

# Second order differential invariants of linear frames

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**Abstract.** The aim of this paper is to characterize all second order tensor-valued and scalar differential invariants of the bundle of linear frames  $F^r X$  over an  $n$ -dimensional manifold  $X$ . These differential invariants are obtained by factorization method and are described in terms of bases of invariants. Second order natural Lagrangians of frames have been characterized explicitly; if  $n = 1, 2, 3, 4$ , the number of functionally independent second order natural Lagrangians is  $N = 0, 6, 33, 104$ , respectively.

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## 1 Introduction

Throughout this paper, a *left  $G$ -manifold* is a smooth manifold endowed with a left action of a Lie group  $G$ . A mapping between two left  $G$ -manifolds transforming  $G$ -orbits into  $G$ -orbits is said to be  *$G$ -equivariant*. As usual, we denote by  $\mathbb{R}$  the field of real numbers. The  $r$ -th differential group  $L_n^r$  of  $\mathbb{R}^n$  is the Lie group of invertible  $r$ -jets with source and target at the origin  $0 \in \mathbb{R}^n$ ; the group multiplication in  $L_n^r$  is defined by the composition of jets. Note that  $L_n^1 = GL_n(\mathbb{R})$ . For generalities on spaces of jets and their mappings, differential groups, their actions, etc., we refer to [6, 11, 13].

Let  $P$  and  $Q$  be two left  $L_n^r$ -manifolds, and  $U$  be an open,  $L_n^r$ -invariant set in  $P$ . A smooth  $L_n^r$ -equivariant mapping  $F : U \rightarrow Q$  is called a *differential invariant*. If  $Q$  is the real line  $\mathbb{R}$ , endowed with the trivial action of  $L_n^r$ , an equivariant mapping  $F$  is called a *scalar invariant*.

Let  $X$  be an  $n$ -dimensional manifold. By an  *$r$ -frame* at a point  $x \in X$  we mean an invertible  $r$ -jet with source  $0 \in \mathbb{R}^n$  and target  $x$ . The set of  $r$ -frames together with its natural structure of a principal  $L_n^r$ -bundle with base  $X$  is denoted by  $F^r X$ , and is called the *bundle of  $r$ -frames* over  $X$ . For  $r = 1$ , we get the *bundle of linear frames*, and write  $F^1 X = FX$ . If  $S$  is a left  $L_n^r$ -manifold, then the bundle with type fibre  $S$ , *associated* with  $F^r X$  is denoted by  $F_S^r X$ .

If  $S$  is a left  $L_n^1$ -manifold, we denote by  $T_n^r S$  the manifold of  $r$ -jets with source  $0 \in \mathbb{R}^n$  and target in  $S$ . For finding differential invariants of frames it is convenient to realize  $FX$  as a bundle with type fibre  $L_n^1$ , associated with itself. Then, the  $r$ -jet prolongation  $J^r FX$  of  $FX$  can be considered as a fibre bundle with type fibre  $T_n^r L_n^1$ , associated with  $F^{r+1}X$ .

For characterizing *natural* Lagrangians on  $J^r FX$ , i.e. Lagrangians invariant with respect to all *diffeomorphisms of  $X$* , it is sufficient to describe all differential invariants defined on the type fiber  $P = T_n^r L_n^1$  of  $J^r FX$ . The aim of this paper is to give explicit characterization of second order natural Lagrangians.

Most of differential invariants with values in  $Q$  appearing in differential geometry correspond with the case when  $Q$  is an  $L_n^1$ -manifold. These differential invariants can be described as follows. Let  $K_n^{r,s}$  be the *kernel* of the canonical group morphism  $\pi_n^{r,s} : L_n^r \rightarrow L_n^s$ , where  $r \geq s$ . If  $L_n^r$  acts on  $Q$  via its subgroup  $L_n^1$ , each *continuous*,  $L_n^r$ -equivariant mapping  $F : U \rightarrow Q$  has the form  $F = f \circ \pi$ , where  $\pi : P \rightarrow P/K_n^{r,1}$  is the quotient projection,  $P/K_n^{r,1}$  is the space of  $K_n^{r,1}$ -orbits, and  $f : P/K_n^{r,1} \rightarrow Q$  is a continuous,  $L_n^1$ -equivariant mapping. Indeed, in this scheme  $P/K_n^{r,1}$  is considered with the quotient topology, but is not necessarily a smooth manifold. The quotient projection  $\pi$  is continuous but not necessarily smooth. If  $P/K_n^{r,1}$  has a smooth structure such that  $\pi$  is a submersion, we call  $\pi$  the basis of differential invariants on  $P$  (for more details of a basis, see [12]). The general concepts on equivariant mappings, related with a normal subgroup of a Lie group, and corresponding assertions with the proofs can be found in [4, 8].

A method based on this observation was first time applied to the problem of finding differential invariants of a linear connection in [8]. The initial problem was reduced to a more simple problem of the classical invariant theory (see e.g. [14, 15]) to describe all  $L_n^1$ -equivariant mappings from  $P/K_n^{r,1}$  to  $L_n^1$ -manifolds. Our aim in this paper is to study invariants of linear frames by the same method, which allows us to simplify expressions of the action of  $L_n^r$  on  $P$ .

In this paper, we first introduce the domain of second order differential invariants with values in  $L_n^1$ -manifolds, which is, according to the prolongation theory of manifolds endowed with a Lie group action (see e.g. [6, 7, 11]), the  $L_n^3$ -manifold  $P = T_n^2 L_n^1$ . Then we describe the *frame* action of  $L_n^3$  on  $T_n^2 L_n^1$ . This action corresponds to the second jet prolongation of the frame lift of a diffeomorphism of  $X$ . Using a tensor decomposition, we also construct the corresponding orbit space of the normal subgroup  $K_n^{3,1}$  of  $L_n^3$ . We show that this orbit space can be identified with Cartesian products of  $L_n^1$  with some tensor spaces over  $\mathbb{R}^n$ ; in this way the differential invariants with values in  $L_n^1$ -manifolds are described in terms of their basis. Note that the second order differential invariants with values in  $L_n^2$ -manifolds can be easily obtained by the same manner.

