Nonholonomic approach of multitime maximum principle

Constantin Udrişte

Abstract. Many science and engineering problems can be formulated as optimization problems that are governed by contact distributions (multitime Pfaff evolution systems) and by cost functionals expressed as multiple integrals or curvilinear integrals. Our paper discuss the contact distribution constrained optimization problems, focussing on a nonholonomic approach of multitime maximum principle. This principle extends the work of Pontryaguin in the ODEs case to include the case of normal PDEs or, more general, the distribution case.

In Section 1 a multitime maximum principle for the case of multiple integral functionals is stated and proved. Section 2 is devoted to the multitime maximum principle for the case of curvilinear integral functionals. Section 3 deals with a multitime maximum principle approach of variational calculus in the case of nonintegrability.

M.S.C. 2000: 93C20, 93C35, 49K20, 49J20, 53C44.
Key words: multitime maximum principle, multiple or curvilinear integral functional, contact distribution constrained optimization.

1 Contact distribution constrained optimization problem with multiple integral functional

Because of their size, complexity, and infinite dimensional nature, the PDE constraints often present significant challenges for optimization principles [1]-[6]. In this context, the papers [9]-[20] formulated a holonomic multitime maximum principle by a scheme which mimics those applied to single-time maximum principle [7], [8]. Some topics in [21]-[23] can be involved in this theory.

Any normal PDE generates a contact distribution. This kind of (holonomic or nonholonomic) distributions are very important in differential geometry and analytical mechanics. Extending our point of view in [19], let us analyze a multitime optimal control problem based on a multiple integral cost functional and a contact distribution constraint:

Balkan Journal of Geometry and Its Applications, Vol.14, No.2, 2009, pp. 101-116.

$$\max_{u(\cdot),x_{t_0}} I(u(\cdot)) = \int_{\Omega_{0,t_0}} X(t,x(t),u(t))\omega \tag{1}$$

subject to

$$dx^i(t) = X^i_\alpha(t,x(t),u(t))dt^\alpha, i = 1,...,n; \alpha = 1,...,m, \tag{2}$$

$$u(t) \in \mathcal{U},\ \forall t \in \Omega_{0,t_0};\ x(0) = x_0,\ x(t_0) = x_{t_0}. \tag{3}$$

Ingredients: $t = (t^\alpha) \in R^m_+$ is the multi-parameter of evolution or *multitime*; $\omega = dt^1 \wedge ... \wedge dt^m$ is the volume form in R^m_+; Ω_{0,t_0} is the parallelepiped fixed by the diagonal opposite points $0 = (0,...,0)$ and $t_0 = (t^1_0,...,t^m_0)$ which is equivalent to the closed interval $0 \leq t \leq t_0$ via the product order on R^m_+; $x : \Omega_{0,t_0} \to R^n$, $x(t) = (x^i(t))$ is a C^1 *state vector*; $u : \Omega_{0,t_0} \to U \subset R^k$, $u(t) = (u^a(t))$, $a = 1,...,k$ is a continuous *control vector*; the *running cost* $X(t,x(t),u(t))\omega$ is a C^1 *Lagrange m-form*; $X^i_\alpha(t,x(t),u(t))$ are C^1 vector fields; the Pfaff equations (2) define a *contact distribution*.

Introducing a *costate variable tensor or Lagrange multiplier tensor* $p = p^\alpha_i(t)\dfrac{\partial}{\partial t^\alpha} \otimes dx^i$, and the $(m-1)$-forms $\omega_\lambda = \dfrac{\partial}{\partial t^\lambda} \rfloor \omega$, we build a new Lagrange m-form

$$L(t,x(t),u(t),p(t)) = X(t,x(t),u(t))\omega + p^\lambda_i(t)[X^i_\alpha(t,x(t),u(t))dt^\alpha - dx^i(t)] \wedge \omega_\lambda.$$

The contact distribution constrained optimization problem (1)-(3) can be changed into another optimization problem

$$\max_{u(\cdot),x_{t_0}} \int_{\Omega_{0,t_0}} L(t,x(t),u(t),p(t))$$

subject to

$$u(t) \in \mathcal{U},\ p(t) \in \mathcal{P},\ \forall t \in \Omega_{0,t_0};\ x(0) = x_0,\ x(t_0) = x_{t_0},$$

where the set $\mathcal{P}$ will be defined later. The *control Hamiltonian m-form*

$$H(t,x(t),u(t),p(t)) = X(t,x(t),u(t))\omega + p^\lambda_i(t)X^i_\alpha(t,x(t),u(t))dt^\alpha \wedge \omega_\lambda$$

$$= (X(t,x(t),u(t)) + p^\alpha_i(t)X^i_\alpha(t,x(t),u(t)))\omega = H_1(t,x(t),u(t),p(t))\omega,$$

i.e.,

$$H = L + p^\lambda_i(t)dx^i(t) \wedge \omega_\lambda \text{ (modified Legendrian duality)},$$

permits to rewrite this new problem as

$$\max_{u(\cdot),x_{t_0}} \int_{\Omega_{0,t_0}} [H(t,x(t),u(t),p(t)) - p^\lambda_i(t)dx^i(t) \wedge \omega_\lambda]$$

subject to

$$u(t) \in \mathcal{U},\ p(t) \in \mathcal{P},\ \forall t \in \Omega_{0,t_0};\ x(0) = x_0, x(t_0) = x_{t_0}.$$

