On stability of Ricci flows based on bounded curvatures

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Abstract. Recognizing the deficiency that C. Guenther's arguments can not solve the stability of Ricci flows because of the Ricci flow equation being not strictly parabolic, our previous paper first studied the stability of Ricci flows based on Killing conditions. In this paper, we consider the stability of Ricci flows, and of quasi-Ricci flows based on bounded curvature conditions, and also obtain some interesting results.

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1 Introduction

It is interesting to investigate the stability of Ricci flows. The study of Ricci flows has been an active field over the past several decades. It is well known that, in the early days of 1983, R. Hamilton [12], drawing inspiration from the work by J. Eells Jr and J. H. Sampson [9], introduced the celebrated Ricci flows as follows

$$\frac{\partial}{\partial t} g = -2Rc[g], \ g(0) = g_0 \tag{1.1}$$

A fundamental and difficult problem in differential geometry is to find a standard metric satisfying some prescribed conditions over a Riemannian manifold. For instance, concerning the celebrated Yamabe problem [20], it is essential to find a metric with a constant scalar curvature; and for the constant Ricci curvature, one needs to solve an Einstein equation. The study of Ricci flows, in general, is exactly to find a standard metric satisfying the given conditions, and to solve Ricci equation. The typical problem related to Ricci flows is the following short-time existence theorem:

Given a compact and smooth Riemannian manifold $(\mathcal{M}^n, g_0)$, there exists a unique smooth solution $g(t)$ defined on a short-time-interval such that $g(0) = g_0$.

It is natural to ask that in which case the long-time existence theorem of Ricci flows is tenable and the solution converges to a constant curvature metric. The

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usual cases in this respect are those with positive curvatures. Moreover, the study of the singularity [18] of solutions to Ricci flows and the estimation [24] of geometric invariants associated with different pinch-conditions have achieved relatively profound and sufficient development.

There are many interesting results related to Ricci flows [4], [11]-[16] since Ricci flows were introduced by R. Hamilton. However, there are many important and interesting problems being open. Anyone of them is the stability problem of Ricci flows. These questions can be written as follows.

Let the solution $g(t)$ of Ricci flows with initial value g_0 converge, and $\tilde{g}_0$ belong to a neighborhood of g_0, then, is it true that the solution $\tilde{g}(t)$ of Ricci flows with initial value $\tilde{g}_0$ converges ?

In [24], Ye studied the stability of Ricci flows with a metric of constant non-zero sectional curvature and he replaced the original Ricci flows by the value-normalized Ricci flows

$$\frac{\partial}{\partial t}g = -2Rc[g] + \frac{2}{n}(\oint Rd\mu)g, \quad g(0) = g_0$$

where $\oint Rd\mu \doteqdot \frac{\int Rd\mu}{\int d\mu}$.

Ye also derived that there exists a C^2-neighborhood $\mathcal{N}(g_0)$ of g_0 such that, for any $\tilde{g}_0 \in \mathcal{N}(g_0)$, the solution of Ricci flows $\tilde{g}(t)$ corresponding to $\tilde{g}_0$ converges to g_0 if g_0 is a Riemannian-Pinched Einstein metric with non-zero scalar curvature. For the stability of Ricci flows of the flat metric, he has not obtained a solution. Following on the heels of Ye's work, C. Guenther etc [10] first introduced center manifolds [6] and maximal regularity theory [17, 21] and derived the stability of Ricci (DeTurck) flows in constant curvature spaces. The maximal regularity theory says that if $\mathcal{A}$ is a suitable quasi-linear differential operator acting on an appropriate function space, and if its linearization $D\mathcal{A}$ at a fixed point has an eigenvalue on the imaginary axis, then the evolution of solutions starting near that fixed point can be described by the presence of exponentially attractive center manifolds.

Since the Ricci flow evolution equation (1.1) is not a strictly parabolic system, the maximal regularity theory can not be applied directly to it. It is known that a strictly parabolic evolution equation, i.e., the DeTurck [7] equation,

$$\frac{\partial}{\partial t}g = -2Rc[g] - P_u(g), \quad g(0) = g_0 \tag{1.2}$$

can replace the Ricci flow equation. In fact, the solution of (1.2) is equivalent to that of (1.1) up to a simple parameter diffeomorphic transform group. Hence, one can study the stability of convergent Ricci flows by virtue of the stability of convergent DeTurck flows.

C. Guenther [10] studied the stability of Ricci flows corresponding to initial value metric with non-zero constant curvature, but in this setting the DeTurck flow does not satisfy maximal regularity theory: no matter what u takes, any stable solution to this equation does not exist. Thus, we will consider the normalized DeTurck equation as follows

$$\frac{\partial}{\partial t}g = -2Rc[g] - P_u(g) + \frac{2}{n}(\oint Rd\mu)g, \quad g(0) = g_0 \tag{1.3}$$

In fact, in this setting, $u = g_0$ is a stable solution of (1.3).

Since a Riemannian manifold of quasi-constant curvatures is the special quasi-Einstein space [26, 27], we generalized naturally the problem (1.2) to the quasi-constant curvature manifold, i.e., we will consider the quasi-DeTurck flows [3] as follows

$$\frac{\partial}{\partial t}g = -2Rc[g] - P_u(g) + 2\frac{R-T}{n-1}g + 2\frac{nT-R}{n-1}\xi\otimes\xi, \quad g(0) = g_0 \tag{1.4}$$

where ξ is a unit vector field, and T is the Ricci principal curvature corresponding to ξ. Notice that the stability here is different from the one that is posed in [28].

We know that the DeTurck flows, given by C. Guenther [10], are in fact obtained by adding $P_u(g)$ to Ricci flows such that all the quadratic terms up to Laplace operator vanishes, and thus they shared the same principal symbols with Laplace operator.

Motivated and inspired by the structure of DeTurck flows, in our previous paper [25], we considered the stability of Ricci flows in terms of Killing conditions, but we adopted the other arguments to derive similar interesting conclusions, this argument successfully avoids DeTurck flows and makes consideration directly to Ricci flows. In this paper [25], we studied the following problem and obtained some interesting conclusions

(a) The stability of the solution of Ricci flows with Killing conditions in a constant curvature space.