These results are subsequently used for extension of the theory of the first order differential invariants of frames in [5], studied in terms of integrals of canonical differential system, to the second order case. Applying factorization method, we give an explicit characterization of second order *scalar invariants* of frames and *Lagrangians*, defined on  $J^2 FX$ , *invariant* with respect to all diffeomorphisms on  $X$ . In Section 8 we also introduce the concept of *canonical odd  $n$ -form* on  $FX$ , where  $n = \dim X$ , which gives an exact description of globally defined invariant object in the role of volume form. In [5], (ordinary)  $n$ -form was considered, which is only available over

orientable manifolds. We also extend the remarks of the authors on the use of differential invariants as Lagrangians over *non-orientable* manifolds. For the theory of odd de Rham forms and odd base forms we refer to [9].

The calculations in this paper rely on the jet description of differential invariants. This is based on the existence of a Lie group (the differential group) whose invariants are exactly the differential invariants. Such description implies, in particular, that the arising theory is comparatively simpler than other versions of the theory of differential invariants.

Using another left action of  $L_n^1$  on itself, called *coframe* action, it is possible to obtain the corresponding *differential invariants of coframes*. Note that they can be obtained by the same method. For differential invariants of coframes, see [3]; it represents an extension of Ph.D. thesis of the first author, devoted to the second order case, to the third order case.

There are several types of invariance of Lagrangians on frame bundles. One of them is invariance with respect to the canonical action of  $L_n^1$  on  $J^r FX$ . All  $L_n^1$ -invariant Lagrangians on  $J^r FX$  are explicitly described in [2].

## 2 Jet prolongations of $L_n^1$ manifolds

In this section, the general prolongation theory of left  $G$ -manifolds is applied to the case of the Lie group  $G = L_n^1 = GL_n(\mathbb{R})$ . We use the prolongation formula derived in [7], and the terminology and notation of the book [11].

Recall that the  $r$ -th differential group  $L_n^r$  of  $\mathbb{R}^n$  is the group of invertible  $r$ -jets with source and target at the origin  $0 \in \mathbb{R}^n$ . The group multiplication in  $L_n^r$  is defined by the composition of jets. Let  $J_0^r \alpha \in L_n^r$ , where  $\alpha = (\alpha^i)$  is a diffeomorphism of a neighborhood  $U$  of the origin  $0 \in \mathbb{R}^n$  into  $\mathbb{R}^n$  such that  $\alpha(0) = 0$ . The *first canonical coordinates*  $a_{j_1}^i, a_{j_1 j_2}^i, \dots, a_{j_1 j_2 \dots j_r}^i$ , where  $1 \leq i \leq n$ ,  $1 \leq j_1 \leq j_2 \leq \dots \leq j_r \leq n$ , on  $L_n^r$  are defined by

$$(2.1) \quad a_{j_1 j_2 \dots j_k}^i (J_0^r \alpha) = D_{j_1} D_{j_2} \dots D_{j_k} \alpha^i(0), \quad 1 \leq k \leq r.$$

We also define the *second canonical coordinates*  $b_{j_1}^i, b_{j_1 j_2}^i, \dots, b_{j_1 j_2 \dots j_r}^i$ , on  $L_n^r$  by

$$b_{j_1 j_2 \dots j_k}^i (J_0^r \alpha) = a_{j_1 j_2 \dots j_k}^i (J_0^r \alpha^{-1}), \quad 1 \leq k \leq r.$$

Indeed,  $a_j^i b_k^j = \delta_k^i$  (the Kronecker symbol).

Let us consider a left  $L_n^1$ -manifold  $S$ , and denote by  $T_n^r S$  the manifold of  $r$ -jets with source  $0 \in \mathbb{R}^n$  and target in  $S$ . According to the general theory of prolongations of left  $G$ -manifolds,  $T_n^r S$  has a (canonical) structure of a left  $L_n^{r+1}$ -manifold. To define this structure, denote by  $t_x$  the translation of  $\mathbb{R}^n$  defined by  $t_x(y) = y - x$ . Consider elements  $q \in T_n^r S$ ,  $q = J_0^r \gamma$ , and  $a \in L_n^{r+1}$ ,  $a = J_0^{r+1} \alpha$ . Denoting  $\bar{\alpha}_x = t_x \circ \alpha \circ t_{-\alpha^{-1}(x)}$ , and  $\bar{\alpha}(x) = J_0^1 \bar{\alpha}_x$  we get an element of the group  $L_n^1$ . Then formula

$$(2.2) \quad a \cdot q = J_0^r (\bar{\alpha} \cdot (\gamma \circ \alpha^{-1}))$$

defines a left action of the differential group  $L_n^{r+1}$  on  $T_n^r S$ . We usually call formula (2.2) the *prolongation formula* for the action of the group  $L_n^1$  on  $S$ . The left  $L_n^{r+1}$ -manifold  $T_n^r S$  is called the  *$r$ -jet prolongation* of the left  $L_n^1$ -manifold  $S$ .

### 3 Frames

Let  $X$  be an  $n$ -dimensional manifold. Recall that an  $r$ -frame at a point  $x \in X$  is an invertible  $r$ -jet with source  $0 \in \mathbb{R}^n$  and target at  $x$ . The set of  $r$ -frames, denoted by  $F^r X$ , will be considered with its natural structure of a principal  $L_n^r$ -bundle over  $X$ . We write  $FX = F^1 X$ ;  $FX$  is the bundle of linear frames.