Suppose that there exists a continuous control $\hat{u}(t)$ defined over the parallelepiped Ω_{0,t_0}, with $\hat{u}(t) \in \mathrm{Int}\,\mathcal{U}$, which is an optimum point in the previous problem. Now consider a variation $u(t,\epsilon) = \hat{u}(t) + \epsilon h(t)$, where h is an arbitrary continuous vector function. Since $\hat{u}(t) \in \mathrm{Int}\,\mathcal{U}$ and a continuous function over a compact set Ω_{0,t_0} is bounded, there exists a number $\epsilon_h > 0$ such that $u(t,\epsilon) = \hat{u}(t) + \epsilon h(t) \in \text{Int}\,\mathcal{U},\ \forall|\epsilon| < \epsilon_h$. This ϵ is used in our variational arguments.

Define the contact distribution corresponding to the control variable $u(t,\epsilon)$, i.e.,

$$dx^i(t,\epsilon) = X^i_\alpha(t, x(t,\epsilon), u(t,\epsilon))dt^\alpha, \forall t \in \Omega_{0,t_0}$$

and $x(0,\epsilon) = x_0$. For $|\epsilon| < \epsilon_h$, we define the function

$$I(\epsilon) = \int_{\Omega_{0,t_0}} X(t, x(t,\epsilon), u(t,\epsilon))\omega.$$

On the other hand, the control $\hat{u}(t)$ must be optimal. Therefore $I(\epsilon) \leq I(0),\ \forall|\epsilon| < \epsilon_h$.

For any continuous tensor function

$$p = (p^\alpha_i) : \Omega_{0,t_0} \to R^{nm},$$

we have

$$\int_{\Omega_{0,t_0}} p^\lambda_i(t)[X^i_\alpha(t, x(t,\epsilon), u(t,\epsilon))dt^\alpha - dx^i(t,\epsilon)] \wedge \omega_\lambda = 0.$$

Necessarily, we must use the Lagrange m-form which includes the variations

$$L(t, x(t,\epsilon), u(t,\epsilon), p(t)) = X(t, x(t,\epsilon), u(t,\epsilon))\omega$$

$$+ p^\lambda_i(t)[X^i_\alpha(t, x(t,\epsilon), u(t,\epsilon))dt^\alpha - dx^i(t,\epsilon)] \wedge \omega_\lambda$$

and the associated function

$$I(\epsilon) = \int_{\Omega_{0,t_0}} L(t, x(t,\epsilon), u(t,\epsilon), p(t)).$$

Suppose that the costate variable p is of class C^1. Also we introduce the control Hamiltonian m-form

$$H(t, x(t,\epsilon), u(t,\epsilon), p(t)) = X(t, x(t,\epsilon), u(t,\epsilon))\omega + p^\lambda_i(t)X^i_\alpha(t, x(t,\epsilon), u(t,\epsilon))dt^\alpha \wedge \omega_\lambda$$

corresponding to the variation. Then we rewrite

$$I(\epsilon) = \int_{\Omega_{0,t_0}} [H(t, x(t,\epsilon), u(t,\epsilon), p(t)) - p^\lambda_i(t)dx^i(t,\epsilon) \wedge \omega_\lambda].$$

To evaluate the multiple integral

$$\int_{\Omega_{0,t_0}} p^\lambda_i(t)dx^i(t,\epsilon) \wedge \omega_\lambda,$$

we integrate by parts, via the formula

$$d(p^\lambda_i x^i \omega_\lambda) = (x^i dp^\lambda_i + p^\lambda_i dx^i) \wedge \omega_\lambda,$$

obtaining

$$\int_{\Omega_{0,t_0}} p^\lambda_i(t)dx^i(t,\epsilon)\wedge\omega_\lambda = \int_{\Omega_{0,t_0}} d(p^\lambda_i(t)x^i(t,\epsilon)\omega_\lambda) - \int_{\Omega_{0,t_0}} dp^\lambda_i(t)x^i(t,\epsilon)\wedge\omega_\lambda.$$

