a_3 & -a_2 & a_1 & a_0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_{12} \\ a_{13} \\ a_{23} \\ a_{123} \end{pmatrix}.$$

Now, if $F \in A$, $F(X) = 0$ implies $a_i = 0$ for any i, i.e. $F = 0$ and Theorem 2.1 is applied. By similar direct computations we obtain the same result for $A = \mathcal{C}l(0,3)$ and $\mathcal{C}l(2,1)$. Recall that the result for $A = \mathcal{C}l(0,3)$ was presented in [3]. □

3 Triple structures

Let us shortly mention that our results have impact to the theory of triple structures, see [4]. First, let us recall that there are four triple structures, the most common is a hypercomplex structure based on affinors I, J, K, such that

$$A = \langle E, I, J, K\rangle,\ I^2 = J^2 = -E,\ K = IJ, IJ = -JI$$

and a hyperproduct structure based on affinors, I, J, K such that

$$A = \langle E, I, J, K\rangle,\ I^2 = J^2 = E,\ K = IJ, IJ = JI.$$

The next two are

$$A = \langle E, I, J, K\rangle,\ I^2 = -E, J^2 = E,\ K = IJ, IJ = -JI$$

called bi–paracomplex or hypercomplex of the second kind and

$$A = \langle E, I, J, K\rangle,\ I^2 = J^2 = -E,\ K = IJ, IJ = JI$$

called bicomplex. The bicomplex and hyperproduct structures are such that the generators commute and they are not based on a Clifford algebra in fact. We will omit it. The other two are based on the Clifford algebras $A = \mathcal{C}l(0,2)$ and $\mathcal{C}l(2,0)$ respectively.

Corollary 3.1. *Let V be a $4n$–dimensional vector space and (V, A) be a hypercomplex structure, i.e. $A = \mathbb{H} = \mathcal{C}l(0,2)$ and V is a A–module. The hypercompex structure V has weak generic rank* 4.

Proof. It is easy to verify that the algebra $\mathcal{C}l(0,2) \subset \mathcal{C}l(0,3)$ is a subalgebra. Let $F \in V$ be the element which acts as a singular map. Then taking into account h_0 and h_1 from (2.1), we have

$$h_0 = \pm h_1$$

and (2.2) with $a_3 = a_{13} = a_{23} = a_{123} = 0$ for $\mathcal{C}l(0,2)$. We conclude that $a_1 = a_2 = a_{12} = a_0 = 0$, i.e. $F = 0$. The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.2. *Let V be an A–module, where $A = \mathcal{C}l(0,2)$ or $\mathcal{C}l(1,1) \cong \mathcal{C}l(2,0)$, respectively. If $\dim(V) = 4n$ then the A–module V based on faithful representation has weak generic rank* 4.

Proof. Let us set $A = \mathcal{C}l(2,0)$. As $\mathcal{C}l(2,0) \subset \mathcal{C}l(3,0)$ is a subalgebra, we restrict the matrix from Theorem 2.2 and obtain

$$\begin{pmatrix} a_0 & a_1 & a_2 & 0 & -a_{12} & 0 & 0 & 0 \\ a_1 & a_0 & a_{12} & 0 & -a_2 & 0 & 0 & -0 \\ a_2 & -a_{12} & a_0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_0 & 0 & a_1 & a_2 & -a_{12} \\ a_{12} & -a_2 & a_1 & 0 & a_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & 0 & a_0 & a_{12} & -a_2 \\ 0 & 0 & 0 & a_2 & 0 & a_{12} & a_0 & a_1 \\ 0 & 0 & 0 & a_{12} & 0 & -a_2 & a_1 & a_0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ 0 \\ a_{12} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which completes the proof according to Theorem 2.1. The case $A = \mathcal{C}l(0,2)$ is dealt with in Corollary 3.1. □

4 Geodesics on Cliffordian manifolds

In this chapter we will work with $A = \mathcal{O}$. If M is an $8m$–dimensional manifold then an almost Clifford manifold M is given by a reduction of the structure group $GL(8m, \mathbb{R})$ of the principal frame bundle over M to $GL(8m, \mathcal{O})$, i.e. in other words, an almost Clifford manifold is a G–structure with structure group $GL(m, \mathcal{O})$. For concrete description of these groups see [2] or [3]. In particular, on the elements of this reduced bundle one can define affinors $F_0, \dots, F_7$ globally. An almost Cliffordian manifold M is given by a reduction of the structure group of the principal frame bundle over M to $GL(m, \mathcal{O})\mathcal{O}(1)$. In this case the above tensor fields $F_0, \dots, F_7$ can by defined only locally.

Definition 4.1. [5] Let M be a smooth manifold such that $\dim(M) = m$. Let A be a smooth ℓ-dimensional ($\ell < m$) vector subbundle in $T^*M \otimes TM$ such that the identity affinor $E = \mathrm{id}_{TM}$ restricted to T_xM belongs to $A_x \subset T_x^*M \otimes T_xM$ at each point $x \in M$. We say that M is equipped with an ℓ-dimensional A–structure.

In this context, an almost Clifford manifold is a manifold equipped with fiber preserving A–module TM and an almost Cliffordian manifold is just A–structure, where $A = \mathcal{O}$. The key point is that the first prolongation of the Lie algebra $\mathfrak{gl}(8m, \mathcal{O})$ is trivial, but the first prolongation of the Lie algebra $\mathfrak{g}$ of the Lie group $GL(8m, \mathcal{O})\mathcal{O}(1)$ is isomorphic to $(\mathbb{R}^{8m})^*$. In particular, there is a distinguish class of connections which share the same torsion (see [1], [3]).