For computing differential invariants of frame bundles it is important to realize  $FX$  as a fibre bundle with type fibre  $L_n^1$ , associated with itself. Thus, the structure group of  $FX$  is the group  $L_n^1 = GL_n(\mathbb{R})$ , with canonical coordinates  $(a_j^i)$ , defined by (2.1). If  $(p_j^i)$  are the canonical coordinates on the type fibre  $L_n^1$  of fibre bundle  $FX$ , then the left action of the structure group  $L_n^1$  of  $FX$  on the type fibre  $L_n^1$  is represented by the group multiplication  $L_n^1 \times L_n^1 \ni (J_0^1 \alpha, J_0^1 \eta) \rightarrow J_0^1(\alpha \circ \eta) \in L_n^1$ . In the canonical coordinates,  $p_j^i(J_0^1(\alpha \circ \eta)) = a_k^i(J_0^1 \alpha) p_j^k(J_0^1 \eta)$ , which can be written simply by

$$(3.1) \quad \bar{p}_j^i = a_k^i p_j^k.$$

(3.1) is called the *frame action* of  $L_n^1$  on itself.

$J^r FX$  denotes the  $r$ -jet prolongation of  $FX$ . It follows from the general theory of jet prolongations of fibre bundles that  $J^r FX$  can be considered as a fibre bundle over  $X$  with type fibre  $T_n^r L_n^1$ , associated with  $F^{r+1} X$ . Equations of the group action of  $L_n^{r+1}$  on  $T_n^r L_n^1$  can be obtained from (2.2) and (3.1).

### 4 The second jet prolongation of the frame action

Now we derive an explicit expression for the action (2.2) of the group  $L_n^3$  on  $T_n^2 L_n^1$ , associated with (3.1).

Let  $U$  be a neighborhood of the origin  $0 \in \mathbb{R}^n$ . Let  $\alpha$  be a diffeomorphism of  $U$  onto  $\alpha(U) \subset \mathbb{R}^n$  such that  $\alpha(0) = 0$ . Then  $\bar{\alpha}(x) = J_0^1 \alpha_x$ , where  $\bar{\alpha}_x = t_x \circ \alpha \circ t_{-\alpha^{-1}(x)}$ . Let  $\gamma : U \rightarrow L_n^1$  be a mapping. For every  $x \in \alpha(U)$  we denote  $\psi(x) = \bar{\alpha}(x) \cdot \gamma(\alpha^{-1}(x))$ , and the dot on the right hand side means the multiplication in the group  $L_n^1$ . In coordinates,

$$(4.1) \quad p_j^i(\psi(x)) = p_j^i(\bar{\alpha}(x) \cdot \gamma(\alpha^{-1}(x))) = a_s^i(\bar{\alpha}(x)) p_j^s(\gamma(\alpha^{-1}(x))).$$

Note that in this formula,

$$(4.2) \quad a_s^i(\bar{\alpha}(x)) = D_s \alpha^i(\alpha^{-1}(x)).$$

Now the chart expression of the frame action is obtained by expressing the  $r$ -jet  $J_0^r \psi = J_0^{r+1} \alpha \cdot J_0^r \gamma$  (2.2) in coordinates. Consider the case  $r = 2$ .

**Lemma 1.** *The group action of  $L_n^3$  on  $T_n^2 L_n^1$  induced by the frame action of  $L_n^1$  on  $L_n^1$  is defined by the equations*

$$(4.3) \quad \begin{aligned} \bar{p}_j^i &= p_j^s a_s^i, \\ \bar{p}_{j,k}^i &= p_{j,t}^s a_s^i b_k^t + p_j^s a_{st}^i b_k^t, \\ \bar{p}_{j,kl}^i &= p_{j,tu}^s a_s^i b_l^u b_k^t + p_{j,t}^s (a_{su}^i b_k^u b_l^t + a_{su}^i b_l^u b_k^t + a_s^i b_{kl}^t) \\ &\quad + p_j^s (a_{stu}^i b_l^u b_k^t + a_{st}^i b_{kl}^t). \end{aligned}$$

*Proof.* First equation is obtained from (4.1) and (4.2), to get the remaining equations, we differentiate (4.1) twice, and then put  $x = 0$ .  $\square$

## 5 Differential invariants with values in $L_n^1$ -manifolds

In this part we are interested in differential invariants  $F : T_n^2 L_n^1 \rightarrow Q$ , where  $Q$  is arbitrary left  $L_n^1$ -manifold. We define the homomorphism

$$\pi_n^{3,1} : L_n^3 \rightarrow L_n^1, \quad \pi_n^{3,1}(a_j^i, a_{jk}^i, a_{jkl}^i) = (a_j^i).$$

Notice that each differential invariant  $F$  with values in  $L_n^1$ -manifold satisfies

$$F(a \cdot q) = \pi_n^{3,1}(a) \cdot F(q),$$

for each  $a \in L_n^3$ ,  $q \in T_n^2 L_n^1$ .

Let  $K_n^{3,1}$  denote the kernel of the canonical homomorphism  $\pi_n^{3,1}$ ;  $K_n^{3,1}$  is normal subgroup of  $L_n^3$  represented by elements in coordinates written as  $(\delta_j^i, a_{jk}^i, a_{jkl}^i)$ . Now we restrict the action (4.3) to the subgroup  $K_n^{3,1}$  of  $L_n^3$ . The following result is fundamental for the discussion of the corresponding orbit spaces.