Moreover, for the quasi-Ricci flows [3]

$$\frac{\partial}{\partial t}g = -2Rc[g] + 2\frac{R-T}{n-1}g + 2\frac{nT-R}{n-1}\xi\otimes\xi, \quad g(0) = g_0 \tag{1.5}$$

we also derived that

(b) The stability of the solution of quasi-Ricci flows with Killing conditions in a quasi constant curvature space.

It is well known that the Killing condition is too strong, thus we wish to eliminate this condition and replace it by curvature conditions. In other words, we study the stability of Ricci flows and quasi-Ricci flows based on bounded curvature conditions, and will get some interesting results.

The organization of this paper is as follows. In Section 2, we will recall some necessary notations and give terminologies. Section 3 is devoted to the proofs of main theorems. The main results are related to the stability of Ricci flows over a constant curvature space and stability of quasi-Ricci flows over a quasi-constant curvature space.

2 Preliminaries

For convenience, we first give some preparatory knowledge. Let $\mathcal{M}$ be a closed connected smooth manifold, and denote by $S_2(\mathcal{M})$ the bundle of symmetric covariant 2-tensors over $\mathcal{M}$, and by $S_2^+(\mathcal{M})$ the subset of the positive definite tensors. In this setting, a smooth Riemannian metric g is an element of $C^\infty(S_2^+(\mathcal{M}))$. On the other hand, we denote briefly by $\mathcal{S}_2 \doteqdot C^\infty(S_2(\mathcal{M}))$ and $\mathcal{S}_2^+ \doteqdot C^\infty(S_2^+(\mathcal{M}))$, and denote also by $\mathcal{S}_2^\mu \doteqdot C^\infty(S_2^\mu(\mathcal{M}))$ the space of all metrics with the same volume element given by g, and by $\mathcal{S}_2^{\mu^+} \doteqdot C^\infty(S_2^{\mu^+}(\mathcal{M}))$ the collection of positive definite tensors of

$C^\infty(S_2^\mu(\mathcal{M}))$. Denote by $\Lambda^p = \Lambda^p(T^*\mathcal{M})$ the $p-$form bundle on $\mathcal{M}$, and denote by $\Omega^p = C^\infty(\Lambda^p)$ the differential $p-$form bundle.

Assume that $\mathcal{D}(\mathcal{M})$ is the smooth diffeomorphic group: $(h,\phi) \mapsto \phi^* h$ acting on $\mathcal{S}_2^+$, and it is easy to check that g is of Einstein if and only if $\phi^* g$ is of Einstein, where g is a Riemannian metric on $\mathcal{M}$ and its volume form is $d\mu$.

Define a map $\delta = \delta_g : \mathcal{S}_2 \to \Omega^1$ by

$$\delta : h \mapsto \delta h = -g^{ij}\nabla_i h_{jk} dx^k \tag{2.1}$$

whose formal adjoint under the L^2 inner product

$$(\cdot,\cdot) \doteqdot \int_M <\cdot,\cdot> d\mu$$

is the map $\delta^* = \delta_g^* : \Omega^1 \to \mathcal{S}_2$ given by $\delta^* : \omega \mapsto \frac{1}{2}\mathcal{L}_{\omega^\sharp} g = \frac{1}{2}(\nabla_i\omega_j + \nabla_j\omega_i)dx^i \otimes dx^j$, where $\omega^\sharp$ is a vector field metrically isomorphic to ω.

Define $G : \mathcal{S}_2^+ \times \mathcal{S}_2 \to \mathcal{S}_2$, by virtue of [7], as

$$(g,u) \mapsto G(g,u) = (u_{ij} - \frac{1}{2}g^{kl}u_{kl}g_{ij})dx^i \otimes dx^j$$

and $P : \mathcal{S}_2^+ \times \mathcal{S}_2^+ \to \mathcal{S}_2$ as

$$(g,u) \mapsto P(g,u) \doteqdot P_u(g) = -2\delta_g^*(u^{-1}\delta_g(G(g,u)))$$

Thus, one can consider the following evolution equation (DeTurck equation)

$$\frac{\partial}{\partial t} g = -2Rc[g] - P_u(g), \quad g(0) = g_0 \tag{2.2}$$

For the sake of convenience, we call $\bar{\mathcal{A}}_u(g)g \doteqdot -2Rc[g] - P_u(g)$ the DeTurck operator, then, the formula (2.2) can be rewritten as

$$\frac{\partial}{\partial t} g = -2Rc[g] - P_u(g) = \bar{\mathcal{A}}_u(g)g, \quad g(0) = g_0.$$

It is well known that the DeTurck operator $\bar{\mathcal{A}}_u(g)g$, in the local sense, can be written as

$$(\bar{\mathcal{A}}_u(g)g)_{ij} = a(x,u,g)_{ij}^{klpq}\frac{\partial^2}{\partial x^p \partial x^q} g_{kl} + b(x,u,\partial u,g)_{ij}^{klp}\frac{\partial}{\partial x^p} g_{kl} + c(x,u,\partial u)_{ij}^{kl} g_{kl},$$

where $a(x,\cdot,\cdot), b(x,\cdot,\cdot,\cdot), c(x,\cdot,\cdot)$ are smooth functions with respect to $x \in \mathcal{M}^n$, respectively, and are analytic with respect to the remaining arguments.

