Finsler metrics of scalar flag curvature and projective invariants

B. Najafi and A. Tayebi

Abstract. In this paper, we define a new projective invariant and call it $\widetilde{W}$-curvature. We prove that a Finsler manifold with dimension $n \geq 3$ is of constant flag curvature if and only if its $\widetilde{W}$-curvature vanishes. Various kinds of projectively flatness of Finsler metrics and their equivalency on Riemannian metrics are also studied.

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1 Introduction

One of the fundamental problems in Finsler geometry is to study and characterize Finsler metrics of constant flag curvature. The best well-known result towards this question is due to Akbar-Zadeh, which classified compact Finsler manifolds with non-positive constant flag curvature [3]. In a 25 year research, initiated by famous Yasuda-Shimada's theorem [16] and finished by Bao-Robless-Shen's theorem [7], Randers metrics of constant flag curvature have been classified.

On the other hand, there are some well-known projective invariants of Finsler metrics namely, Douglas curvature [5][6][10], Weyl curvature, generalized Douglas - Weyl curvature [4][13] and another projective invariant which is due to Akbar-Zadeh [1]. In [21], Weyl introduces a projective invariant for Riemannian metrics. Then Douglas extendes Weyl's projective invariant to Finsler metrics [10]. Finsler metrics with vanishing projective Weyl curvature are called *Weyl metrics*. In [18], Z. Szabó proves that Weyl metrics are exactly Finsler metrics of scalar flag curvature.

In [3], Akbar-Zadeh introduces the non-Riemannian quantity $\mathbf{H}$ which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. This is a positively homogeneous scalar function of degree zero on the slit tangent bundle, and recently has been studied [11][12]. Akbar-Zadeh proves that for a Weyl manifold of dimension $n \geq 3$, the flag curvature is constant if and only if $\mathbf{H} = 0$. The natural question is: Is there any projectively invariant quantity which characterizes Finsler metrics of constant flag curvature?

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In this paper, using Akbar-Zadeh's method in [1], we define a new projective invariant and call it $\widetilde{W}$-curvature (see the equation (3.14)). We show that the $\widetilde{W}$-curvature is another candidate for characterizing Finsler metrics of constant flag curvature. More precisely, we prove the following

Theorem 1.1. *Let (M, F) be a Finsler manifold with dimension $n \geq 3$. Then F is of constant flag curvature if and only if $\widetilde{W} = 0$.*

By Akbar-Zadeh's theorem and Theorem 1.1, we have the following

Corollary 1.1. *Let (M, F) be a Finsler manifold with dimension $n \geq 3$. Suppose that F is of scalar flag curvature. Then $\mathbf{H} = 0$ if and only if $\widetilde{W} = 0$.*

Throughout this paper, we use the Berwald connection on Finsler manifolds [19][20]. The h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

2 Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \cup_{x \in M} T_xM$ the tangent space of M. A *Finsler metric* on M is a function $F : TM \to [0, \infty)$ which has the following properties: (i) F is C^∞ on TM_0; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, and (iii) for each $y \in T_xM$, the following quadratic form g_y on T_xM is positive definite,

$$g_y(u, v) := \frac{1}{2}\left[F^2(y + su + tv)\right]|_{s,t=0}, \quad u, v \in T_xM.$$

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where $G^i(x, y)$ are local functions on TM_0 satisfying $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$, for all $\lambda > 0$. Functions G^i are given by

$$G^i := \frac{1}{4} g^{il}\{2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\}y^j y^k, \tag{2.1}$$

where g_{ij} is the vertical Hessian of $F^2/2$ and g^{ij} denotes its inverse. $\mathbf{G}$ is called the associated *spray* to (M, F). The projection of an integral curve of $\mathbf{G}$ is called a *geodesic* in M. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$ [14].

For a vector v^i vertical and horizontal covariant derivative with respect to Berwald connection are given by

$$v^i_{,k} = \dot{\partial}_k v^i, \quad v^i_{|k} = d_k v^i + G^i_{jk} v^j,$$

where $d_k = \partial_k - G^m_k \dot{\partial}_m$, $\partial_k = \frac{\partial}{\partial x^k}$, $\dot{\partial}_k = \frac{\partial}{\partial y^k}$, $G^i_k = \dot{\partial}_k G^i$ and $G^i_{jk} = \dot{\partial}_j G^i_k$.