Now we apply the Stokes integral formula

$$\int_{\Omega_{0,t_0}} d(p^\lambda_i(t)x^i(t,\epsilon)\omega_\lambda) = \int_{\partial\Omega_{0,t_0}} \delta_{\alpha\beta}p^\alpha_i(t)x^i(t,\epsilon)n^\beta(t)d\sigma,$$

where $(n^\beta(t))$ is the unit normal vector to the boundary $\partial\Omega_{0,t_0}$. Substituting, we find

$$I(\epsilon) = \int_{\Omega_{0,t_0}} [H(t,x(t,\epsilon),u(t,\epsilon),p(t)) + dp^\lambda_j(t)x^j(t,\epsilon)\wedge\omega_\lambda]$$

$$-\int_{\partial\Omega_{0,t_0}} \delta_{\alpha\beta}p^\alpha_i(t)x^i(t,\epsilon)n^\beta(t)d\sigma.$$

Differentiating with respect to ϵ, it follows

$$I'(\epsilon) = \int_{\Omega_{0,t_0}} [H_{x^j}(t,x(t,\epsilon),u(t,\epsilon),p(t)) + dp^\lambda_j(t)\wedge\omega_\lambda]x^j_\epsilon(t,\epsilon)$$

$$+\int_{\Omega_{0,t_0}} H_{u^a}(t,x(t,\epsilon),u(t,\epsilon),p(t))h^a(t) - \int_{\partial\Omega_{0,t_0}} \delta_{\alpha\beta}p^\alpha_i(t)x^i_\epsilon(t,\epsilon)n^\beta(t)d\sigma.$$

Evaluating at $\epsilon = 0$, we find

$$I'(0) = \int_{\Omega_{0,t_0}} [H_{x^j}(t,x(t),\hat{u}(t),p(t)) + dp^\lambda_j(t)\wedge\omega_\lambda]x^j_\epsilon(t,0)$$

$$+\int_{\Omega_{0,t_0}} H_{u^a}(t,x(t),\hat{u}(t),p(t))h^a(t) - \int_{\partial\Omega_{0,t_0}} \delta_{\alpha\beta}p^\alpha_i(t)x^i_\epsilon(t,0)n^\beta(t)d\sigma.$$

where $x(t)$ is the state variable corresponding to the optimal control $\hat{u}(t)$.

We need $I'(0) = 0$ for all $h(t) = (h^a(t))$. On the other hand, the functions $x^i_\epsilon(t,0)$ are involved in the Cauchy-Pfaff problem

$$dx^i_\epsilon(t,0) = X_{\alpha x^j}(t,x(t,0),u(t))x^j_\epsilon(t,0)dt^\alpha + X_{\alpha u^a}(t,x(t,0),u(t))h^a(t)dt^\alpha,$$

$$t\in\Omega_{0,t_0},\ x_\epsilon(0,0) = 0$$

and hence they depend on $h(t)$. To overpass this confusion, we define $\mathcal{P}$ as the set of solutions of the boundary value problem

$$\operatorname{div} p_j(t) = -H_{1x^j}(t,x(t),\hat{u}(t),p(t)),\ \forall t\in\Omega_{0,t_0}, \tag{4}$$

$$\delta_{\alpha\beta}p^\alpha_j(t)n^\beta(t)|_{\partial\Omega_{0,t_0}} = 0\,(\text{orthogonality or tangency}).$$

Therefore

$$H_{1u^a}(t,x(t),\hat{u}(t),p(t)) = 0,\ \forall t\in\Omega_{0,t_0}. \tag{5}$$

Moreover

$$dx^j(t) = \frac{\partial H_1}{\partial p^\alpha_j}(t, x(t), \hat{u}(t), p(t))dt^\alpha, \ \forall t \in \Omega_{0,t_0}, \ x(0) = x_0. \tag{6}$$

Remarks. (i) The algebraic system (5) describes the critical points of the Hamiltonian m-form with respect to the control variable. (ii) The PDEs (4), the contact distribution (6) and the condition (5) are Euler-Lagrange distributions associated to the new Lagrangian. (iii) Any ODE or PDE, written in the normal form, generates a contact distribution.

Summarizing the previous reasonings we obtain a *multitime maximum principle* similar to the *single-time Pontryaguin maximum principle.*

Theorem 1. (Multitime maximum principle; necessary conditions) *Suppose that the problem of maximizing the functional (1) subject to the contact distribution constraint (2) and to the conditions (3), with X, X^i_α of class C^1, has an interior solution $\hat{u}(t) \in \mathcal{U}$ which determines the contact distribution (2). Then there exists a C^1 costate $p(t) = (p^\alpha_i(t))$ defined over Ω_{0,t_0} such that the relations (4), (5), (6) hold.*

Theorem 2. (Sufficient conditions) *Consider the problem of maximizing the functional (1) subject to the contact distribution constraint (2) and to the conditions (3), with X, X^i_α of class C^1. Suppose that an interior solution $\hat{u}(t) \in \mathcal{U}$ and the corresponding contact distribution (2) satisfy the relations (4), (5), (6). If for the resulting costate variable $p(t) = (p^\alpha_i(t))$ the control Hamiltonian $H(t, x, u, p)$ is jointly concave in (x, u) for all $t \in \Omega_{0,t_0}$, then $\hat{u}(t)$ and the corresponding contact distribution achieve the unique global maximum of (1).*

