

On generalized Einstein metrics

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Abstract. Recall that the usual Einstein metrics are those for which the first Ricci contraction of the covariant Riemann curvature tensor is proportional to the metric. Assuming the same type of restrictions but instead on the different contractions of Thorpe tensors, one gets several natural generalizations of Einstein's condition. In this paper, we study some properties of these classes of metrics.

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1 Introduction

Let (M, g) be a Riemannian manifold of dimension n and R be its Riemann covariant curvature tensor. For each even integer $2p$, $2 \leq 2p \leq n$, J. A. Thorpe [9] introduced a generalization of the Gauss-Kronecker curvature tensor as follows: For each $m \in M$ and for u_i, v_j tangent vectors at m we set

$$R_{2p}(u_1, \dots, u_{2p}, v_1, \dots, v_{2p}) = \frac{2^{-p}}{(2p)!} \sum_{\alpha, \beta \in S_{2p}} \epsilon(\alpha)\epsilon(\beta) R(u_{\alpha(1)}, u_{\alpha(2)}, v_{\beta(1)}, v_{\beta(2)}) \dots \\ \dots R(u_{\alpha(2p-1)}, u_{\alpha(2p)}, v_{\beta(2p-1)}, v_{\beta(2p)}),$$

where S_{2p} denotes the group of permutations of $\{1, \dots, 2p\}$ and, for $\alpha \in S_{2p}$, $\epsilon(\alpha)$ is the signature of α . In particular, R_2 is the Riemann covariant curvature tensor R . If the dimension n is even, $R_n(e_1, \dots, e_n, e_1, \dots, e_n)$ is the Gauss-Bonnet integrand of (M, g) , $\{e_1, \dots, e_n\}$ being an orthonormal basis of $T_m M$.

It is evident that the tensor R_{2p} is alternating in the first $(2p)$ -variables, alternating in the last $(2p)$ variables, and is invariant under the operation of interchanging the first $2p$ variables with the last $2p$. Hence it is a symmetric $(2p, 2p)$ double form. Using the exterior product of double forms, the tensors R_{2p} can be written in the following compact form [3, 8, 5],

$$R_{2p} = \frac{2^p}{(2p)!} R^p,$$

where R^p denotes the exterior product of R with itself p times. The full Ricci contraction of R_{2p} is, up to a constant, the $(2p)$ -th Gauss-Bonnet curvature of (M, g) [6, 5]. For $0 < k < 2p$, we shall say that a metric is (k, p) -Einstein if the k^{th} Ricci contraction of its Thorpe curvature tensor R_{2p} is proportional to g^{2p-k} . This condition generalizes the usual Einstein condition obtained for $p = k = 1$, and it implies that the metric is $2p$ -Einstein in the sens of [6]. In particular, in the compact case the (k, p) -Einstein metrics are for any k critical metrics of the total $2p$ -th Gauss-Bonnet curvature functional once restricted to metrics of unit volume [6].

The paper is divided into three parts. In the first one, we recall some useful facts about double forms and by the way we complete and improve two results of [5]. These results are then used in the second part where we study some geometric properties of (k, p) -Einstein metrics. We emphasize in the second part the special class of $(1, p)$ -Einstein manifolds, these manifolds behave in many directions like the usual 4-dimensional Einstein manifolds. In the last part, we discuss and generalize a related but more subtle generalization of Einstein metrics suggested by a work of Thorpe.

2 Preliminaries

Let (M, g) be a smooth Riemannian manifold of dimension n . A double form of degree p is a section of the bundle $\mathcal{D}^{p,p} = \Lambda^{*p}M \otimes \Lambda^{*p}M$ where $\Lambda^{*p}M$ denotes the bundle of differential forms of degree p on M . Double forms are abundant in Riemannian geometry: the metric, Ricci and Einstein tensors are symmetric double forms of degree 1, The Riemann curvature tensor, Weyl curvature tensor are symmetric double forms of order two, Gauss-Kronecker and Weitzenböck curvature tensors are examples of symmetric double forms of higher order.

The usual exterior product of differential forms extends in a natural way to double forms. The resulting product is some times called Kulkarni-Nomizu product of double forms. In particular, $\frac{k}{2}$ times the exterior product of the metric with itself $\frac{k}{2}g^2$ is the curvature tensor of a Riemannian manifold of constant curvature k , the repeated products of the Riemann curvature tensor $RR\dots R = R^q$ determine the generalized Gauss-Kronecker tensors defined above. If we denote by $E^{(r,r)}$ the trace free double forms of order r , then we have the following orthogonal decomposition of the bundle $D^{(p,p)}$ for $1 \leq p \leq n$:

$$(2.1) \quad D^{p,p} = E^{p,p} \oplus gE^{p-1,p-1} \oplus g^2E^{p-2,p-2} \oplus \dots \oplus g^pE^{0,0}.$$

The following proposition is a direct consequence of corollary 2.4 in [5].