On the other hand, the right hand of (2.2) and Laplacian operator have the same symbol. It is easy to see that, for any $u \in \mathcal{S}_2^+$, the equation (2.2) is strictly parabolic, and its unique solution g provides a unique solution $\phi_t^* g$ of (1.1) with initial value g_0, where the diffeomorphisms ϕ_t are generated by integrating the vector field

$$V^i \doteqdot g^{ij}u_{jk}^{-1}g^{kl}g^{pq}(\nabla_p u_{ql} - \frac{1}{2}\nabla_l u_{pq})$$

Assume that $(\mathcal{M}, g)$ is a Riemannian manifold, and denote by $\Delta = g^{ij}\nabla_i\nabla_j$ the Laplace operator. Let Δ_l be the Licherowicz-Laplace operator such that $\Delta_l : \mathcal{S}_2 \to \mathcal{S}_2$ given by

$$\Delta_l h_{ji} = \Delta_l h_{ji} + 2R_{jpqi}h^{pq} - R_j^k h_{ki} - R_i^k h_{jk}.$$

Lemma 2.1 ([10]). *Let $g \in \mathcal{S}_2^+, h \in \mathcal{S}_2$, and define $H \doteqdot tr_g h \doteqdot g^{ji}h_{ji}$, $divh_k \doteqdot \nabla^p h_{kp}$. Let $\tilde{g} = g + \epsilon h$, and denote by $\tilde{\Gamma}, \tilde{R}, d\tilde{\mu}$ the Christoffel coefficient, curvature tensor, volume element of $\tilde{g}$, respectively. Then, one arrives at*

$$\begin{aligned}
\frac{\partial}{\partial\epsilon}\tilde{\Gamma}^k_{ij}(\tilde{g})|_{\epsilon=0} &= \frac{1}{2}g^{kl}(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij});\\
\frac{\partial}{\partial\epsilon}\tilde{R}^l{}_{ijk}(\tilde{g})|_{\epsilon=0} &= \frac{1}{2}(\nabla_i\nabla_k h^l_j - \nabla_i\nabla^l h_{jk} - \nabla_j\nabla_k h^l_i + \nabla_j\nabla^l h_{ik}\\
&\quad + R^l{}_{ijm}h^m_k - R^m{}_{ijk}h^l_m);\\
\frac{\partial}{\partial\epsilon}d\tilde{\mu}(\tilde{g})|_{\epsilon=0} &= \frac{1}{2}Hd\mu;\\
\frac{\partial}{\partial\epsilon}\tilde{g}^{ij}|_{\epsilon=0} &= -g^{ik}g^{jl}h_{kl} = -h^{ij};\\
\frac{\partial}{\partial\epsilon}(\mathcal{L}_X\tilde{g})_{ij}|_{\epsilon=0} &= X^k\nabla_k h_{ij} + h_{ik}\nabla_j X^k + h_{jk}\nabla_i X.
\end{aligned}$$

Let $\Theta_0, \Theta_1, \Xi_0, \Xi_1$ be the Banach spaces such that there holds[2]: $\Xi_0 = h^{0+\sigma} \supset \Theta_0 = h^{0+\rho} \supset \Xi_1 = h^{2+\sigma} \supset \Theta_1 = h^{2+\rho}$, where $0 < \sigma < \rho < 1$, $h^{r+\rho}(r \in \mathbb{N}, \rho \in (0,1))$ is the special little Hölder space. Assume that $\|\cdot\|_{r+\rho}$ is the Hölder norm of $C^r(M, \mathcal{S}_2)$. Taking $\theta = \frac{\rho-\sigma}{2} \in (0,1)$, by using [2, 8, 22], one gets $\Theta_0 \cong (\Xi_0, \Xi_1)_\theta$ and $\Theta_1 \cong (\Xi_0, \Xi_1)_{1+\theta}$.

For the given $0 < \epsilon \ll 1$ and $\frac{1}{2} \leq \beta < \alpha < 1$, let

$$G_\beta = G_\beta(u,\epsilon) = \{g \in (\Theta_0, \Theta_1)_\beta : g > \epsilon u\}, \quad G_\alpha = G_\alpha(u,\epsilon) = G_\beta \cap (\Theta_0, \Theta_1)_\alpha$$

where $g > \epsilon u$ implies that it holds $g(X,X) > \epsilon$ for any X satisfying $|X|^2_u = 1$.

Moreover, for any $g \in G_\beta$, $\bar{\mathcal{A}}_u(g)$ can be regarded as a linear operator acting on $h^{2+\sigma}$. Denote by $\bar{\mathcal{A}}_{\Xi_1}(g) : \Xi_1 \subseteq \Xi_0 \to \Xi_0$ the unbounded linear operator on Ξ_0, its dense domain $D(\bar{\mathcal{A}}_{\Xi_1}(g)) = \Xi_1$. Make corresponding changes and denote by $\bar{\mathcal{A}}_{\Theta_1}(g) : \Theta_1 \subseteq \Theta_0 \to \Theta_0$ the unbounded linear operator whose dense domain $D(\bar{\mathcal{A}}_{\Theta_1}(g)) = \Theta_1$. At the same time, the functions $g \mapsto \bar{\mathcal{A}}_{\Theta_1}(g)$ and $g \mapsto \bar{\mathcal{A}}_{\Xi_1}(g)$ define the analytic maps given by $G_\alpha \to L(\Theta_1, \Theta_0)$, $G_\beta \to L(\Xi_1, \Xi_0)$, where $L(\Theta_1, \Theta_0)$ is the vector space of all bounded linear operators from Θ_1 to Θ_0, and for any $g \in G_\beta$, $\bar{\mathcal{A}}_u(g)$ is the minimal generator of a strongly continuous analytic semigroup.

Theorem 2.1 ([10, 17, 21, 25]). *Let $\Theta_1 \subset \Theta_0$ be a continuous dense inclusion of a Banach space. For a given $0 < \beta < \alpha < 1$, suppose that Θ_α and Θ_β are the corresponding interpolation space. For the following equation*

$$\frac{\partial}{\partial t}g = \bar{\mathcal{A}}(g)g, \quad g(0) = g_0 \tag{2.3}$$

where $\bar{\mathcal{A}}(\cdot) \in C^k(G_\beta, L(\Theta_1, \Theta_0))$, and k is a positive integer, $G_\beta \subset \Theta_\beta$ is an open subset. Assume that there exist a pair Banach space $\Xi_0 \supset \Xi_1$ and a prolongation $\tilde{\mathcal{A}}(\cdot)$ of $\bar{\mathcal{A}}(\cdot)$ to domain $D(\tilde{\mathcal{A}}(\cdot))$ that are dense in Ξ_0. In addition, for any $g \in G_\alpha = G_\beta \cap \Theta_\alpha$, then there holds

· $\tilde{\mathcal{A}}(g) \in L(\Xi_1, \Xi_0)$ generates a strongly continuous semigroup on $L(\Xi_0, \mathbb{R}) \doteqdot L(\Xi_0)$;