**Lemma 2.** *The group action of  $K_n^{3,1}$  on  $T_n^2 L_n^1$  induced by the frame action of  $L_n^1$  on  $L_n^1$  is defined by the equations*

$$(5.1) \quad \begin{aligned} \bar{p}_j^i &= p_j^i, \\ \bar{p}_{j,k}^i &= p_{j,k}^i + p_j^s a_{sk}^i, \\ \bar{p}_{j,kl}^i &= p_{j,kl}^i + p_{j,l}^s a_{sk}^i + p_{j,k}^s a_{sl}^i + (p_{j,t}^i + p_j^s a_{st}^i) b_{kl}^t + p_j^s a_{skl}^i. \end{aligned}$$

**Proof.** We take  $a_j^i = b_j^i = \delta_j^i$  in (4.3).  $\square$

**Corollary 1.** *The action (5.1) is free.*

Now we describe *orbits* of the group actions (5.1). Let us introduce some notation. Using the second canonical coordinates on  $T_n^2 L_n^1$ , we denote by  $q_j^i$  the inverse matrix of the matrix  $p_j^i$ ; thus,  $q_j^i : T_n^2 L_n^1 \rightarrow \mathbb{R}$  are functions such that  $q_j^i p_j^s = \delta_j^i$ .

We also use the special notation for *symmetrization* and *antisymmetrization* of indexed families of functions through selected indices. Symmetrization (resp. antisymmetrization) in some indices  $j, k, l, \dots$  is denoted by writing a bar (resp. a tilde) over these indices, i.e., by writing  $\bar{j}, \bar{k}, \bar{l}, \dots$  (resp.  $\tilde{j}, \tilde{k}, \tilde{l}, \dots$ ).

First, we state some auxiliary assertions on the Young decomposition of tensors of type  $(0, 3)$ . Let us have a tensor  $\Delta = \Delta_{jkl}$  and let  $n$  be the dimension of the underlying vector space. We define

$$(5.2) \quad \begin{aligned} \mathcal{S}\Delta &= \frac{1}{6}(\Delta_{jkl} + \Delta_{ljk} + \Delta_{klj} + \Delta_{jlk} + \Delta_{lkj} + \Delta_{kjl}), \\ \mathcal{Q}\Delta &= \frac{1}{3}(\Delta_{jkl} + \Delta_{kjl} - \Delta_{lkj} - \Delta_{klj}) + \frac{1}{3}(\Delta_{jkl} + \Delta_{lkj} - \Delta_{kjl} - \Delta_{ljk}), \\ \mathcal{A}\Delta &= \frac{1}{6}(\Delta_{jkl} + \Delta_{ljk} + \Delta_{klj} - \Delta_{jlk} - \Delta_{lkj} - \Delta_{kjl}). \end{aligned}$$

Note that (5.2) can be equivalently written by

$$(5.3) \quad \mathcal{S}\Delta = \Delta_{\bar{j}\bar{k}\bar{l}}, \quad \mathcal{Q}\Delta = \frac{1}{3}(2\Delta_{jkl} - \Delta_{ljk} - \Delta_{klj}), \quad \mathcal{A}\Delta = \Delta_{\bar{j}\bar{k}\bar{l}}.$$

**Lemma 3.** (a) If  $n \geq 3$ , a tensor  $\Delta = \Delta_{jkl}$  has a unique decomposition

$$\Delta = \mathcal{S}\Delta + \mathcal{Q}\Delta + \mathcal{A}\Delta,$$

such that  $\mathcal{S}(\mathcal{S}\Delta) = \mathcal{S}\Delta$ ,  $\mathcal{Q}(\mathcal{Q}\Delta) = \mathcal{Q}\Delta$ ,  $\mathcal{A}(\mathcal{A}\Delta) = \mathcal{A}\Delta$ ,  $\mathcal{S}(\mathcal{Q}\Delta) = \mathcal{Q}(\mathcal{S}\Delta) = \mathcal{A}(\mathcal{Q}\Delta) = \mathcal{Q}(\mathcal{A}\Delta) = \mathcal{S}(\mathcal{A}\Delta) = \mathcal{A}(\mathcal{S}\Delta) = 0$ .

(b) If  $n = 2$ , a tensor  $\Delta = \Delta_{jkl}$  has a unique decomposition

$$\Delta = \mathcal{S}\Delta + \mathcal{Q}\Delta,$$

such that  $\mathcal{S}(\mathcal{S}\Delta) = \mathcal{S}\Delta$ ,  $\mathcal{Q}(\mathcal{Q}\Delta) = \mathcal{Q}\Delta$ ,  $\mathcal{S}(\mathcal{Q}\Delta) = \mathcal{Q}(\mathcal{S}\Delta) = 0$ .

*Proof.* These assertions can be verified by a direct computation.  $\square$

**Corollary 2.** For tensor  $\Delta = \Delta_{jkl}$  symmetric in the indices  $k, l$  there is a unique decomposition

$$(5.4) \quad \Delta = \mathcal{S}\Delta + \mathcal{Q}\Delta,$$

such that  $\mathcal{S}(\mathcal{S}\Delta) = \mathcal{S}\Delta$ ,  $\mathcal{Q}(\mathcal{Q}\Delta) = \mathcal{Q}\Delta$ ,  $\mathcal{S}(\mathcal{Q}\Delta) = \mathcal{Q}(\mathcal{S}\Delta) = 0$ .