Proposition 2.1. *If $n < 2p$, then*

$$(2.2) \quad D^{p,p} = g^{2p-n}D^{n-p,n-p}$$

In particular, if $\omega = \sum_{i=0}^p g^i \omega_{p-i} \in D^{p,p}$ is the decomposition of a double form ω following the orthogonal splitting (2.1), then

$$(2.3) \quad \omega_k = 0 \text{ for } n-p < k \leq p.$$

Proof. Straightforward. Just set $j = 2p - n$ and discuss following the cases where j is even or odd using corollary 2.4 of [5]. \square

Remark. For a given double form ω of order p , we evaluated in theorem 3.7 of [5], the components ω_k with respect to the orthogonal splitting (2.1). The author forgot to include in the assumptions of that formula the condition $n \geq 2p$. This condition is necessary as if not some coefficients will vanish in the proof. Also, it is clear that the coefficients $(n - 2k)!$ and $(n - p - k)!$ in the final formula are undefined if we do not assume $n \geq 2p$.

However, if $n < 2p$, the previous proposition asserts that ω is divisible by some power of the metric g and hence some components vanish as above, the other components can be determined by applying theorem 3.7 to the quotient double form.

The following proposition improves lemma 5.7 of [5] and its proof below completes a missing part in the proof of the same lemma.

Proposition 2.2. *Let $\omega = \sum_{i=0}^p g^i \omega_{p-i}$ be a double form of order p and $1 \leq k < p$. Then $c^k \omega$ is proportional to the metric g^{p-k} if and only if*

$$\omega_r = 0 \quad \text{for } 1 \leq r \leq p - k.$$

Proof. Formula (12) in [5] shows that

$$c^k \omega = \sum_{i=k}^p \alpha_{ik} g^{i-k} \omega_{p-i},$$

where $\alpha_{i0} = 1$ for $0 \leq i \leq p$. For $1 \leq k \leq i \leq p$ we have

$$\alpha_{ik} = \frac{i!}{(i-k)!} \prod_{j=1}^{j=k} (n - 2p + i + j).$$

Remark that all the coefficients α_{ik} are nonnegative for $0 \leq k \leq i \leq p$, furthermore, $\alpha_{ik} = 0$ if and only if $2p > n$ and $i < 2p - n$. In such a case we do have $\omega_{p-i} = 0$ by proposition 2.1.

Now, suppose $\omega_r = 0$ for $1 \leq r \leq p - k$, then $\omega_{p-i} = 0$ for $k \leq i \leq p - 1$ and therefore $c^k \omega = \alpha_{ip} g^{p-k} \omega_0$ is proportional to the metric.

Conversely, suppose $c^k \omega$ is proportional to the metric g^{p-k} , then $\alpha_{ik} \omega_{p-i} = 0$ for $k \leq i \leq p - 1$. If $\alpha_{ik} = 0$ then $2p > n$ and $i < 2p - n$ so that $\omega_{p-i} = 0$. In case $\alpha_{ik} \neq 0$, we have $\omega_{p-i} = 0$.

So in both cases, we have $\omega_{p-i} = 0$ for $k \leq i \leq p - 1$, that is $\omega_r = 0$ for $1 \leq r \leq p - k$. \square

3 Generalized Einstein metrics

Recall that a Riemannian manifold is said to be Einstein if the first contraction of its Riemann curvature tensor R is proportional to the metric, that is $cR = \lambda g$. Einstein metrics are also known to be the critical metrics of the total scalar curvature functional once restricted to metrics of unit volume. The gradient of the above functional

is the Einstein tensor.

In [6], we considered the critical metrics of the total Gauss-Bonnet curvature functionals once restricted to metrics with unit volume. The resulting metrics are called $(2k)$ -Einstein and are characterized by the condition that the contraction of order $(2k - 1)$ of Thorpe's tensor R^k is proportional to the metric, that is

$$(3.1) \quad c^{2k-1} R^k = \lambda g.$$

Special classes of the previous metrics are defined by:

Definition 3.1. Let $0 < p < 2q < n$, we say that a Riemannian n -manifold is (p, q) -Einstein if the p -th contraction of Thorpe's tensor of order q is proportional to the metric g^{2k-p} , that is

$$(3.2) \quad c^p R^q = \lambda g^{2q-p}.$$

We recover the usual Einstein manifolds for $p = q = 1$ and the previous $(2q)$ -Einstein condition for $p = 2q - 1$.