· $\Theta_0 \cong (\Xi_0, D(\tilde{\mathcal{A}}(g)))_\theta, \Theta_1 \cong (\Xi 0, D(\tilde{\mathcal{A}}(g)))_{1+\theta}, \theta \in (0,1)$, where $(\cdot,\cdot)_\theta$ are the continuous interpolations [8, 22];

· $\bar{\mathcal{A}}(g)$ is identical to $\tilde{\mathcal{A}}(g)$ on $D(\bar{\mathcal{A}}) \subset \Theta_0$;

$\cdot\ \Xi_1 \hookrightarrow \Theta_\beta \hookrightarrow \Xi_0$ *is a continuous dense inclusion and there exists* $c > 0, \delta \in (0,1)$ *such that for any* $\eta \in \Xi_1$ *there holds*

$$\|\eta\|_{\Theta_\beta} \leq c\|\eta\|_{\Xi_0}^{1-\delta}\|\eta\|_{\Xi_1}^{\delta}.$$

Let $\hat{g} \in G_\alpha$ be a fixed point of (1.2) and the spectral decompositions[1] $\sum$ of the linearization operator $D\bar{\mathcal{A}}|_g$ be of $\sum = \sum_s \cup \sum_{cu}$, where $\sum_s \subset \{z : Rez < 0\}$, $\sum_{cu} \subset \{z : Rez \geq 0\}$ and $\sum_{cu} \cap i\mathbb{R} \neq \varnothing$, then it holds

(1) If one denotes by $S(\lambda)$ the eigenspace of $\lambda \in \sum_{cu}$, then Θ_α admits the decomposition $\Theta_\alpha = \Theta_\alpha^s \oplus \Theta_\alpha^{cu}$ for all $\alpha \in [0,1]$, where $\Theta_\alpha^{cu} = \bigoplus_{\lambda \in \sum_{cu}} S(\lambda)$;

(2) For any $r \in \mathbb{N}$, there exists $d_r > 0$ such that for all $d \in (0, d_r]$, there is a bounded C^r map $\varphi = \varphi_d^r : B(\Theta_1^{cu}, \hat{g}, d) \to \Theta_1^s$ with $\varphi(\hat{g}) = 0$ and $D\varphi(\hat{g}) = 0$. The image of φ lies in the closed ball $\bar{B}(\Theta_1^s, \hat{g}, d)$, and its graph is a C^r manifold $\mathcal{M}_{loc}^{cu} \doteqdot \{(\gamma, \varphi(\gamma)) : \gamma \in B(\Theta_1^{cu}), \hat{g}, d\} \subset \Theta_1$ satisfying the following

$$T_{\hat{g}}\mathcal{M}_{loc}^{cu} \cong \Theta_1^{cu}.$$

If $\sum_{cu} \subset i\mathbb{R}$, we call $\mathcal{M}_{loc}^{cu}$ a local center manifold [6] and a local center unstable manifold otherwise;

(3) There are constants $C_\alpha > 0$ $(\alpha \in (0,1))$ independent of $\hat{g}$ and constant $\omega > 0$ and $\hat{d} \in (0, d_0]$ such that for each $d \in (0, \hat{d}]$, one arrives at

$$\|\pi^s g(t) - \varphi(\pi^{cu} g(t))\|_{\Theta_1} \leq \frac{C_\alpha}{t^{1-\alpha}} e^{-\omega t} \|\pi^s g(0) - \varphi(\pi^{cu} g(0))\|_{\Theta_\alpha}$$

for all solutions $g(t)$ with $g(0) \in B(\Theta_\alpha, \hat{g}, d)$ and all times $t \geq 0$ such that the solution $g(t)$ remains in $B(\Theta_\alpha, \hat{g}, d)$, where π^s, π^{cu} say the projections from $B(\Theta_\alpha, \hat{g}, d)$ onto Θ_α^s, and Θ_α^{cu}, respectively.

3 Main theorems

We first state some necessary notations and terminologies in this subsections. In fact, we know that for Ricci flows one can write down by virtue of Lemma 2.1 the following

$$DR_{ji} = \frac{1}{2}(\Delta h_{ji} + \nabla_j \nabla_i H - \nabla_j \mathrm{div} h_i - \nabla_i \mathrm{div} h_j + 2R_{jpqi} g^{pq} - R_{jl} h_i^l - R_{il} h_j^l),$$

then, one gets

$$-2[D(Rc)]_{ji} = \Delta h_{ji} - \nabla_j X_i - \nabla_i X_j + S_{ji}$$

where X_j is of the 1-form defined by $X_j = g^{pq} \nabla_p h_{qj} - \frac{1}{2} \nabla_j (g^{pq} h_{pq})$, and $S_{ji} = 2R_{jpqi} g^{pq} - R_{jl} h_i^l - R_{il} h_j^l$.

Moreover, let V, W be two vector bundles over a manifold $\mathcal{M}^n$, and $\mathcal{L} : C^\infty(V) \to C^\infty(W)$ be a linear differential operator with order k. Denote by $\mathcal{L}(\nu) \doteqdot \sum_{|\alpha| \leq k} \mathcal{L}_\alpha \partial^\alpha \nu$, where $\mathcal{L}_\alpha \in \mathrm{hpm}(V, W)$ is a bundle homomorphism for each multi-index α. If $\xi \in C^\infty(T^* M^n)$, then we call $\sigma[\mathcal{L}](\xi) = \sum_{|\alpha| \leq k} \mathcal{L}_\alpha (\Pi_j \xi^{\alpha j})$ the total symbol of $\mathcal{L}$ in the direction ξ. We also call $\hat{\sigma}[\mathcal{L}](\xi) = \sum_{|\alpha|=k} \mathcal{L}_\alpha(\Pi_j \xi^{\alpha j})$ the principal symbol of $\mathcal{L}$ in the direction ξ.

A linear partial differential operator $\mathcal{L}$ is said to be elliptic if its principal symbol $\hat{\sigma}[\mathcal{L}](\xi)$ is an isomorphism where $\xi \neq 0$. A nonlinear operator L is said to be elliptic if its linearization DL is elliptic.