Finally, we introduce the following functions on  $T_n^2 L_n^1$ :

$$(5.5) \quad \begin{aligned} I_{jk}^i(p_b^a, p_{b,c}^a, p_{b,cd}^a) &= q_j^l p_{l,\bar{k}}^i, \\ I_{jkl}^i(p_b^a, p_{b,c}^a, p_{b,cd}^a) &= 2q_j^s p_{s,kl}^i - q_k^s p_{s,lj}^i - q_l^s p_{s,jk}^i \\ &\quad - 3(q_j^t p_{t,\bar{l}}^m q_{\bar{m}}^s p_{s,\bar{k}}^i + q_j^t p_{t,\bar{k}}^m q_{\bar{m}}^s p_{s,\bar{l}}^i) \\ &\quad + p_{s,m}^i (2q_j^s q_{\bar{k}}^t p_{t,\bar{l}}^m - q_k^s q_{\bar{l}}^t p_{t,\bar{j}}^m - q_l^s q_{\bar{j}}^t p_{t,\bar{k}}^m). \end{aligned}$$

It is obvious that the functions  $I_{jk}^i$  are antisymmetric in indices  $j, k$ , and the functions  $I_{jkl}^i$  are symmetric in indices  $k, l$ , which gives us  $I_{jk}^i + I_{kj}^i = 0$ ,  $I_{jkl}^i - I_{jlk}^i = 0$ , respectively. Moreover, we have the identity  $I_{jkl}^i + I_{ljk}^i + I_{klj}^i = 0$ .

**Lemma 4.**  $K_n^{3,1}$ -orbits in  $T_n^2 L_n^1$  induced by the frame action of  $L_n^1$  on  $L_n^1$  is defined by the equations

$$p_j^i = c_j^i, \quad I_{jk}^i(p_b^a, p_{b,c}^a, p_{b,cd}^a) = c_{jk}^i, \quad I_{jkl}^i(p_b^a, p_{b,c}^a, p_{b,cd}^a) = c_{jkl}^i,$$

where  $c_j^i, c_{jk}^i, c_{jkl}^i \in \mathbb{R}$  are arbitrary constants such that  $\det c_j^i \neq 0$ .

*Proof.* Consider the action (5.1) of  $K_n^{3,1}$  on  $T_n^2 L_n^1$ , induced by the frame action of  $L_n^1$  on  $L_n^1$ , in standard notation given by  $\bar{p}_j^i = a_k^i p_j^k$ . Rewrite this action in the form

$$(5.6) \quad \bar{p}_j^i = p_j^i, \quad \bar{p}_{j,k}^i = p_{j,k}^i + p_j^s a_{sk}^i, \quad \bar{p}_{j,kl}^i = p_{j,kl}^i + \chi_{j,kl}^i + p_j^s a_{skl}^i,$$

where the functions

$$(5.7) \quad \chi_{j,kl}^i = p_{j,l}^s a_{sk}^i + p_{j,k}^s a_{sl}^i + (p_{j,t}^i + p_j^s a_{st}^i) b_{kl}^t$$

are symmetric in the indices  $k, l$ . From (5.6) we get

$$(5.8) \quad \bar{q}_j^i = q_j^i, \quad a_{sk}^i = q_{\bar{s}}^j (\bar{p}_{j,\bar{k}}^i - p_{j,\bar{k}}^i), \quad a_{skl}^i = q_{\bar{s}}^j (\bar{p}_{j,\bar{k}\bar{l}}^i - p_{j,\bar{k}\bar{l}}^i - \chi_{j,\bar{k}\bar{l}}^i).$$

Substituting the second equation of (5.8) to (5.6) we have

$$q_s^j (\bar{p}_{j,k}^i - p_{j,k}^i) = q_{\bar{s}}^j (\bar{p}_{j,\bar{k}}^i - p_{j,\bar{k}}^i),$$

which means that we compare the tensor on the left hand side with its symmetric part. It gives us  $q_j^l \bar{p}_{l,\bar{k}}^i = q_j^l p_{l,\bar{k}}^i$ . Thus, for the functions  $I_{jk}^i$ , defined by (5.5), we have

$$(5.9) \quad I_{jk}^i(\bar{p}_b^a, \bar{p}_{b,c}^a, \bar{p}_{b,cd}^a) = I_{jk}^i(p_b^a, p_{b,c}^a, p_{b,cd}^a),$$

and the functions  $I_{jk}^i$  are invariant with respect to the action (5.1) of  $K_n^{3,1}$  on  $T_n^2 L_n^1$ .

Again, substituting (5.8) to (5.6) we have

$$q_j^s (\bar{p}_{s,kl}^i - p_{s,kl}^i - \chi_{s,kl}^i) = q_j^s (\bar{p}_{s,\bar{k}\bar{l}}^i - p_{s,\bar{k}\bar{l}}^i - \chi_{s,\bar{k}\bar{l}}^i).$$

It means that we compare the tensor

$$(5.10) \quad \Delta_{jkl}^i = q_j^s (\bar{p}_{s,kl}^i - p_{s,kl}^i - \chi_{s,kl}^i),$$

symmetric in the subscripts  $k, l$ , on the left hand side, with its symmetric part  $\mathcal{S}\Delta = \Delta_{j\bar{k}\bar{l}}^i$ . Using decomposition (5.4) for the tensor  $\Delta$  given by (5.10), we get

$$(5.11) \quad \mathcal{Q}\Delta = 0.$$

Applying (5.3) to (5.11), and using (5.7), and (5.8), after long calculation we obtain that (5.11) is equivalent to

$$(5.12) \quad I_{jkl}^i(\bar{p}_b^a, \bar{p}_{b,c}^a, \bar{p}_{b,cd}^a) = I_{jkl}^i(p_b^a, p_{b,c}^a, p_{b,cd}^a),$$

which means that the functions  $I_{jkl}^i$ , defined by (5.5), are invariant with respect to the action (5.1) of  $K_n^{3,1}$  on  $T_n^2 L_n^1$ .  $\square$

## 6 Basis of the second order invariants

Now, from the assertions on equivariant mappings of manifolds (see [4, 8]) we can obtain the exact characteristics of basis of differential invariants with values in  $L_n^1$ -manifolds.