Applying $(2q - p - 1)$ contractions to the previous equation (3.2) we get

$$c^{2q-1} R^q = \frac{(2q-p)!(n-1)!}{(n-2q+p)!} \lambda g,$$

that is the $(2q)$ -Einstein condition above. Therefore we have for all p with $0 < p < 2q$:

$$(p, q)\text{-Einstein} \Rightarrow (2q)\text{-Einstein}.$$

In particular, the (p, q) -Einstein metrics are all critical metrics for the total Gauss-Bonnet curvature functional of order $2q$ once restricted to metrics with unit volume. Also, Schur's theorem in [6] implies that the function λ in (3.2) is then a constant. It is evident that for all $p \geq 1$, (p, q) -Einstein implies $(p+1, q)$ -Einstein. In particular, the metrics with constant q -sectional curvature (that is the sectional curvature of R^q is constant) are (p, q) -Einstein for all p .

On the other hand, the (p, q) -Einstein condition neither implies nor is implied by the $(p, q+1)$ -condition as shown by the following examples:

Let M be a 3-dimensional non-Einstein Riemannian manifold and T^k be the k -dimensional flat torus, $k \geq 1$, then the Riemann curvature tensor R of the Riemannian product $N = M \times T^k$ satisfies $R^q = 0$ for $q \geq 2$. In particular, N is (p, q) -Einstein for all $p \geq 0$ and $q \geq 2$ but it is not $(1, 1)$ -Einstein.

On the other hand, let M be a 4-dimensional Ricci-flat but not flat manifold (for example a K3 surface endowed with the Calabi-Yau metric), then the Riemannian product $N = M \times T^k$ is $(1, 1)$ -Einstein but not $(q, 2)$ -Einstein for any q with $0 \leq q \leq 3$.

It results directly from theorem 5.6 of [5] and the previous proposition the following:

Theorem 3.2. Let $0 < p < 2q$. The following statements are equivalent for a Riemannian manifold (M, g) of dimension $n \geq p + 2q$:

1. The manifold (M, g) is (p, q) -Einstein.

2. The divergence free tensor $R_{(p,q)} = * \frac{g^{n-2q-p}}{(n-2q-p)!} R^q$ is proportional to the metric g^p .
3. The components ω_r of R^q (with respect to the orthogonal splitting (2.1) for Thorpe's tensor R^q) vanish for $1 \leq r \leq 2q - p$.

The divergence free tensor $R_{(p,q)}$ used in the second part of the previous proposition (which is nothing but the (p, q) -curvature tensor [6]) generalizes the Einstein tensor obtained for $p = q = 1$.

It results from the second part of the above proposition that

Corollary 3.3. *For a $(p + 2q)$ -dimensional manifold, being (p, q) -Einstein is equivalent to constant q -sectional curvature.*

Recall that constant q -sectional curvature means that the sectional curvature of Thorpe's tensor R_{2q} is constant.

The previous result is the analogous of the fact that in dimension 3, Einstein manifolds are those with constant sectional curvature.

The last part of the previous theorem generalizes a well known characterization of Einstein manifolds ($p = q = 1$) by the vanishing of the component ω_1 in the orthogonal decomposition for the Riemann curvature tensor R .

Using generalized Avez-type formulas we proved in [4] the following

Theorem 3.4 ([4]). *Let (M, g) be a Riemannian manifold of even dimension $n = 2k + 2$. Suppose the metric g is $(2k - 2, k)$ -Einstein then the Gauss-Bonnet integrand h_n of (M, g) equals*

$$h_n = \frac{h_{n-2}}{n(n-1)} Scal.$$

In particular,

$$\chi(M) = ch_{n-2} \int_M Scal \, dvol.$$

Where c is a positive constant, h_{n-2} is a constant : the $(2k - 2, k)$ -Einstein constant. Precisely it is determined by the condition $c^{2k-2} R^k = \frac{(n-2)!h_{n-2}}{2n(n-1)} g^2$, and $Scal$ denotes as usual the scalar curvature.

The previous theorem shows that for a $(2k - 2, k)$ -Einstein manifold of dimension $2k + 2$ with $h_{n-2} \neq 0$, the integral of the scalar curvature is a topological invariant like in the two dimensional case.