In other words, one arrives at the following

$$\hat{\sigma}[-2D(Rc)](\xi)(h) = |\xi|^2 h$$

This implies that the linearized Ricci flow in view of Killing 1-form is elliptic. In this setting, we call $\mathcal{A}(g)g = -2Rc[g] + \frac{2}{n}(\oint Rd\mu)g$ the Ricci operator. The volume-normalized Ricci flow [24] can be rewritten as

$$\frac{\partial}{\partial t} g = \mathcal{A}(g)g, \quad g(0) = g_0 \tag{3.1}$$

In the following subsection, we will also pay our attention to the linearization of the Ricci operator.

Lemma 3.1. *Assume that $\mathcal{M}^n$ is a compact manifold of constant curvature, then the linearized Ricci operator $\mathcal{A}(g)g$ at g_0 is as follows*

$$[(D\mathcal{A}(g)g)|_{g_0} h]_{ji} = \Delta h_{ji} + 2R_{jpqi}h^{pq} - \frac{2R}{n^2} g_{ji} \oint_{\mathcal{M}_n} H d\mu + (\mathcal{L}_{X^\sharp} g)_{ji} \tag{3.2}$$

where $H = g^{ji}h_{ji}$.

Proof. According to Definition and Lemma 2.1, one has

$$[D\mathcal{A}_u(g)]|_{g_0} h = -2DRc|_{g_0} + \frac{2}{n} D(\oint Rd\mu)|_{g_0} g + \frac{2}{n} \oint H d\mu h.$$

$$\begin{aligned}(-2DRc|_{g_0} h)_{ji} &= \Delta h_{ji} - \nabla_j(g^{pq}\nabla_p h_{pi}) - \nabla_i(g^{pq}\nabla_p h_{pj}) \\ &\quad + \nabla_j\nabla_i(g^{pq}h_{pq}) + 2g^{pq}R^r{}_{pji}h_{rq} - g^{pq}R_{jp}h_{iq} - g^{pq}R_{ip}h_{jq}.\end{aligned}$$

Since $(\mathcal{M}^n, g_0)$ is of an Einstein manifold, we get

$$(-2DRc|_{g_0} h)_{ji} = \Delta h_{ji} + 2g^{pq}R^r{}_{pji}h_{rq} - \frac{2R}{n} h_{ji} + (\mathcal{L}_{X^\sharp} g)_{ji}.$$

$$\frac{2}{n} D(\oint Rd\mu)g = \frac{2}{n}[\oint (\frac{1}{2}(R - \oint Rd\mu)H - < Rc, h >)d\mu]g = -\frac{2}{n}[\oint < Rc, h > d\mu]g.$$

Thus, we have

$$\begin{aligned}(D\mathcal{A}_u(g)|_{g_0} h)_{ji} &= \Delta h_{ji} + 2R_{jpqi}h_{pq} - \frac{2R}{n} h_{ji} - \frac{2}{n}[\oint < Rc, h > d\mu] g_{ji} \\ &+ \frac{2R}{n} h_{ji} + (\mathcal{L}_{X^\sharp} g)_{ji} \\ &= \Delta h_{ji} + 2R_{jpqi}h^{pq} - \frac{2R}{n^2}[\frac{\int_{\mathcal{M}_n} H d\mu}{\int_{\mathcal{M}_n} d\mu}] g_{ji} + (\mathcal{L}_{X^\sharp} g)_{ji}.\end{aligned}$$

This ends the proof of Lemma 3.1. □

Notice that g_0 is a stable point of (1.5) for Quasi-Constant curvature spaces [3, 24]. Then, we can state and derive the main conclusions in the next subsection.

Theorem 3.1. *Let $\mathcal{M}^n$ be a compact manifold of constant curvatures, and $\|Rm\| \le 2\Lambda(\Lambda-1)$, where $\Lambda = \inf\limits_h\{\frac{\int_{\mathcal{M}^n}|\nabla h|^2}{\int_{\mathcal{M}^n}|h|^2}\}$, h is of a (0,2)-type tensor. Θ is of a closure of $\mathcal{S}_2^\mu(\supset \mathcal{S}_2^{\mu^+})$ in the sense of $\|\cdot\|_{2+\rho}$ for a fixed $\rho \in (0,1)$, then there holds the following*

(1) $T_{g_0}\mathcal{S}_2^{\mu^+} \cong \Theta$ has a decomposition: $T_{g_0}\mathcal{S}_2^{\mu^+} = \Theta^s \oplus \Theta^c$;

(2) For each $r \in \mathbb{N}$, there exists a C^r-center manifold $\mathcal{M}^c_{loc}$ that is tangential to Θ^c in an neighborhood $\mathcal{O}_r$ of g_0 on Θ and is locally invariant for solutions of (3.1) as long as they remain in $\mathcal{O}_r$;

(3) There exist positive constants C and ω, and neighborhoods $\mathcal{O}'_r$ of g_0 in Θ such that

$$\|\pi^s\tilde{g}(t) - \varphi(\pi^c\tilde{g}(t))\|_{2+\rho} \le Ce^{-\omega t}\|\pi^s\tilde{g}(0) - \varphi(\pi^c\tilde{g}(0))\|_{2+\rho}$$

for all solution $\tilde{g}(t)$ of (3.1) and all times $t \ge 0$ such that $\tilde{g}(t) \in \mathcal{O}'_r$.