In the paper [3] the author claims that a linear connection ∇ on M is almost Cliffordian if and only if the covariant derivatives of local canonical basis $(F_0, \dots F_7)$ of action of $\mathcal{O}$ on TM are expressed as follows:

$$\nabla F_\alpha = \sum_{b=1}^{6} \nu_{\alpha\beta} \otimes F_\beta,$$

where $\alpha, \beta = 1, \dots, 6$ and $\nu_{\alpha\beta}$ are 1–forms on M. In the same paper the author proves that if the linear connection $\tilde{\nabla}$ is almost Cliffordian, then the set of all Cliffordian connections on M is given by

$$\nabla_X = \tilde{\nabla}_X + \frac{1}{8}\sum_{i=1}^{6}(\tilde{\nabla}_X F_i) - \frac{1}{8}(\tilde{\nabla}_X F_7)F_7 + \frac{1}{2}\sum_{i=1}^{6}\nu_i(X)F_i + \frac{1}{8}(F_0\otimes F_0 + F_7\otimes F_7 - \Lambda)Q_X,$$

where Q is an arbitrary tensor field of type $(1,2)$, $X \in C^\infty(M)$ and Λ is defined by

$$\Lambda = \sum_{i=1}^{6} F_i \otimes F_i.$$

Suppose that the almost Cliffordian structure is locally paralelizable, i.e. in each coordinate neighborhood there exists a canonical local basis $(F_1, \dots, F_6)$ of V satisfying

$$\nabla_x F_i = 0, \text{ where } i = 1, \dots, 6.$$

Finally, if the connections ∇ and $\tilde{\nabla}$ share the same torsion and Cliffordian structure is paralelizable then

$$\nabla = \bar{\nabla} + \xi\otimes I + I\otimes\xi - \sum_{j=1}^{6}[(\xi\circ F_j)\otimes F_j) - F_j\otimes(\xi\circ F_j)] + (\xi\circ F_7)\otimes F_7 - F_7\otimes(\xi\circ F_7), \tag{4.1}$$

where ξ is a 1–form on M (for more see again [3]).

We call the connections ∇ of the form (4.1) A–planar connections and denote them by $[\nabla]_A$, where $A = \mathcal{O}$. The theory of planar curves can be found in [8] and [9].

Definition 4.2. Let M be a smooth manifold equipped with an A–structure and a linear connection ∇. A smooth curve $c : \mathbb{R} \to M$ is called to be A–planar if

$$\nabla_{\dot{c}}\dot{c} \in A(\dot{c}).$$

Theorem 4.1. *Let $(M, \mathcal{O})$ be a Cliffordian manifold, such that the Cliffordian structure is paralelizable. A curve $c : \mathbb{R} \to M$ is A–planar with respect to at least one A–planar connection ∇ from the expression (4.1) on M if and only if $c : \mathbb{R} \to M$ is a geodesic of some connection from (4.1).*

Proof. Consider an A–planar curve $c : \mathbb{R} \to M$, such that $\nabla_{\dot{c}}\dot{c} \in A(\dot{c})$, where ∇ is from (4.1), i.e. $\nabla_{\dot{c}}\dot{c} = \sum_{i=0}^{7}\varphi_i(\dot{c})F_i(\dot{c})$. Then

$$\begin{aligned}
\bar{\nabla}_{\dot{c}}\dot{c} &= \nabla_{\dot{c}}\dot{c} + 2\xi(\dot{c})\dot{c} - \sum_{j=1}^{6}[2\xi(\dot{c})F_j(\dot{c})] + (\xi(\dot{c})F_7(\dot{c})) \\
\bar{\nabla}_{\dot{c}}\dot{c} &= \sum_{i=0}^{7}\varphi_i(\dot{c})F_i(\dot{c}) + 2\xi(\dot{c})\dot{c} - \sum_{j=1}^{6}[2\xi(\dot{c})F_j(\dot{c})] + (\xi(\dot{c})F_7(\dot{c})) \\
\bar{\nabla}_{\dot{c}}\dot{c} &= \sum_{i=1}^{\dim A}(2\varphi_i^1(\dot{c}) + \xi_i)F_i(\dot{c}).
\end{aligned}$$

The system of equations $2\varphi_i^1(\dot{c}) + \xi_i(x) = 0$ has a solution, i.e. there exists $\varphi_i^1 \in \Omega^1(M)$ such that the curve c is a geodesic curve of connection $\bar{\nabla}$. The rest of the proof is easy to see. □

Theorem 4.2. *[5] Let* $(M, A, [\nabla]_A)$, $(M', A', [\nabla]_{A'})$ *be smooth manifolds of dimension* m *equipped with* A*–structure with the class of* A*–planar connections and* A'*–structure with the class of* A*–planar connections of the same generic rank* $\ell \leq 2m$*, where* A*,* A' *are unitary associative algebras. If* $f : M \to M'$ *maps* A*–planar curves on* A'*–planar curves then* f *is a morphism of the* A*–structures, i.e* $f^*A' = A$*, where* f^* *denotes the pullback of* f*.*

Theorem 4.3. *Let* $(M, V, [\nabla])$, $(M', V', [\nabla]')$ *be almost Cliffordian manifolds of dimension* $8m$*,* $m > 2$ *equipped with the class of* A*–connections with the same torsion. If* $f : M \to M'$ *maps* A*–planar curves on* A'*–planar curves then* f *is a morphism of the* A*–structures, i.e* $f^*A' = A$*.*

Proof. Almost Cliffordian manifolds are A–structures, where $A = \mathcal{O}$. According to Theorem 2.1, the generic rank is 8 and the rest of the proof follows from Theorem 4.2. □

Acknowledgments. The first author was supported by the grant GA ČR, grant no. 201/09/0981, and the second author by the grant no. FSI-S-11-3.

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