Using the same formulas one can prove the following:

Theorem 3.5. *Let (M, g) be a conformally flat $2k$ -Einstein manifold of even dimension $n = 2k + 2 \geq 4$. Then the Gauss-Bonnet integrand h_n of (M, g) equals*

$$h_n = \frac{h_{n-2}}{n(n-1)} Scal.$$

In particular,

$$\chi(M) = ch_{n-2} \int_M Scal \, dvol.$$

Where c is a positive constant, h_{n-2} is a constant : the $2k$ -th Einstein constant. Precisely it is determined by the condition $c^{2k-1}R^k = \frac{(n-2)(n-3)!h_{n-2}}{n}g$, and $Scal$ is the scalar curvature.

Proof. Recall that for a conformally flat metric we have $R = Ag$ where A is the Schouten tensor. On the other hand, for a $2k$ Einstein metric we have $c^{2k-1}R^k = \lambda g$ with $\lambda = \frac{(n-2)h_{n-2}}{n}$. These two facts together with the generalized Avez type formula of [4] show that

$$\begin{aligned} h_{2k+2} &= (2k-1)\langle \lambda g, A \rangle - \langle \lambda g, cR \rangle + h_{2k}h_2 \\ &= (2k-1)\lambda cA - \lambda c^2R + h_{2k}h_2 \\ &= \frac{(n-2k)(n-2k-1)}{n(n-1)}h_{2k}h_2. \end{aligned}$$

This completes the proof of the theorem. \square

We bring the attention of the reader to a recent work of Lima and Santos [7] where they prove interesting results about the moduli space of $2k$ Einstein structures.

Hyper $2k$ -Einstein metrics.

For q fixed, the $(1, q)$ -Einstein condition is the strongest among all the other (p, q) conditions. In the rest of this section we shall emphasize some properties of this particular condition.

Definition 3.6. Let $0 < 2q < n$, we shall say that a Riemannian n -manifold is hyper $(2q)$ -Einstein if the first contraction of Thorpe's tensor R^q is proportional to the metric g^{2q-1} , that is a $(1, q)$ -Einstein manifold.

The next proposition provides topological and geometrical obstructions to the existence of hyper $(2q)$ -Einstein metrics:

Theorem 3.7. Let $k \geq 1$ and (M, g) be a hyper $(2k)$ -Einstein manifold of dimension $n \geq 4k$. Then the Gauss-Bonnet curvature h_{4k} of (M, g) is nonnegative. Furthermore, $h_{4k} \equiv 0$ if and only if (M, g) is k -flat.

In particular, a compact hyper $(2k)$ -Einstein manifold of dimension $n = 4k$ has its Euler-Poincaré characteristic nonnegative. Furthermore, it is zero if and only if the metric is k -flat.

Recall that k -flat means that the sectional curvature of R^k is identically zero. The previous proposition generalizes a well known obstruction to the existence of Einstein metrics in dimension four due to Berger.

Proof. For a hyper $(2k)$ -Einstein metric, the components R_r^q vanish for $1 \leq r \leq 2k-1$ by the previous proposition. Next, for $n \geq 4k$, the generalized Avez formula, see corollary 6.5 in [5], shows that

$$(3.3) \quad h_{4k} = \frac{1}{(n-4k)!} \{n! \|R_0^k\|^2 + (n-2k)! \|R_{2k}^k\|^2\}.$$

Where R_i^k denotes the component of R^k with respect to the orthogonal decomposition (2.1). In particular, $h_{4k} \geq 0$ and it vanishes if and only if $R_i^k = 0$ for all $0 \leq i \leq 2k$, that is $R^k \equiv 0$.

If $n = 4k$, then h_n is up to a positive constant the Gauss-Bonnet integrand. This completes the proof. \square

A double form ω is said to be harmonic if it is closed and co-closed that is $D\omega = \delta\omega = 0$. Where D is the second Bianchi map (or equivalently, the vector valued exterior derivative) and $\delta = c\tilde{D} + \tilde{D}c$ its "formal adjoint", \tilde{D} being the adjoint second Bianchi map, see [3, 6].

Proposition 3.8. *Let $k \geq 1$, then for a hyper $(2k)$ -Einstein manifold, Thorpe's curvature tensor R^k is a harmonic double form.*

Proof. Since R satisfies the second Bianchi identity then $DR^k = 0$. On the other hand,

$$\delta R^k = (c\tilde{D} + \tilde{D}c)R^k = \tilde{D}(cR^k) = \tilde{D}(\lambda g^{2k-1}) = 0.$$

\square

The converse in the previous proposition is not generally true. One needs to impose extra conditions on the manifold, similar to the ones in [2], in order to ensure the generalized Einstein condition.