In [1], Akbar-Zadeh cosideres a non-Riemannian quantity $\mathbf{H}$ which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. This is a positively homogeneous scalar function of degree zero on the slit tangent bundle. The quantity $\mathbf{H} = H_{ij}dx^i \otimes dx^j$ is defined as the covariant derivative of $\mathbf{E}$ along geodesics, where $E_{ij} = \frac{1}{2}\dot{\partial}_m G^m_{ij}$ [12]. More precisely $H_{ij} := E_{ij|m}y^m$. In local coordinates, we have

$$\begin{aligned} 2H_{ij} = \ & y^m \frac{\partial^4 G^k}{\partial y^i \partial y^j \partial y^k \partial x^m} - 2G^m \frac{\partial^4 G^k}{\partial y^i \partial y^j \partial y^k \partial y^m} \\ - \ & \frac{\partial G^m}{\partial y^i} \frac{\partial^3 G^k}{\partial y^j \partial y^k \partial y^m} - \frac{\partial G^m}{\partial y^j} \frac{\partial^4 G^k}{\partial y^i \partial y^k \partial y^m}. \end{aligned}$$

The Riemannian curvature tensor of Berwald connection are given by

$$K^i{}_{hjk} = d_j G^i_{hk} + G^m_{hk} G^i_{mj} - d_k G^i_{hj} + G^m_{hj} G^i_{mk}.$$

Let $K^i_{jk} = K^i{}_{0jk}$ and $K^i_k = K^i{}_{0k}$. Then we have

$$K^i_{jk} = \frac{1}{3}\{\dot{\partial}_j K^i_k - \dot{\partial}_k K^i_j\}.$$

Then, the Riemann curvature operator of Berwald connection at $y \in T_xM$ is defined by $\mathbf{K}_y = K^i{}_k dx^k \otimes \frac{\partial}{\partial x^i}|_x : T_xM \to T_xM$, which is a family of linear maps on tangent spaces. The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry, which is first introduced by L. Berwald [8]. For a flag $P = \text{span}\{y, u\} \subset T_xM$ with flagpole y, the *flag curvature* $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(P, y) := \frac{\mathbf{g}_y(u, \mathbf{K}_y(u))}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2}.$$

When F is Riemannian, $\mathbf{K} = \mathbf{K}(P)$ is independent of $y \in P$, which is just the sectional curvature of P in Riemannian geometry. We say that a Finsler metric F is *of scalar curvature* if for any $y \in T_xM$, the flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ is a scalar function on the slit tangent space TM_0. If $\mathbf{K} = constant$, then F is said to be of *constant flag curvature*.

The projective Weyl curvature is defined as follows

$$W^i_{jkl} := K^i_{jkl} - \frac{1}{1-n^2}\Big\{ -\delta^i_j(\tilde{K}_{kl} - \tilde{K}_{lk}) - \delta^i_k \tilde{K}_{jl} + \delta^i_l \tilde{K}_{jk} - y^i \dot{\partial}_j(\tilde{K}_{kl} - \tilde{K}_{lk})\Big\}$$

where $\tilde{K}_{jk} := nK_{jk} + K_{kj} + y^r \dot{\partial}_j K_{kr}$. As it is well known, a Finsler metric is of scalar flag curvature if and only if $W^i_{jkl} = 0$.

3 C-projective Weyl curvature

Let $\phi : F^n \to \bar{F}^n$ be a diffeomorphism. We call ϕ a projective mapping if there exists a positive homogeneous scalar function $P(x, y)$ of degree one satisfying

$$\bar{G}^i = G^i + Py^i.$$

In this case, P is called the projective factor ([17]). Under a projective transformation with projective factor P, the Riemannian curvature tensor of Berwald connection change as follows

$$\bar{K}^i{}_{hjk} = K^i{}_{hjk} + y^i \dot{\partial}_h Q_{jk} + \delta^i_h Q_{jk} + \delta^i_j \dot{\partial}_h Q_k - \delta^i_k \dot{\partial}_h Q_j, \tag{3.1}$$

where $Q_i = d_i P - PP_i$ and $Q_{ij} = \dot{\partial}_i Q_j - \dot{\partial}_j Q_i$. A projective transformation with projective factor P is said to be *C-projective* if $Q_{ij} = 0$.