Remark 3.1. *Theorem 3.1 in [25] is a generalization of Theorem 3.1 in [3] with symmetric conditions being replaced by Killing conditions. In [3], we considered the stability of DeTurck flows, but the stability of Ricci flows here is studied here. On the other hand, Theorem 3.1 in this paper is also a further generalization of Theorem 3.1 in [25] with Killing conditions being replaced by curvature conditions.*

Proof. We take $\mathcal{S}_2^\mu \doteqdot C^\infty(S_2^\mu(\mathcal{M}))$ as the space of all metrics with the same volume element given by g_0. By [19], one knows that the elements in $\mathcal{S}_2^+$ can be changed into those in $\mathcal{S}_2^\mu$ by using homothetic deformations and the tangent space $T\mathcal{S}_2^\mu$ of $\mathcal{S}_2^\mu$ consists of all zero-trace elements in $\mathcal{S}_2$, then on $T\mathcal{S}_2^\mu$, there holds $H = 0$. On the other hand, it is well known that g_0 is a stable point of (3.1), then formula (3.2) can be simplified as

$$\begin{aligned}\frac{\partial}{\partial t}h_{ji} &= \mathcal{L}h_{ji} = \Delta h_{ji} + 2R_{jpqi}h^{pq} - (\mathcal{L}_{X^\sharp}g)_{ji}\\ &= \Delta h_{ji} + 2R_{jpqi}h^{pq} - \nabla_j X_i - \nabla_i X_j\end{aligned}$$

Since

$$\begin{aligned}\int_{\mathcal{M}^n} h^{ji}R_{jpqi}h^{pq}h_{ij}d\mu &= \int_{\mathcal{M}^n} R_{jpqi}g^{pl}g^{qm}h_{lm}h_{ji}d\mu\\ &= \int_{\mathcal{M}^n} R_{j\,i}^{\,lm}h_{lm}h_{ji}d\mu\\ &\le \int_{\mathcal{M}^n} 2\Lambda(\Lambda-1)h_{lm}h_{ji}d\mu\\ &\le 2\Lambda(\Lambda-1)(\int_{\mathcal{M}^n} h_{lm}^2 d\mu)^{\frac{1}{2}}(\int_{\mathcal{M}^n} h_{ji}^2 d\mu)^{\frac{1}{2}}\\ &= 2\Lambda(\Lambda-1)\|h\|^2_{L^2}\end{aligned} \tag{3.3}$$

By using the hypothesis and formula (3.3), it is not hard to see by a direct computation that there holds the following

$$(\mathcal{L}h, h) \le -2\int_{\mathcal{M}^n}|\nabla h|^2 d\mu + 2\Lambda(\Lambda-1)\|h\|^2_{L^2} + 2(\int_{\mathcal{M}^n}|\nabla h|^2 d\mu)^{\frac{1}{2}}(\int_{\mathcal{M}^n}|h|^2 d\mu)^{\frac{1}{2}} \le 0,$$

where $h \in \mathcal{S}_2^0$ is of an non-zero element. Considering the operator $\mathcal{L}$ acting on $\mathcal{S}_2^\mu$, it is not hard to see by virtue of [10] that Theorem 3.1 is tenable. □

Theorem 3.2. *Assume that* $(\mathcal{M}^n, g_0)$ *is a quasi-constant curvature space,* $\|Rm\| \leq \frac{1}{2}\Lambda(\Lambda-1)$*, where* $\Lambda = \inf_h\{\frac{\int_{\mathcal{M}}|\nabla h|^2}{\int_{\mathcal{M}}|h|^2}\}$*,* ξ *is a unit vector field and its corresponding Ricci principal curvature* T *satisfies* $T \geq n-1$*. For a fixed* $\rho \in (0,1)$*, let* Θ *be a closure of* S_2^μ *in the sense of* $\|\cdot\|_{2+\rho}$*. Then it holds*

(1) $T_{g_0}S_2^{\mu^+} \cong \Theta$ *has the following decomposition:* $T_{g_0}S_2^{\mu^+} = \Theta^s \oplus \Theta^c$*;*

(2) There exists a constant $d_0 > 0$ *such that for all* $d \in (0, d_0]$*, there is a bounded* C^∞ *map* $\psi : B(\Theta^c, g_0, d) \to \Theta^s$ *satisfying* $\psi(g_0) = 0, D\psi(g_0) = 0$*, the image of* ψ *dependent on the closed ball* $\bar{B}(\Theta^s, g_0, d)$ *and its graph* $M_{loc}^c = \{(\gamma, \psi(\gamma)) : \gamma \in B(\Theta^c, g_0, d)\} \subset \Theta_1$ *satisfying* $T_{g_0}M_{loc}^c \cong \Theta^c$*;*

(3) There are constants $C > 0, \omega > 0$ *and* $d_* \in (0, d_0]$ *such that for each* $d \in (0, d_*]$*, one arrives at*

$$\|\pi^s\tilde{g}(t) - \psi(\pi^c\tilde{g}(t))\|_{2+\rho} \leq Ce^{-\omega t}\|\pi^s\tilde{g}(0) - \psi(\pi^c\tilde{g}(0))\|_{2+\rho}$$

for all solutions $\tilde{g}(t)$ *of the quasi-Ricci flow (1.5) with* $\tilde{g}(0) \in B(\Theta, g_0, d)$ *and all times* $t \geq 0$*, where* π^s, π^c *denote the projections onto* Θ^s, Θ^c *respectively.*

Remark 3.2. *Similar to Remark 3.1, Theorem 3.2 can be regarded as a generalization of Theorem 3.2 in [25] and of Theorem 3.1 in [3]. In this note, we use the curvature conditions to replace the symmetry conditions of (0,2) tensors given in [3], and the Killing conditions posed in [25] to consider the stability of Ricci flows not that of DeTurck flows.*

Proof. By a similar argument in [25], we now denote firstly by $\mathcal{A}(g)g$ of (1.5) at g_0, and then consider the linearization of the right-hand (1.5), we have

$$\frac{\partial}{\partial t}h_{ji} = (D(\mathcal{A}_u(g))|_{g_0}h)_{ji} = -2(DRc|_{g_0}h)_{ji} + 2D(\frac{R-T}{n-1}g_{ji} + \frac{nT-R}{n-1}\xi_i\xi_j)|_{g_0}.$$

and gets, by using Lemma 2.1 and [5], the following

$$\begin{aligned} DR|_{g_0} &= -\triangle H + \nabla^p\nabla^q h_{pq} - <h, Rc>; \\ DT|_{g_0} &= D(\xi^i\xi^j R_{ji}) = -m^i\xi^j R_{ij} - \xi^i m^j R_{ij} + \xi^i\xi^j DR_{ij} \\ &= -m^i\xi^j R_{ij} - \xi^i m^j R_{ij} + \frac{1}{2}\xi^i\xi^j(\nabla^p\nabla_i h_{jp} + \nabla^p\nabla_j h_{ip} - \triangle h_{ij} - \nabla_i\nabla_j H), \end{aligned}$$