For $n \geq 4k$, following [3, 8], an n -dimensional Riemannian manifold is said to be k -conformally flat if Thorpe's tensor R^k is divisible by the metric g . Equivalently, the component $(R^k)_{2k}$ of R^k with respect to the orthogonal splitting (2.1) vanishes. The proof of the following proposition is straightforward if one uses theorem 3.2

Proposition 3.9. *A hyper $(2k)$ -Einstein manifold which is k -conformally flat has constant k -sectional curvature.*

The previous proposition generalizes a similar result about usual Einstein conformally flat manifolds obtained for $k = 1$.

4 Thorpe's condition

In this section we consider different generalizations of the Einstein condition suggested by the work of Thorpe [9].

Recall that in dimension four, Einstein metrics are characterized by the invariance of their curvature tensor under the Hodge star operator in the sense that $*R = R$, where R is seen as a double form and $*$ is the generalized Hodge star operator [5]. In higher even dimensions we proved a similar characterization:

Proposition 4.1 ([5]). *A Riemannian manifold (M, g) of dimension $n = 2p \geq 4$ is Einstein if and only if its Riemann curvature tensor R satisfies*

$$(4.1) \quad *g^{p-2}R = g^{p-2}R.$$

For $(4k)$ -dimensions, Thorpe considered the following similar condition but on the generalized Thorpe's tensor instead, namely $*R^k = R^k$.

For a $(4k)$ -dimensional compact and oriented Riemannian manifold, he proved that the previous condition implies

$$(4.2) \quad \chi \geq \frac{(k!)^2}{(2k)!} |p_k|.$$

Furthermore, $\chi = 0$ if and only if the manifold is k -flat. Where p_k denotes the k^{th} Pontrjagin number of the manifold.

If $R^k = \sum_{i=0}^{2k} g^i \omega_{2k-i}$ is the decomposition of R^k following the orthogonal splitting (2.1), then Thorpe's condition $*R^k = R^k$ is equivalent [5] to the vanishing of ω_r for r odd and $1 \leq r \leq 2k-1$. This in turn can be seen to be equivalent to the condition that $c^r R^k$ is divisible per the metric (that is $c^r R^k = g\bar{\omega}$) for r odd and $1 \leq r \leq 2k-1$. These restrictions on R^q are guaranteed in arbitrary even dimension $n = 2p \geq 4q$ if one assumes the following generalized Thorpe type condition, see theorem 5.8 in [5]

$$(4.3) \quad *g^{p-2q} R^q = g^{p-2q} R^q.$$

The previous generalization of Thorpe's condition forces the Gauss-Bonnet curvature h_{4k} to be nonnegative in dimensions $n \geq 4k$ (theorem 6.7 of [5]).

It results from the previous discussion that for a $(4k)$ -dimensional manifold, the hyper $(2k)$ -Einstein condition is stronger than Thorpe's condition $*R^k = R^k$. In higher arbitrary even dimensions, the (p, q) -Einstein condition implies the generalized Thorpe's condition $*g^{p-2q} R^q = g^{p-2q} R^q$. In particular, theorem 3.7 is then a consequence of the above result of Thorpe in the special case where $n = 4k$. Also, theorem 6.7 of [5] implies the conclusions of theorem 3.7 for even dimensions higher than $4k$.

On the other hand Thorpe's condition and its generalization are actually complicated expressions once written in terms of the contractions of R^q as it is shown below.

The proof of the following proposition is a direct application of formula (15) of [5]

Proposition 4.2. *A Riemannian manifold of dimension $n = 2p \geq 4q$ satisfies the generalized Thorpe's condition (that is $*g^{p-2q} R^q = g^{p-2q} R^q$) if and only if*

$$(4.4) \quad \sum_{r=1}^{2q} \frac{(-1)^r (p-2q+1)!}{r!(p-2q+r)!} g^{r-1} c^r R^q = 0.$$

In particular, if $n = 4q$, Thorpe's condition is equivalent to

$$(4.5) \quad \sum_{r=1}^{2q} \frac{(-1)^r}{(r!)^2} g^{r-1} c^r R^q = 0.$$

Final remarks and open questions

The model spaces for (p, q) -Einstein manifolds are the spaces with constant q -th Thorpe's sectional curvature (that is $(0, q)$ -Einstein). It would be then interesting to classify Riemannian manifolds of constant k -th Thorpe's sectional curvature.

Another interesting natural question is to generalize Hamilton's Ricci flow techniques to the various generalized Ricci curvatures introduced here in this paper.

Finally, we bring the attention of the reader to different generalizations of the Einstein condition namely quasi-Einstein due first to Chaki and Maity and then generalized by several authors, see for instance [1] and the references therein.

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