First, let us denote by  $S_n^0$  the vector subspace of the tensor product  $\otimes^2 \mathbb{R}^{n*} = \mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$ , defined in the canonical coordinates on  $\mathbb{R}$  by the equations

$$x_{jk} + x_{kj} = 0.$$

Similarly,  $S_n^1$  denotes the vector subspace of the tensor product  $\otimes^3 \mathbb{R}^{n*} = \mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$ , defined by the equations

$$x_{jkl} - x_{jlk} = 0, \quad x_{jkl} + x_{ljk} + x_{klj} = 0.$$

We can summarize the discussion of Section 5 in the following theorem, describing differential invariants on  $T_n^2 L_n^1$  with values in  $L_n^1$ -manifolds.

**Theorem 1.** (a) *The frame action defines on  $T_n^2 L_n^1$  the structure of a left principal  $K_n^{3,1}$ -bundle.*

(b) *The quotient space  $T_n^2 L_n^1 / K_n^{3,1}$  is canonically isomorphic to the space  $L_n^1 \times (\mathbb{R}^n \otimes S_n^0) \times (\mathbb{R}^n \otimes S_n^1)$ .*

*Proof.* (a) Since we have already proved that the action (5.1) of  $K_n^{3,1}$  on  $T_n^2 L_n^1$  is free (see Corollary 1), in order to show that  $T_n^2 L_n^1$  is a principal  $K_n^{3,1}$ -bundle it remains to show that the equivalence " $J_0^2 \gamma \sim J_0^2 \bar{\gamma}$ " if and only if there exists an element  $J_0^3 \alpha \in K_n^{3,1}$  such that  $J_0^2 \bar{\gamma} = J_0^3 \alpha \cdot J_0^2 \gamma$  is a closed submanifold in  $T_n^2 L_n^1 \times T_n^2 L_n^1$ . But using (5.1) with help of (5.9) and (5.12), we see that this submanifold is defined by the equations

$$\begin{aligned} p_j^i(J_0^2 \bar{\gamma}) - p_j^i(J_0^2 \gamma) &= 0, \\ I_{jk}^i(J_0^2 \bar{\gamma}) - I_{jk}^i(J_0^2 \gamma) &= 0, \\ I_{jkl}^i(J_0^2 \bar{\gamma}) - I_{jkl}^i(J_0^2 \gamma) &= 0, \end{aligned}$$

and is therefore closed.

(b) Let  $J_0^2 \gamma \in T_n^2 L_n^1$  and let  $[J_0^2 \gamma]$  be the corresponding class of  $J_0^2 \gamma$  in quotient  $T_n^2 L_n^1 / K_n^{3,1}$ . We set

$$(6.1) \quad \begin{aligned} p_j^i([J_0^2 \gamma]) &= p_j^i(J_0^2 \gamma), \\ I_{jk}^i([J_0^2 \gamma]) &= I_{jk}^i(J_0^2 \gamma), \\ I_{jkl}^i([J_0^2 \gamma]) &= I_{jkl}^i(J_0^2 \gamma). \end{aligned}$$

Relations (5.1), (5.9), and (5.12) imply that coordinates defined by (6.1) do not depend on a representant of given class, and for two different classes there are different sets of numbers. It means that factor projection  $\pi : T_n^2 L_n^1 \rightarrow T_n^2 L_n^1 / K_n^{3,1}$  can be expressed by

$$\pi = (p_j^i, I_{jk}^i, I_{jkl}^i).$$

Thus, canonical isomorphism of bundles maps the class from  $T_n^2 L_n^1 / K_n^{3,1}$  with coordinates  $(p_j^i, I_{jk}^i, I_{jkl}^i)$ , to the element of  $L_n^1 \times (\mathbb{R}^n \otimes S_n^0) \times (\mathbb{R}^n \otimes S_n^1)$  with the same canonical coordinates.  $\square$

Theorems 1 says that every second order differential invariant of frames factorizes through the corresponding bundle projection. Consider the components of the isomorphisms defined by  $p_j^i : T_n^2 L_n^1 \rightarrow L_n^1$ ,  $I_{jk}^i : T_n^2 L_n^1 \rightarrow \mathbb{R}^n \otimes S_n^0$ , and  $I_{jkl}^i : T_n^2 L_n^1 \rightarrow \mathbb{R}^n \otimes S_n^1$ . We have the following results.

**Corollary 3.** *The mappings  $p_j^i, I_{jk}^i, I_{jkl}^i$  represent a basis of second order invariants of frames with values in left  $L_n^1$ -manifold.*

## 7 Basis of scalar invariants

Note that if  $G$  is a Lie group,  $K$  is its normal subgroup, and  $P$  is a  $G$ -manifold then the quotient manifold  $(P/K)/(G/K)$  is canonically isomorphic with  $P/G$  (see [1]).

This means that for finding scalar invariants of Lie group  $G$ , we can equivalently factorize  $P$  by normal subgroup  $K$  and subsequently by the factor group  $G/K$ . Thus, to obtain scalar invariants of  $L_n^3$  on  $T_n^2 L_n^1$  it is sufficient to consider  $L_n^1$ -equivariant mappings defined on  $T_n^2 L_n^1 / K_n^{3,1}$ .