Let X be a projective vector field on a Finsler manifold (M, F). Let the vector field X in a local coordinate (x^i) on M be written in the form $X = X^i(x)\partial_i$. Then the complete lift of X is denoted by $\hat{X}$ and locally defined by $\hat{X} = X^i \partial_i + y^j \partial_j X^i \dot{\partial}_i$. Suppose that $\pounds_{\hat{X}}$ stands for Lie derivative with respect to the complete lift of X. Then we have

$$\pounds_{\hat{X}} G^i = P y^i,$$
$$\pounds_{\hat{X}} G^i_k = \delta^i_k P + y^i P_k,$$
$$\pounds_{\hat{X}} G^i_{jk} = \delta^i_j P_k + \delta^i_k P_j + y^i P_{jk},$$

$$\pounds_{\hat{X}} G^i_{jkl} = \delta^i_j P_{kl} + \delta^i_k P_{jl} + \delta^i_l P_{kj} + y^i P_{jkl}, \tag{3.2}$$

$$\pounds_{\hat{X}} K^i_{jkl} = \delta^i_j (P_{l|k} - P_{k|l}) + \delta^i_l P_{j|k} - \delta^i_k P_{j|l} + y^i \dot{\partial}_j (P_{l|k} - P_{k|l}). \tag{3.3}$$

Since $Q_{ij} = P_{i|j} - P_{j|i}$, we have

$$\pounds_{\hat{X}} K^i_{jkl} = \delta^i_j Q_{lk} + \delta^i_l P_{j|k} - \delta^i_k P_{j|l} + y^i \dot{\partial}_j Q_{lk}. \tag{3.4}$$

We have

$$\dot{\partial}_j P_{k|l} = P_{jk|l} - P_r G^r_{jkl}. \tag{3.5}$$

Contracting i and k in (3.4), we get

$$\pounds_{\hat{X}} K_{jl} = P_{l|j} - nP_{j|l} + P_{jl|s} y^s. \tag{3.6}$$

Consequently

$$\pounds_{\hat{X}} (y^r \dot{\partial}_l K_{jr}) = -(n+1) P_{jl|s} y^s. \tag{3.7}$$

Hence

$$P_{jl|s} y^s = -\frac{1}{n+1} L(\hat{X})(y^r \dot{\partial}_l K_{jr}), \tag{3.8}$$

and

$$\pounds_{\hat{X}} (K_{jl} + \frac{1}{n+1} y^r \dot{\partial}_l K_{jr}) = P_{l|j} - nP_{j|l}, \tag{3.9}$$

$$\pounds_{\hat{X}} (K_{lj} + \frac{1}{n+1} y^r \dot{\partial}_j K_{lr}) = P_{j|l} - nP_{l|j}. \tag{3.10}$$

Using (3.9) and (3.10), one can obtain

$$P_{j|l} = \frac{1}{1-n^2}\pounds_{\hat{X}}\Big\{K_{lj} + \frac{1}{n+1}y^r\dot{\partial}_j K_{lr} + nK_{jl} + \frac{n}{n+1}y^r\dot{\partial}_l K_{jr}\Big\}. \tag{3.11}$$

If $Q_{ij}=0$, then (3.4) reduces to the following

$$\pounds_{\hat{X}} K^i_{jkl} = \delta^i_l P_{j|k} - \delta^i_k P_{j|l}. \tag{3.12}$$

Using (3.11) and eliminating $P_{j|l}$ from (3.12), we are led to the following tensor

$$\begin{aligned}\widetilde{W}^i_{jkl} := K^i_{jkl} - & \frac{1}{1-n^2}\delta^i_l\Big\{\tilde{K}_{jk} + \frac{n}{n+1}y^r(\dot{\partial}_k K_{jr} - \dot{\partial}_j K_{kr})\Big\}\\ + & \frac{1}{1-n^2}\delta^i_k\Big\{\tilde{K}_{jl} + \frac{n}{n+1}y^r(\dot{\partial}_l K_{jr} - \dot{\partial}_j K_{lr})\Big\}.\end{aligned} \tag{3.13}$$