Proof. Let us have in mind that we must maximize the functional (1) subject to the evolution system (2) and the conditions (3). We fix a pair $(\hat{x}, \hat{u})$, where $\hat{u}$ is a candidate optimal control and $\hat{x}$ is a candidate optimal state. Calling $\hat{I}$ the values of the functional for $(\hat{x}, \hat{u})$, let us prove that

$$\hat{I} - I = \int_{\Omega_{0,t_0}} (\hat{X} - X)\omega \geq 0,$$

where the strict inequality holds under strict concavity. Denoting $\hat{H} = H(\hat{x}, \hat{p}, \hat{u})$ and $H = H(x, \hat{p}, u)$, we find

$$\hat{I} - I = \int_{\Omega_{0,t_0}} \Big((\hat{H} - \hat{p}^\lambda_i d\hat{x}^i \wedge \omega_\lambda) - (H - \hat{p}^\lambda_i dx^i \wedge \omega_\lambda) \Big).$$

Integrating by parts, we obtain

$$\hat{I} - I = \int_{\Omega_{0,t_0}} \Big((\hat{H} + \hat{x}^i d\hat{p}^\lambda_i \wedge \omega_\lambda) - (H + x^i d\hat{p}^\lambda_i \wedge \omega_\lambda) \Big)$$

$$+ \int_{\partial\Omega_{0,t_0}} (\delta_{\alpha\beta}\hat{p}^\alpha_i(t)x^i(t)n^\beta(t) - \delta_{\alpha\beta}\hat{p}^\alpha_i(t)\hat{x}^i(t)n^\beta(t))d\sigma.$$

Taking into account that any admissible sheet has the same initial and terminal conditions as the optimal sheet, we derive

$$\hat{I} - I = \int_{\Omega_{0,t_0}} \Big((\hat{H} - H) + (\hat{x}^i - x^i)d\hat{p}^\lambda_i \wedge \omega_\lambda \Big).$$

The definition of concavity implies

$$\int_{\Omega_{0,t_0}} \Big((\hat{H} - H) + (\hat{x}^i - x^i) d\hat{p}_i^\lambda \wedge \omega_\lambda\Big)$$

$$\geq \int_{\Omega_{0,t_0}} \left((\hat{x}^i - x^i)(\frac{\partial \hat{H}}{\partial x^i} + \frac{\partial \hat{p}_i^\alpha}{\partial t^\alpha}\omega) + (\hat{u}^a - u^a)\frac{\partial \hat{H}}{\partial u^a} \right) = 0.$$

This last equality follows from that all "ˆ" variables satisfy the conditions of the multitime maximum principle. In this way, $\hat{I} - I \geq 0$.

Theorem 3. (Sufficient conditions) *Consider the problem of maximizing the functional (1) subject to the contact distribution constraint (2) and to the conditions (3), with X, X_α^i of class C^1. Suppose that an interior solution $\hat{u}(t) \in \mathcal{U}$ and the corresponding contact distribution satisfy the relations (4), (5), (6). Giving the resulting costate variable $p(t) = (p_i^\alpha(t))$, we define $M(t,x,p) = H(t,x,\hat{u}(t),p)$. If $M(t,x,p)$ is concave in x for all $t \in \Omega_{0,t_0}$, then $\hat{u}(t)$ and the corresponding contact distribution achieve the unique global maximum of (1).*

Remark. The Theorems 2 and 3 can be extended immediately to *incave functionals.*

Examples. 1) We consider the problem

$$\max_{u(\cdot),\, x_1} I(u(\cdot)) = -\int_{\Omega_{0,1}} (x(t) + u_1(t)^2 + u_2(t)^2) dt^1 \wedge dt^2$$

subject to

$$dx(t) = u_\alpha(t) dt^\alpha,\ \alpha = 1,2,\ x(0,0) = 0,\ x(1,1) = x_1 = \text{free}.$$

This problem means to find an optimal control $u = (u_1, u_2)$ to bring the (PDE) dynamical system from the origin $x(0,0) = 0$ at two-time $t^1 = 0, t^2 = 0$ to a terminal point $x(1,1) = x_1$, which is unspecified, at two-time $t^1 = 1, t^2 = 1$, such as to maximize the objective functional. The control Hamiltonian 2-form is

$$H(x(t), u(t), p(t)) = -(x(t) + u_1(t)^2 + u_2(t)^2) dt^1 \wedge dt^2 + p^\lambda(t)(u_\alpha(t) dt^\alpha) \wedge \omega_\lambda$$

$$= -(x(t) + u_1(t)^2 + u_2(t)^2) dt^1 \wedge dt^2 + p^\alpha(t) u_\alpha(t) dt^1 \wedge dt^2$$