Let us define some functions  $\mathcal{I}_{jk}^i, \mathcal{I}_{jkl}^i$ , on  $T_n^2 L_n^1$ , by

$$(7.1) \quad \begin{aligned} \mathcal{I}_{jk}^i &= q_i^i p_k^s p_{j,s}^l, \\ \mathcal{I}_{jkl}^i &= q_r^i (2p_k^s p_l^t p_{j,st}^r - p_l^s p_j^t p_{k,st}^r - p_j^s p_k^t p_{l,st}^r \\ &\quad - \frac{3}{2} q_m^u p_{u,s}^r (p_k^s p_l^t p_{j,t}^m + p_l^s p_k^t p_{j,t}^m) - \frac{5}{2} p_{j,s}^m p_k^s p_{l,m}^r \\ &\quad + \frac{1}{2} p_j^s p_{k,m}^r p_{l,s}^m + 2p_{j,m}^r p_k^s p_{l,s}^m). \end{aligned}$$

We have the following

**Theorem 2.** *The functions  $\mathcal{I}_{jk}^i, \mathcal{I}_{jkl}^i$  represent a basis of second order scalar invariants of frames.*

*Proof.* The group  $L_n^1 \simeq L_n^3 / K_n^{3,1}$  acts in factor space  $T_n^2 L_n^1 / K_n^{3,1}$ , where the functions  $\mathcal{I}_{jk}^i$  and  $\mathcal{I}_{jkl}^i$  live, by

$$(7.2) \quad \bar{I}_{jk}^i = a_r^i b_j^s b_k^t I_{st}^r, \quad \bar{I}_{jkl}^i = a_r^i b_j^s b_k^t b_l^u I_{stu}^r,$$

respectively. Using relations  $a_r^i = q_r^m \bar{p}_m^i$ , and  $b_j^s = \bar{q}_j^v p_v^s$ , obtained from (3.1), in (7.2), we have

$$\bar{q}_i^a \bar{p}_b^j \bar{p}_c^k \bar{I}_{jk}^i = q_r^a p_b^s p_c^t I_{st}^r, \quad \bar{q}_i^a \bar{p}_b^j \bar{p}_c^k \bar{p}_d^l \bar{I}_{jkl}^i = q_r^a p_b^s p_c^t p_d^u I_{stu}^r,$$

which describes  $L_n^1$ -invariant objects in  $T_n^2 L_n^1 / K_n^{3,1}$ . Using (5.5), we get

$$q_r^a p_b^s p_c^t I_{st}^r = \mathcal{I}_{bc}^a, \quad q_r^a p_b^s p_c^t p_d^u I_{stu}^r = \mathcal{I}_{bcd}^a,$$

where  $\mathcal{I}_{bc}^a, \mathcal{I}_{bcd}^a$  are given by (7.1).  $\square$

Using factorization method, we are allowed to determine the number of independent  $L_n^3$ -invariant functions on  $T_n^2 L_n^1$  as the dimensions of the corresponding factor spaces. Thus, the number of functionally independent invariants  $\mathcal{I}_{jk}^i, \mathcal{I}_{jkl}^i$  is given by

$$\dim(T_n^1 L_n^1 / L_n^2) = \frac{1}{2} n^2 (n-1), \quad \dim(T_n^2 L_n^1 / L_n^3) = \frac{1}{3} n^2 (n^2 - 1),$$

respectively. For instance for  $n = 1, 2, 3, 4$ , the number of independent  $L_n^3$ -invariant functions on  $T_n^2 L_n^1$  is  $N = 0, 6, 33, 104$ , respectively.

Let us consider a left action of the general linear group  $L_n^1$  on the real line  $\mathbb{R}$  by

$$L_n^1 \times \mathbb{R} \ni (a, t) \mapsto |\det a^{-1}| \cdot t \in \mathbb{R}.$$

The real line, endowed with this action, is an  $L_n^1$ -manifold, denoted by  $\widetilde{\mathbb{R}}$ .

We also introduce the function  $\mathcal{I}_0$  defined on  $T_n^2 L_n^1$  by

$$T_n^2 L_n^1 \ni q \mapsto \mathcal{I}_0(q) = |\det q_j^i(q)| \in \mathbb{R}.$$

**Lemma 5.** *The function  $\mathcal{I}_0 : T_n^2 L_n^1 \rightarrow \widetilde{\mathbb{R}}$  is a differential invariant.*

*Proof.* Obviously, the function  $\mathcal{I}_0$  is smooth, and for every  $a \in L_n^3$ , and every  $q \in T_n^2 L_n^1$  we have  $\mathcal{I}_0(a \cdot q) = |\det a^{-1}| \cdot \mathcal{I}_0(q)$ .  $\square$

**Corollary 4.** *Every differential invariant  $\mathcal{I}$  defined on  $T_n^2 L_n^1$ , with values in  $\widetilde{\mathbb{R}}$ , is the product of some scalar invariant and the function  $\mathcal{I}_0$ .*

## 8 Canonical odd $n$ -form on $FX$

In order to compare our results with [5], we recall in this Section the concept of volume form needed for integration on not necessarily orientable manifold.

Any chart  $(U, \varphi)$ ,  $\varphi = (x^i)$ , on  $X$ , induces the *fibred* chart  $(V, \psi)$ ,  $\psi = (x^i, x_j^i)$ , on  $FX$ . For every frame  $\Xi \in V$  we have  $\det x_j^i(\Xi) \neq 0$ , and we can define some other coordinates  $y_k^j$  of  $\Xi$  by setting  $x_j^i y_k^j = \delta_k^i$ . We define a function  $V \ni \Xi \mapsto |\det y_j^i(\Xi)| \in \mathbb{R}$ , associated with the chart  $(V, \psi)$ . With a chart  $(V, \psi)$  we also associate the object

$$(8.1) \quad \tilde{\omega}_{(V, \psi)} = |\det y_j^i| \cdot \tilde{\varphi} \otimes dx^1 \wedge dx^2 \wedge \dots \wedge dx^n,$$

where  $\tilde{\varphi}$  is a *field of odd scalars* on  $X$ , associated with  $(U, \varphi)$  (see [9]). It is easily seen that (8.1) represent a globally defined odd base form on  $FX$ ; we denote this form by  $\tilde{\omega}$ , and call it *canonical odd  $n$ -form on  $X$* .