Since $y^j y^r\dot{\partial}_k K_{jr}=0$, if we put $\widetilde{W}^i_k := \widetilde{W}^i_{jkl}y^j y^l$, then we have

$$\widetilde{W}^i{}_k = K^i{}_k - \frac{1}{1-n^2}\Big\{y^i\tilde{K}_{0k} - \delta^i_k\tilde{K}_{00}\Big\}. \tag{3.14}$$

The tensor $\widetilde{W}^i_k$ is said to be *C-projective Weyl curvature* or $\widetilde{W}$-curvature. According to the way we construct $\widetilde{W}$, it is easy to see that $\widetilde{W}$ is C-projective invariant tensor. A Finsler metric F is called *C-projective Weyl metric* if its C-projective Weyl-curvature vanishes. First, we prove that the class of Weyl metrics contains the class of C-projective Weyl metrics.

Theorem 3.1. *Let F be a C-projective Weyl metric. Then F is a Weyl metric.*

Proof. By assumption, we have the following

$$K^i_k - \frac{1}{1-n^2}\Big\{y^i\tilde{K}_{0k} - \delta^i_k\tilde{K}_{00}\Big\} = 0. \tag{3.15}$$

Contracting (3.15) with y_i implies that

$$F^2\tilde{K}_{0k} - y_k\tilde{K}_{00} = 0. \tag{3.16}$$

Hence

$$\tilde{K}_{0k} = F^{-2}y_k\tilde{K}_{00}. \tag{3.17}$$

Plugging (3.17) into (3.15), we get

$$K^i_k = \frac{1}{1-n^2}\tilde{K}_{00}h^i_k, \tag{3.18}$$

which means that F is of scalar flag curvature. Hence, F is a Weyl metric. □

4 Proof of Theorem 1.1

To prove Theorem 1.1, we need to find the $\widetilde{W}$-curvature of Weyl metrics.

Proposition 4.1. *Let F be a Finsler metric of scalar flag curvature λ. Then $\widetilde{W}$-curvature is given by*

$$\widetilde{W}^i_k = \frac{1}{3}F^2 y^i \lambda_k, \tag{4.1}$$

where $\lambda_k := \dot{\partial}_k \lambda$.

Proof. By assumption, the Riemannian curvature of Berwald connection is in the following form.

$$\begin{aligned} K^i_{jkl} = {} & \lambda(\delta^i_k g_{jl} - \delta^i_l g_{jk}) + \lambda_j F(\delta^i_k F_l - \delta^i_l F_k) + \frac{1}{3}F^2(h^i_k \lambda_{jl} - h^i_l \lambda_{jk}) \\ + {} & \frac{1}{3}\lambda_l F(2\delta^i_k F_j - 2\delta^i_j F_k - g_{jk}\ell^i) \\ - {} & \frac{1}{3}F\lambda_k(2\delta^i_l F_j - 2\delta^i_j F_l - g_{jl}\ell^i). \end{aligned} \tag{4.2}$$

where $\lambda_{ij} = \dot{\partial}_j \lambda_i$. Hence, we have

$$K^i_k = \lambda F^2 h^i_k. \tag{4.3}$$

Then, we get the following relations.

$$\begin{aligned} K_{jl} &= (n-1)(\lambda g_{jl} + FF_l\lambda_j) + \frac{n-2}{3}(F^2\lambda_{jl} + 2FF_j\lambda_l), \\ K_{00} &= \lambda(n-1)F^2, \quad \tilde{K}_{00} = \lambda(n^2-1)F^2, \\ K_{k0} &= \lambda(n-1)FF_k + \frac{2n-1}{3}F^2\lambda_k, \\ K_{0k} &= \lambda(n-1)FF_k + \frac{n-2}{3}F^2\lambda_k, \\ \tilde{K}_{0k} &= (n^2-1)(\lambda FF_k + \frac{1}{3}F^2\lambda_k). \end{aligned} \tag{4.4}$$

Plugging (4.3) and (4.4) into (3.14), we get the result. □

Lemma 4.1. *Let (M, F) be a C-projective Weyl manifold with dimension $n \geq 3$. Then F is of constant flag curvature.*

Proof. By Theorem 3.1 and Proposition 4.1, we have

$$\widetilde{W}^i_k = \frac{1}{3}F^2 y^i \lambda_k.$$

From assumption, we get $\lambda_k = 0$. It means that F is of isotropic flag curvature. The result follows by Schur's Lemma. □

Now, let us consider the case F being of constant flag curvature.