$$= H_1(x(t), u(t), p(t)) dt^1 \wedge dt^2.$$

Since

$$\frac{\partial H_1}{\partial u_\alpha} = -2u_\alpha + p^\alpha,\ \frac{\partial^2 H_1}{\partial u_\alpha^2} = -2 < 0,\ \frac{\partial^2 H_1}{\partial u_\alpha \partial u_\beta} = 0,$$

the critical point $p^\alpha = 2u_\alpha$ is a maximum point. Then the PDE $\frac{\partial p^\alpha}{\partial t^\alpha} = -\frac{\partial H_1}{\partial x}$ reduces to $\frac{\partial p^1}{\partial t^1} + \frac{\partial p^2}{\partial t^2} = 1$. Under the transversality condition

$$p^1(t) n^1(t) + p^2(t) n^2(t)|_{\partial \Omega_{0,1}} = 0,$$

this PDE has an infinity of solutions.

Consequently the optimal control $u(t) = (u_1(t), u_2(t))$ is solution of PDE $\frac{\partial u_1}{\partial t^1} + \frac{\partial u_2}{\partial t^2} = 1$ satisfying the boundary conditions $u_1(0, t^2) = u_1(1, t^2) = 0$, $u_2(t^1, 0) = u_2(t^1, 1) = 0$.

Case 1. If the Pfaff equation $dx(t) = u_\alpha(t)dt^\alpha$ is completely integrable, i.e., $\frac{\partial u_1}{\partial t^2} = \frac{\partial u_2}{\partial t^1}$, then the components of the optimal control are harmonic functions. Also the dynamical system $dx = u_1(t)dt^1 + u_2(t)dt^2$, equivalent to the PDE system $\frac{\partial x}{\partial t^\alpha}(t) = u_\alpha(t)$, gives $x(t) - x(0) = \int_{\Gamma_{0,t}} u_1(s)ds^1 + u_2(s)ds^2$.

Case 2. If the Pfaff equation $dx(t) = u_\alpha(t)dt^\alpha$ is not completely integrable, then $x(t) - x(0) = \int_{\Gamma_{0,t}} u_1(s)ds^1 + u_2(s)ds^2$ depends on the path $\Gamma_{0,t} : s^\alpha = s^\alpha(\tau)$, i.e.,

$$\frac{\partial x}{\partial t^\alpha}(t(\tau)) - u_\alpha(t(\tau)) = \mu_\alpha(\tau),\ \mu_\alpha(\tau)\frac{dt^\alpha}{d\tau}(\tau) = 0.$$

2) We consider the problem

$$\max_{u(\cdot),\, x_1} I(u(\cdot)) = -\frac{1}{2}x(1,1)^2 - \frac{1}{2}\int_{\Omega_{0,1}} (u_1(t)^2 + u_2(t)^2)dt^1 \wedge dt^2$$

subject to

$$dx(t) = -u_\alpha(t)dt^\alpha,\ \alpha = 1, 2,\ x(0,0) = 1.$$

This problem means to find an optimal control $u = (u_1, u_2)$ to bring the (PDE) dynamical system from the point $x(0,0) = 1$ at two-time $t^1 = 0, t^2 = 0$ to a terminal point $x(1,1) = x_1$, at two-time $t^1 = 1, t^2 = 1$, such as to maximize the objective functional. The control Hamiltonian 2-form is defined by

$$H_1(x(t), u(t), p(t)) = -\frac{1}{2}(u_1(t)^2 + u_2(t)^2) - p^\alpha(t)u_\alpha(t).$$

Since

$$\frac{\partial H_1}{\partial u_\alpha} = -u_\alpha - p^\alpha,\ \frac{\partial^2 H_1}{\partial u_\alpha^2} = -1 < 0,\ \frac{\partial^2 H_1}{\partial u_\alpha \partial u_\beta} = 0,$$

the critical point $p^\alpha = -u_\alpha$ is a maximum point. Then the PDE $\frac{\partial p^\alpha}{\partial t^\alpha} = -\frac{\partial H}{\partial x} = 0$ reduces to $\frac{\partial p^1}{\partial t^1} + \frac{\partial p^2}{\partial t^2} = 0$. The transversality condition implies

$$p^1(t)n^1(t) + p^2(t)n^2(t)|_{\partial\Omega_{0,1}} = 0.$$

Consequently the optimal control $u(t) = (u_1(t), u_2(t))$ is solution of PDE $\frac{\partial u_1}{\partial t^1} + \frac{\partial u_2}{\partial t^2} = 0$ satisfying the boundary conditions $u_1(0, t^2) = u_1(1, t^2) = 0$, $u_2(t^1, 0) = u_2(t^1, 1) = 0$.