This form has the following properties:

1. For each frame field  $\zeta : W \rightarrow FX$ , where  $W$  is an open set on  $X$ , the pullback  $\zeta^* \tilde{\omega}$  is an odd volume form on  $W$ .
2. The construction of  $\tilde{\omega}$  does not depend on orientability of base manifold  $X$ . In the case of orientable and oriented manifolds  $X$ ,  $\tilde{\omega}$  is equivalent to an (ordinary)  $n$ -form on  $FX$ .
3. The form  $\tilde{\omega}$  is *diff  $X$ -invariant*, i.e.  $(F\alpha)^* \tilde{\omega} = \tilde{\omega}$  for all diffeomorphisms  $\alpha$  of  $X$ , where  $F\alpha$  is the canonical lift of  $\alpha$  to  $FX$ .

It should be pointed out that odd  $n$ -forms  $\zeta^* \tilde{\omega}$  may be used as volume forms for integration on the base manifold  $X$ . In particular, these forms naturally appears as a components of Lagrangians for variational problems for frame fields. We discuss these questions in the subsequent Section.

## 9 Second order natural Lagrangians of frames

Our aim in this Section is to characterize all Lagrangians on  $J^2 FX$ , invariant with respect to all diffeomorphisms of  $X$ . First we recall main concepts to this purpose.

We present basic definitions in full generality (for odd base forms). If the underlying manifold  $X$  is orientable, odd base forms may be replaced by ordinary forms.

Let us denote by  $\mu$  the bundle projection  $\mu : FX \rightarrow X$ . The canonical jet projection  $\mu^2 : J^2 FX \rightarrow X$  is, for every  $J_x^2 \zeta \in J^2 FX$ , defined by  $\mu^2(J_x^2 \zeta) = x$ . A second order Lagrangian for  $FX$  is any  $\mu^2$ -horizontal  $n$ -form  $\lambda$  defined on the second jet prolongation  $J^2 FX$  of  $FX$ . In a chart  $(V, \psi)$ ,  $\psi = (x^i, x_j^i)$ , on  $FX$ , and the

associated chart  $(V^2, \psi^2)$ ,  $\psi^2 = (x^i, x_j^i, x_{j,k}^i, x_{j,kl}^i)$ , on  $J^2FX$ , a Lagrangian  $\lambda$  has an expression

$$\lambda = \mathcal{L} \cdot \tilde{\varphi} \otimes \omega_0,$$

where  $\omega_0 = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$ ,  $\tilde{\varphi}$  is a field of odd scalars, and  $\mathcal{L} : V^2 \rightarrow \mathbb{R}$  is the component of  $\lambda$  with respect to  $(V, \psi)$  (the Lagrange function associated with  $(V, \psi)$ ).

We say that a second order Lagrangian  $\lambda$  is *natural*, if for every diffeomorphism  $\alpha : W \rightarrow X$ , where  $W$  is an open set in  $X$ , the canonical lift  $F\alpha$  of  $\alpha$  to  $FX$  is an invariance transformation of  $\lambda$ , i.e.,

$$(J^2F\alpha)^*\lambda = \lambda$$

on the corresponding open set in  $J^2FX$ .

The following theorem is an application of a general result to the structure we consider in this paper (see [10, 11]).

**Theorem 3.** *Let  $X$  be an  $n$ -dimensional manifold. There exists a one-to-one correspondence between natural Lagrangians on  $J^2FX$  and differential invariants  $\mathcal{I} : T_n^2 L_n^1 \rightarrow \tilde{\mathbb{R}}$ .*

We denote by  $\mathcal{A}_{\text{diff}X}$  the algebra of diff  $X$ -invariant functions on  $J^2FX$ . Define in any chart  $(V, \psi)$ ,  $\psi = (x^i, x_j^i)$ , on  $FX$ , functions  $\mathcal{L}_{jk}^i, \mathcal{L}_{jkl}^i$ , by

$$(9.1) \quad \begin{aligned} \mathcal{L}_{jk}^i &= y_l^i x_k^s x_{j,s}^l, \\ \mathcal{L}_{jkl}^i &= y_r^i (2x_k^s x_l^t x_{j,st}^r - x_l^s x_j^t x_{k,st}^r - x_j^s x_k^t x_{l,st}^r \\ &\quad - \frac{3}{2} y_m^u x_{u,s}^r (x_k^s x_l^t x_{j,t}^m + x_l^s x_k^t x_{j,t}^m) - \frac{5}{2} x_{j,s}^m x_k^s x_{l,m}^r \\ &\quad + \frac{1}{2} x_j^s x_{k,m}^r x_{l,s}^m + 2x_{j,m}^r x_k^s x_{l,s}^m) \end{aligned}$$

(compare with (7.1)). The functions  $\mathcal{L}_{jk}^i, \mathcal{L}_{jkl}^i$ , in coordinates expressed by (9.1), are globally defined functions on  $J^2FX$ .

**Corollary 5.** *The functions  $\mathcal{L}_{jk}^i, \mathcal{L}_{jkl}^i \in \mathcal{A}_{\text{diff}X}$ , and every function  $\mathcal{L} \in \mathcal{A}_{\text{diff}X}$  can be locally written as a differentiable function of the functions  $\mathcal{L}_{jk}^i, \mathcal{L}_{jkl}^i$ , defined by (9.1).*

The following theorem is an immediate consequence of the invariance theory.

**Theorem 4.** *Every natural Lagrangian  $\lambda$  on  $J^2FX$  is of the form*

$$\lambda = \mathcal{L} \tilde{\omega},$$

where  $\mathcal{L} \in \mathcal{A}_{\text{diff}X}$  and  $\tilde{\omega}$  is canonical odd  $n$ -form of  $FX$ .

**Remark 1.** Restricting ourselves to the *first order* natural Lagrangians, Theorem 4 corresponds with the known results published in [5].

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