Lemma 4.2. *Let F be a Finsler metric of constant flag curvature $\mathbf{K} = \lambda$. Then F is C-projective Weyl metric.*

Proof. If F is of constant flag curvature λ, then (4.2) reduces to the following

$$K^i_{jkl} = \lambda(g_{jl}\delta^i_k - g_{jk}\delta^i_l). \tag{4.5}$$

Hence

$$K_{jl} = \lambda(1-n)g_{jl}, \quad \tilde{K}_{jk} = \lambda(1-n^2)g_{jl}. \tag{4.6}$$

Plugging (4.6) into (3.13), we obtain $\widetilde{W}^i_{jkl} = 0$ and consequently $\widetilde{W}^i_k = 0$. □

5 Reduction in Riemannian manifolds

As mentioned before, in Finsler metrics F^n of scalar flag curvature with $(n \geq 3)$, we have this equivalence $\widetilde{W} = 0$ if and only if $\mathbf{H} = 0$. Observing C-projective invariancy of $\widetilde{W}$-curvature, one can conjecture that $\mathbf{H}$-curvature must be C-projective invariant too. Here, we prove that this is true. By definition, $H_{ij} = E_{ij|s}y^s$. Under a projective transformation with the projective factor P, we have the following relations:

$$\begin{aligned}
\bar{E}_{ij} &= E_{ij} + \frac{n+1}{2}P_{ij}, \\
y^l\bar{d}_l &= y^l d_l - 2Py^m\dot{\partial}_m, \\
\bar{E}_{mj}\bar{G}^m_i &= E_{mj}G^m_i + PE_{ij} + \frac{n+1}{2}(P_{mj}G^m_i + PP_{ij}).
\end{aligned}$$

Now, we can prove the following

Proposition 5.1. $\mathbf{H}$*-curvature is C-projective invariant.*

Proof. Under a projective transformation, we have

$$\begin{aligned}
\bar{H}_{ij} &= \bar{E}_{ij|l}y^l \\
&= y^l\bar{d}_l\bar{E}_{ij} - \bar{E}_{mj}\bar{G}^m_i - \bar{E}_{im}\bar{G}^m_j \\
&= (y^l d_l\bar{E}_{ij} - 2Py^m\dot{\partial}_m\bar{E}_{ij}) - \bar{E}_{mj}\bar{G}^m_i - \bar{E}_{im}\bar{G}^m_j \\
&= y^l d_l E_{ij} + \frac{n+1}{2}y^l d_l P_{ij} + 2PE_{ij} + (n+1)PP_{ij} - \bar{E}_{mj}\bar{G}^m_i - \bar{E}_{im}\bar{G}^m_j \\
&= y^l d_l E_{ij} - E_{mj}G^m_i - E_{im}G^m_j + \frac{n+1}{2}(y^l d_l P_{ij} - P_{mj}G^m_i - P_{im}G^m_j) \\
&= H_{ij} + \frac{n+1}{2}(y^l d_l P_{ij} - P_{mj}G^m_i - P_{im}G^m_j). \qquad (5.1)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
y^l\dot{\partial}_i Q_{jl} &= y^l d_l P_{ij} - P_{mj}G^m_i - y^l d_j P_{il} \\
&= y^l d_l P_{ij} - P_{mj}G^m_i - P_{mi}G^m_j \qquad (5.2)
\end{aligned}$$

Plugging (5.2) into (5.1) yields

$$\bar{H}_{ij} = H_{ij} + \frac{n+1}{2} y^l \dot{\partial}_i Q_{jl}. \tag{5.3}$$

We deal with C-projective mapping, i.e., $Q_{ij} = 0$. Hence $\bar{H}_{ij} = H_{ij}$. This completes the proof. □

A locally projectively flat Finsler manifold (M, F) with the projective factor P is said to be locally C-projectively flat if P satisfies $Q_{ij} = 0$, this means F is locally C-projectively related to a locally Minkowskian metric.