Case 1. If the Pfaff equation $dx(t) = -u_\alpha(t)dt^\alpha$ is completely integrable, i.e., $\frac{\partial u_1}{\partial t^2} = \frac{\partial u_2}{\partial t^1}$, then the components of the optimal control are harmonic functions.

Also the dynamical system $dx = -u_1(t)dt^1 - u_2(t)dt^2$, equivalent to the PDE system $\frac{\partial x}{\partial t^\alpha}(t) = -u_\alpha(t)$, gives $x(t) - x(0) = -\int_{\Gamma_{0,t}} u_1(s)ds^1 + u_2(s)ds^2$.

Case 2. If the Pfaff equation $dx(t) = -u_\alpha(t)dt^\alpha$ is not completely integrable, then $x(t) - x(0) = -\int_{\Gamma_{0,t}} u_1(s)ds^1 + u_2(s)ds^2$ depends on the path $t^\alpha = t^\alpha(\tau)$, i.e.,

$$\frac{\partial x}{\partial t^\alpha}(t(\tau)) + u_\alpha(t(\tau)) = \mu_\alpha(\tau),\ \mu_\alpha(\tau)\frac{dt^\alpha}{d\tau}(\tau) = 0.$$

2 Contact distribution constrained optimization problem with curvilinear integral cost functional

The cost functionals of mechanical work type are very important for applications [10]-[20]. Extending our point of view in [19], let us analyze a multi-time optimal control problem formulated using as cost functional a curvilinear integral and a contact distribution constraint:

$$\max_{u(\cdot),x_{t_0}} J(u(\cdot)) = \int_{\Gamma_{0,t_0}} X^0_\alpha(t, x(t), u(t))dt^\alpha \tag{7}$$

subject to

$$dx^i(t) = X^i_\alpha(t, x(t), u(t))dt^\alpha,\ i = 1, ..., n;\ \alpha = 1, ..., m, \tag{8}$$

$$u(t) \in \mathcal{U}, \forall t \in \Omega_{0,t_0};\ x(0) = x_0,\ x(t_0) = x_{t_0}. \tag{9}$$

Ingredients: $t = (t^\alpha) \in R^m_+$ is the multi-parameter of evolution or *multitime*; Γ_{0,t_0} is a C^1 curve joining the diagonal opposite points $0 = (0, ..., 0)$ and $t_0 = (t^1_0, ..., t^m_0)$ in Ω_{0,t_0}; $x : \Omega_{0,t_0} \to R^n$, $x(t) = (x^i(t))$ is a C^2 *state vector*; $u : \Omega_{0,t_0} \to U \subset R^k$, $u(t) = (u^a(t))$, $a = 1, ..., k$ is a continuous *control vector*; the *running cost* $\eta = X^0_\alpha(t, x(t), u(t))dt^\alpha$ is a *nonautonomous Lagrangian 1-form*; the vector fields $X^i_\alpha(t, x(t), u(t))$ are of class C^1; the Pfaff equations (8) define a *contact distribution.*

Introducing a *costate* 1*-form or Lagrange multiplier* 1*-form* $p = p_i(t)dx^i$, we obtain a new Lagrange 1-form

$$L(t, x(t), u(t), p(t)) = X^0_\alpha(t, x(t), u(t))dt^\alpha + p_i(t)[X^i_\alpha(t, x(t), u(t))dt^\alpha - dx^i(t)].$$

The contact distribution constrained optimization problem (7)-(9) can be replaced by another optimization problem

$$\max_{u(\cdot),\,x_{t_0}} \int_{\Gamma_{0,t_0}} L(t, x(t), u(t), p(t))$$

subject to

$$u(t) \in \mathcal{U},\ p(t) \in \mathcal{P},\ \forall t \in \Omega_{0,t_0};\ x(0) = x_0,\ x(t_0) = x_{t_0},$$

where the set $\mathcal{P}$ will be defined later. If we use the *control Hamiltonian* 1*-form*

$$H(t, x(t), u(t), p(t)) = X^0_\alpha(t, x(t), u(t))dt^\alpha + p_i(t)X^i_\alpha(t, x(t), u(t))dt^\alpha,$$

$$H = L + p_i dx^i \text{ (modified Legendrian duality)},$$

we can rewrite

$$\max_{u(\cdot), x_{t_0}} \int_{\Gamma_{0,t_0}} [H(t, x(t), u(t), p(t)) - p_i(t) dx^i(t)]$$

subject to

$$u(t) \in \mathcal{U},\ p(t) \in \mathcal{P},\ \forall t \in \Omega_{0,t_0};\ x(0) = x_0,\ x(t_0) = x_{t_0}.$$

Suppose that there exists a continuous control $\hat{u}(t)$ defined over Ω_{0,t_0} with $\hat{u}(t) \in \mathrm{Int}\,\mathcal{U}$ which is optimum in the previous problem. Now consider a variation $u(t,\epsilon) = \hat{u}(t) + \epsilon h(t)$, where h is an arbitrary continuous vector function. Since $\hat{u}(t) \in \mathrm{Int}\,\mathcal{U}$ and a continuous function over a compact set Ω_{0,t_0} is bounded, there exists a number $\epsilon_h > 0$ such that $u(t,\epsilon) = \hat{u}(t) + \epsilon h(t) \in \mathrm{Int}\,\mathcal{U},\ \forall |\epsilon| < \epsilon_h$. This ϵ is used in the next variational arguments.