where m is of the variation of ξ. Thus, by a direct computation similar to [3], we know

$$\begin{aligned} 2D(\frac{R-T}{n-1}g_{ij})|_{g_0} &= \frac{2}{n-1}(-\triangle H + \nabla^p\nabla^q h_{pq} - <h, Rc>)g_{ij} \\ &+ \frac{2}{n-1}[m^k\xi^l R_{kl} + \xi^k m^l R_{kl} \\ &- \frac{1}{2}\xi^k\xi^l(\nabla^p\nabla_k h_{lp} + \nabla^p\nabla_l h_{kp} - \triangle h_{kl} - \nabla_k\nabla_l H)]g_{ij} \\ &+ \frac{2}{n-1}(R-T)h_{ij}, \end{aligned}$$

and

$$\begin{aligned}
2D(\frac{nT-R}{n-1}\xi_i\xi_j)|_{g_0} &= \frac{2nDT}{n-1}\xi_i\xi_j - \frac{2DR}{n-1}\xi_i\xi_j + \frac{2}{n-1}(nT-R)D(\xi_i\xi_j) \\
&= \frac{2n}{n-1}[-m^k\xi^l R_{kl} - \xi^k m^l R_{kl} + \frac{1}{2}\xi^k\xi^l(\nabla^p\nabla_k h_{lp} + \nabla^p\nabla_l h_{kp} \\
&- \triangle h_{kl} - \nabla_k\nabla_l H)]\xi_i\xi_j - \frac{2\xi_i\xi_j}{n-1}(-\triangle H + \nabla^p\nabla^q h_{pq} \\
&- < h, Rc >) + \frac{2}{n-1}(nT-R)D(\xi_i\xi_j).
\end{aligned}$$

Since $\mathcal{M}$ is a quasi-constant curvature space, and by a direct computation, then we have

$$\begin{aligned}
-2[D(Rc)(h)]_{ji} &= \triangle h_{ji} + 2R_{ipqj}h^{pq} - (\mathcal{L}_{X^\sharp}g)_{ji} \\
&- 2(\frac{R-T}{n-1})h_{ji} - (\frac{nT-R}{n-1})(\xi_i\xi_k h_j^k + \xi_j\xi_k h_i^k),
\end{aligned}$$

where $X = X(g,h)$ is of 1-form defined as $X_k = g^{pq}\nabla_p h_{qk} - \frac{1}{2}\nabla_k(g^{pq}h_{pq})$.

From these formulae above and [25], then we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}h_{ji} &= \triangle h_{ji} + 2R_{ipqj}h^{pq} - (\mathcal{L}_{X^\sharp}g)_{ji} \\
&+ \frac{2}{n-1}(-\Delta H + \nabla^p\nabla^q h_{pq} - \frac{R-T}{n-1}H - \frac{nT-R}{n-1}h^{pq}\xi_p\xi_q)(g_{ji} - \xi_j\xi_i) \\
&+ \frac{\xi^k\xi^l}{n-1}(-2h_{kl} + \nabla^p\nabla_k h_{lp} + \nabla^p\nabla_l h_{kp} - \Delta h_{kl} - \nabla_k\nabla_l H)(g_{ji} - n\xi_j\xi_i)
\end{aligned} \tag{3.4}$$

Adopting similar arguments in Theorem 3.1, we now take $\mathcal{S}_2^\mu \doteqdot C^\infty(S_2^\mu(\mathcal{M}))$ as the space of all metrics with the same volume element given by g_0, and by [19], one knows that the elements in $\mathcal{S}_2^+$ can be changed into those in $\mathcal{S}_2^\mu$ by using homothetic deformations, and the tangent space $T\mathcal{S}_2^\mu$ of $\mathcal{S}_2^\mu$ consists of all trace-zero elements in $\mathcal{S}_2$. Then, on $T\mathcal{S}_2^\mu$, there holds $H = 0$. Thus, from (3.4), we arrive at

$$\begin{aligned}
\frac{\partial}{\partial t}h_{ij} &= \frac{2}{n-1}(\nabla^p\nabla^q h_{pq} - \frac{nT-R}{n-1}h^{pq}\xi_p\xi_q)(g_{ij} - \xi_i\xi_j) - (\mathcal{L}_{X^\sharp}g)_{ji} \\
&+ \frac{\xi^k\xi^l}{n-1}(-2h_{kl} + \nabla^p\nabla_k h_{lp} + \nabla^p\nabla_l h_{kp} - \triangle h_{kl})(g_{ij} - n\xi_i\xi_j) \\
&+ \triangle h_{ij} - \frac{2(R-2T)}{(n-1)(n-2)}h_{ij} \\
&+ \frac{2(nT-R)}{(n-1)(n-2)}(g_{ij}\xi_p\xi_q h^{pq} - g_{iq}\xi_p\xi_j h^{pq} - g_{pj}\xi_i\xi_q h^{pq})
\end{aligned} \tag{3.5}$$

Considering the acting on equation (3.5) with ξ^i and ξ^j, we have

$$\xi^i\xi^j(\frac{\partial}{\partial t}h_{ij} - 2\triangle h_{ij} - 2h_{ij} + \frac{2T}{n-1}h_{ij} + \nabla^p\nabla_i h_{jp} + \nabla^p\nabla_j h_{ip} - (\mathcal{L}_{X^\sharp}g)_{ji}) = 0.$$

Since ξ is of arbitrary, this implies that

$$\frac{\partial}{\partial t}h_{ij} - 2\triangle h_{ij} - 2h_{ij} + \frac{2T}{n-1}h_{ij} + \nabla^p\nabla_i h_{jp} + \nabla^p\nabla_j h_{ip} - (\mathcal{L}_{X^\sharp}g)_{ji} = 0 \tag{3.6}$$

We compute attentively and simplify (3.6) as follows

$$\begin{aligned} \frac{\partial}{\partial t} h_{ij} &= 2\triangle h_{ij} + 2h_{ij} - \frac{2T}{n-1} h_{ij} \\ &\quad + g^{tp}(R^k{}_{tip} h_{jk} + R^k{}_{tjp} h_{ik} + R^k{}_{tij} h_{kp} + R^k{}_{tji} h_{kp}) + 2(\mathcal{L}_{X^\sharp} g)_{ji}. \end{aligned}$$