Example. Let Θ be the Funk metric on the Euclidean unit ball $B^n(1)$, i.e.,

$$\Theta(x,y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - <x,y>^2)} + <x,y>}{1-|x|^2}, \quad y \in T_x B^n(1) \simeq \mathbb{R}^n,$$

where $<,>$ and $|.|$ denotes the Euclidean inner product and norm on $\mathbb{R}^n$, respectively. For a constant vector $a \in \mathbb{R}^n$, let F be the Finsler metric given by

$$F := \{1 + <a,x> + \frac{<a,y>}{\Theta}\}\{\Theta + \Theta_{x^k} x^k\}. \tag{5.4}$$

In [15], Shen proves that F is projectively flat with projective factor $P = \Theta$. A direct computation shows that $Q_{ij} = 0$. Hence, F is locally C-projectively flat. Moreover, Shen proves that F is of constant flag curvature $\mathbf{K} = 0$.

Every locally Minkowskian metric has vanishing **H**-curvature. It is well known that every locally projectively flat Finsler metric is of scalar flag curvature. In the case of locally C-projectively flat Finsler metrics we have the following

Corollary 5.1. *Let F be a locally C-projectively flat Finsler metric. Then F is of constant flag curvature.*

In studying the subgroups of the group of projective transformations, Akbar-Zadeh considers projective vector fields satisfying $P_{ij} = 0$ and calls this kind of vector fields, restricted projective vector field [1]. The condition $P_{ij} = 0$ means that the projective factor P is linear, which is always true in Riemannian manifolds. Hence, in Riemannian manifolds, every projective transformation is restricted.

Let us define locally restricted projectively flatness similar to C-projectively flatness. Note that Finsler metric given in Example 1 is not locally restricted projectively flat. In fact, a restricted projective vector field with $P = a_i(x)y^i$ is C-projective vector field, if $a_i(x)$ is gradient, that is $P = d\sigma$ for some scalar function on the underlying manifold.

Using (3.11) and eliminating $P_{j|l}$ from (3.4), Akbar-Zadeh introduces the following tensor

$$\begin{aligned} {}^*W^i_{jkl} := K^i_{jkl} &- \frac{1}{n^2-1}\Big\{\delta^i_k(nK_{jl} + K_{lj}) - \delta^i_l(nK_{jk} + K_{kj})\Big\} \\ &- \frac{1}{n+1}\delta^i_j(K_{kl} - K_{lk}). \end{aligned} \tag{5.5}$$

Under a C-projective mapping, we have

$$ {}^*\overline{W}^{\,i}_{jkl} = {}^*W^i_{jkl} + 2\delta^i_k \dot{\partial}_l Q_j - 2\delta^i_l \dot{\partial}_j Q_k. \tag{5.6} $$

This means that ${}^*W^i_{jkl}$ is not a C-projective invariant. In fact, ${}^*W^i_{jkl}$ is a restricted projective invariant. We call ${}^*W^i_{jkl}$ *restricted projective Weyl-curvature*. The geometric importance of the restricted projective Weyl-curvature is to characterize Finsler metrics of constant flag curvature, i.e., a Finsler metric F^n with $(n \geq 3)$ is of constant flag curvature if and only if F has vanishing restricted projective Weyl-curvature ([2] page 209).

Now let F be a Riemannian metric. By Beltrami's well-known theorem, locally projectively flat Riemannian manifolds are exactly Riemannian manifolds of constant sectional curvature. Summarizing up, we get the following reduction theorem in Riemannian manifolds.

Theorem 5.1. *Let (M, F) be Riemannian manifold with dimension $n \geq 3$. Then the following are equivalent.*

1. *F is locally projectively flat.*
2. *F is locally restricted projectively flat.*
3. *F is locally C-projectively flat.*

This is not true in generic Finslerian manifolds. The non-equivalence between these kind of projective mappings in Finsler manifolds reveals the complexity of Finsler spaces.

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Authors' addresses:

Behzad Najafi
Faculty of Science, Department of Mathematics,
Shahed University, Tehran, Iran.
E-mail: najafi@shahed.ac.ir

Akbar Tayebi
Faculty of Science, Department of Mathematics,
Qom University, Qom, Iran.
E-mail: akbar.tayebi@gmail.com