Let us consider an arbitrary vector function $h(t)$ and define the contact distribution corresponding to the control variable $u(t,\epsilon)$, i.e.,

$$dx^i(t,\epsilon) = X^i_\alpha(t, x(t,\epsilon), u(t,\epsilon)) dt^\alpha, \forall t \in \Omega_{0,t_0},\ x(0,\epsilon) = x_0.$$

For $|\epsilon| < \epsilon_h$, we define the function

$$J(\epsilon) = \int_{\Gamma_{0,t_0}} X^0_\alpha(t, x(t,\epsilon), u(t,\epsilon)) dt^\alpha.$$

On the other hand, the control $\hat{u}(t)$ is supposed to be optimal. Therefore $J(\epsilon) \leq J(0),\ \forall |\epsilon| < \epsilon_h$.

For any continuous 1-form $p = (p_i) : \Omega_{0,t_0} \to R^n$, we have

$$\int_{\Gamma_{0,t_0}} p_i(t)[X^i_\alpha(t, x(t,\epsilon), u(t,\epsilon)) dt^\alpha - dx^i(t,\epsilon)] = 0.$$

The variations determine the Lagrange 1-form

$$L(t, x(t,\epsilon), u(t,\epsilon), p(t)) = X^0_\alpha(t, x(t,\epsilon), u(t,\epsilon)) dt^\alpha$$

$$+ p_i(t)[X^i_\alpha(t, x(t,\epsilon), u(t,\epsilon)) dt^\alpha - dx^i(t,\epsilon)]$$

and the function

$$J(\epsilon) = \int_{\Gamma_{0,t_0}} L(t, x(t,\epsilon), u(t,\epsilon), p(t)).$$

Suppose that the costate p is of class C^1. Also we introduce the control Hamiltonian 1-form

$$H(t, x(t,\epsilon), u(t,\epsilon), p(t)) = X^0_\alpha(t, x(t,\epsilon), u(t,\epsilon)) dt^\alpha + p_i(t) X^i_\alpha(t, x(t,\epsilon), u(t,\epsilon)) dt^\alpha.$$

Then we rewrite

$$J(\epsilon) = \int_{\Gamma_{0,t_0}} [H(t, x(t,\epsilon), u(t,\epsilon), p(t)) - p_i(t) dx^i(t,\epsilon)].$$

To evaluate the curvilinear integral

$$\int_{\Gamma_{0,t_0}} p_i(t)dx^i(t,\epsilon),$$

we integrate by parts, via

$$d(p_i x^i) = x^i dp_i + p_i dx^i,$$

obtaining

$$\int_{\Gamma_{0,t_0}} p_i(t)dx^i(t,\epsilon) = (p_i(t)x^i(t,\epsilon))|_0^{t_0} - \int_{\Gamma_{0,t_0}} x^i(t,\epsilon)dp_i(t).$$

Substituting, we get the function

$$J(\epsilon) = \int_{\Gamma_{0,t_0}} [H(t,x(t,\epsilon),u(t,\epsilon),p(t)) + dp_j(t)x^j(t,\epsilon)] - (p_i(t)x^i(t,\epsilon))|_0^{t_0}.$$

It follows

$$J'(\epsilon) = \int_{\Gamma_{0,t_0}} [H_{x^j}(t,x(t,\epsilon),u(t,\epsilon),p(t)) + dp_j(t)]x^j_\epsilon(t,\epsilon)$$

$$+\int_{\Gamma_{0,t_0}} H_{u^a}(t,x(t,\epsilon),u(t,\epsilon),p(t))h^a(t) - p_i(t_0)x^i_\epsilon(t_0,\epsilon) + p_i(0)x^i_\epsilon(0,\epsilon).$$

Evaluating at $\epsilon = 0$, we find

$$J'(0) = \int_{\Gamma_{0,t_0}} [H_{x^j}(t,x(t),\hat{u}(t),p(t)) + dp_j(t)]x^j_\epsilon(t,0)$$

$$+\int_{\Gamma_{0,t_0}} H_{u^a}(t,x(t),\hat{u}(t),p(t))h^a(t) - p_i(t_0)x^i_\epsilon(t_0,0),$$

where $x(t)$ is the state variable corresponding to the optimal control $\hat{u}(t)$. We need $J'(0) = 0$ for all $h(t) = (h^a(t))$. Therefore we are forced to define $\mathcal{P}$ via the contact distribution terminal value problem

$$dp_j(t) = -\frac{\partial H}{\partial x^j}(t,x(t),\hat{u}(t),p(t)),\ \forall t \in \Gamma_{0,t_0};\ p_j(t_0) = 0. \tag{10}$$