According to the proof of Theorem 3.1, it is not hard to derive that there holds

$$\begin{aligned} (\mathcal{L}h, h) &\leq 2\int_{\mathcal{M}^n} \triangle h \cdot h d\mu + 2\int_{\mathcal{M}^n} (1 - \frac{T}{n-1}) h^2 d\mu \\ &+ 2\Lambda(\Lambda - 1)\|h\|^2_{L^2} + 4\|\nabla h\|_{L^2}\|h\|_{L^2} \\ &\leq 2(\int_{\mathcal{M}^n} \nabla(\nabla h \cdot h) d\mu - \|\nabla h\|^2) + 2(1 - \frac{T}{n-1})\|h\|^2 \\ &+ 2\Lambda(\Lambda - 1)\|h\|^2_{L^2} + 4\|\nabla h\|_{L^2}\|h\|_{L^2} \\ &\leq -2\int_{\mathcal{M}^n} |\nabla h|^2 d\mu + 2\Lambda(\Lambda - 1)\|h\|^2_{L^2} \\ &+ 4\|\nabla h\|_{L^2}\|h\|_{L^2} - 2(\frac{T}{n-1} - 1)\|h\|^2 \leq 0, \end{aligned}$$

where $h \in \mathcal{S}_2^0$ is of a non-zero element. Considering the operator $\mathcal{L}$ acting on $\mathcal{S}_2^\mu$, it is not hard to see by virtue of [10] that Theorem 3.2 is tenable. □

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References

[1] M. Berger, D. Ebin, *Some decompositions of the space of symmetric tensors on a Riemannian manifold,* J. Differential Geom., **3** (1969), 379-392.

[2] P. Butzer, H. Johnen, *Lipschitz spaces on compact manifolds,* J. Funct. Anal., 7 (1971), 242-266.

[3] Q. H. Cai, P. B. Zhao, *Stability of quasi-DeTurck flows in Riemannian manifolds of quasi-constant sectional curvatures,* Chinese Annals of Mathematics, Series A, 29, 1 (2008), 97-106.

[4] H. D. Cao, *Recent developments on the Ricci flow,* Bull. Amer. Math. Soc. 31, 1 (1999), 59-74.

[5] B. Chow, D. Knopf, *The Ricci flow: An introduction,* American Mathematical Society, 2005.

[6] G. Da Prato, A. Lunardi, *Stability, instability and center manifold theorem for fully nonlinear autonomous parabolic equations in Banach space,* Arch. Rational Mech. Anal., 101, 2 (1988), 115 -141.

[7] M. Deturck DE, *Deforming metrics in the direction of their Ricci tensors,* J. Diff. Geom., 18 (1983), 157-162.

[8] G. Dore, A. Favini, *On the equivalence of certain interpolation methods,* Boll. Un. Mat. Ital., 4, B (7) (1984), 1227-1238.

[9] J. Eells Jr, J. H. Sampson, *Harmonic mappings of Riemannian manifolds,* Amer. J. Math., 64 (1964), 109-160.

[10] C. Guenther, J. Isenberg and D. Knopf, *Stability of the Ricci flow at Ricci-flat metrics,* Comm. Anal. Geom., 10 (2002), 741-777.

[11] R. Hamilton, *The inverse function theorem of Nash and Moser,* Bull. Amer. Math. Soc. (N.S.), 7, 1 (1982), 65-222.

[12] R. Hamilton, *Three manifolds with positive Ricci curvature,* J. Differential Geom., 18 (1983), 255-306.

[13] R. Hamilton, *Four manifolds with positive Ricci curvature,* J. Differential Geom., 24 (1986), 153-179.

[14] R. Hamilton, *The Ricci flow on surfaces, mathematics and general relativity (Santa Cruz, CA, 1986),* Contemp. Math., 71, Amer. Math. Soc. Providence, 1988, 237-262.

[15] R. Hamilton, *The formation of singularities in the Ricci flow,* Surveys in Differential Geometry., 2 (1995), 7-136.

[16] R. Hamilton, *Non-singular solutions of the Ricci flow on three-manifolds,* Comm. Anal. Geom., 7, 4 (1999), 695-729

[17] D. Henry, *Geometric theorem of semilinear parabolic equations,* Springer Verlag, Berlin, 1981

[18] J. Isenberg, M. Jackson, *Ricci flows of locally homogeneous geometries on closed manifolds,* J. Diff. Geom., 35 (1992), 723-741

[19] J. Moser, *On the volume element of a manifold,* Trans. Amer. Math. Soc., 120 (1965), 286-294.

[20] R. Schoen, *Conformal definition of a Riemannian to constant scalar curvature,* J. Diff. Geom., 20 (1984), 479-495.

[21] G. Simonett, *Center manifolds for quasilinear reaction-diffusion systems,* Differential Integral Equations, 8, 4 (1995), 753-796.

[22] H. Triebel, *Interpolation theory, function spaces, differential operators,* Johann Ambro. BarthVerlag Heidelberg, 1995.

[23] K. Yano, *Differential geometry on complex and almost complex spaces,* Perfamon Press Let, 1965

[24] R. Ye, *Ricci flows, Einstein metrics and space forms,* Trans. Amer. Math. Soc., 1993 (2), 871-896.

[25] P. B. Zhao, Q. H. Cai, *Stability of Ricci flows based on Killing conditions,* J. of the Korean Math. Soc., 46, 6 (2009), 1193-1206.

[26] P. B. Zhao, H. Z. Song, *Quasi-Einstein hypersurfaces in a hyperbolic space,* Chinese Quarterly J. of Mathematics, 13, 2 (1998), 49-52.

[27] P. B. Zhao, X. P. Yang, *On quasi Einstein field equations,* Northeast Math. J., 21, 4 (2005), 411-420.

[28] P. B. Zhao, X. P. Yang, *On stationary hypersurfaces in Euclidean spaces,* Acta Mathematica Scientia, Series B, 26, 2 (2006), 349-357.

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