Consequently

$$H_{u^a}(t,x(t),\hat{u}(t),p(t)) = 0,\ \forall t \in \Gamma_{0,t_0}. \tag{11}$$

Moreover

$$dx^j(t) = \frac{\partial H}{\partial p_j}(t,x(t),\hat{u}(t),p(t)),\ \forall t \in \Gamma_{0,t_0};\ x(0) = x_0. \tag{12}$$

Remarks. (i) The algebraic system (11) describes the common critical points of the functions H_α with respect to the control variable. (ii) The contact distributions (10), (12) and the relation (11) are Euler-Lagrange distributions associated to the new Lagrangian 1-form.

Summarizing, we obtain a new variant of multitime maximum principle.

Theorem 4. (Simplified multitime maximum principle; necessary conditions) *Suppose that the problem of maximizing the functional (7) subject to the contact distribution constraint (8) and to the conditions (9), with* X^0_α, X^i_α *of class* C^1*, has an interior solution* $\hat{u}(t) \in \mathcal{U}$ *which determines the contact distribution (8). Then there exists a* C^1 *costate* $p(t) = (p_i(t))$ *defined over* Γ_{0,t_0} *such that the relations (10), (11), (12) hold.*

Theorem 5. (Sufficient conditions) *Consider the problem of maximizing the functional (7) subject to the contact distribution constraint (8) and to the conditions (9), with* X^0_α, X^i_α *of class* C^1*. Suppose that an interior solution* $\hat{u}(t) \in \mathcal{U}$ *and the corresponding contact distribution satisfy the relations (10), (11), (12). If for the resulting costate variable* $p(t) = (p_i(t))$ *the control Hamiltonian 1-form* $H(t, x, u, p)$ *is jointly concave in* (x, u) *for all* $t \in \Gamma_{0,t_0}$*, then* $\hat{u}(t)$ *and the corresponding contact distribution achieve the unique global maximum of (7).*

Theorem 6. (Sufficient conditions) *Consider the problem of maximizing the functional (7) subject to the contact distribution constraint (8) and to the conditions (9), with* X^0_α, X^i_α *of class* C^1*. Suppose that an interior solution* $\hat{u}(t) \in \mathcal{U}$ *and the corresponding contact distribution satisfy the relations (10), (11), (12). Giving the resulting costate variable* $p(t) = (p_i(t))$*, we define the 1-form* $M(t, x, p) = H(t, x, \hat{u}(t), p)$*. If the 1-form* $M(t, x, p)$ *is concave in* x *for all* $t \in \Gamma_{0,t_0}$*, then* $\hat{u}(t)$ *and the corresponding contact distribution achieve the unique global maximum of (7).*

Remark. The Theorems 5 and 6 can be extended immediately to *incave functionals.*

Example. Let $t = (t^1, t^2) \in \Omega_{0,1}$, where $0 = (0, 0), 1 = (1, 1)$ are diagonal opposite points in $\Omega_{0,1}$. Denote by $\Gamma_{0,1}$ a piecewise C^1 curve joining the points 0 and 1. We consider the problem

$$\max_{u(\cdot),\, x_1} J(u(\cdot)) = -\int_{\Gamma_{0,1}} (x(t) + u_\beta(t)^2)dt^\beta$$

subject to

$$dx(t) = u_\alpha(t)dt^\alpha,\ \alpha = 1, 2,\ x(0, 0) = 0,\ x(1, 1) = x_1 = \text{free}.$$

This problem means to find an optimal control $u = (u_1, u_2)$ to bring the Pfaff dynamical system from the origin $x(0, 0) = 0$ at two-time $t^1 = 0, t^2 = 0$ to a terminal point $x(1, 1) = x_1$, which is unspecified, at two-time $t^1 = 1, t^2 = 1$, such as to maximize the objective functional.

The control Hamiltonian 1-form is

$$H(x(t), u(t), p(t)) = (-(x(t) + u_\beta(t)^2) + p(t)u_\beta(t))dt^\beta.$$

We accept that it is enough to work with the components

$$H_\beta(x(t), u(t), p(t)) = -(x(t) + u_\beta(t)^2) + p(t)u_\beta(t).$$

Since

$$\frac{\partial H_\beta}{\partial u_\beta} = -2u_\beta + p,\ \frac{\partial^2 H_\beta}{\partial u_\beta^2} = -2 < 0,$$

the critical point $u_1 = u_2 = \dfrac{p}{2}$ is a maximum point. Then the Pfaff equation $dp(t) = -\dfrac{\partial H}{\partial x}$ is a completely integrable equation $dp(t) = dt^1 + dt^2$. Also, since the point $x(1,1) = x_1$ is unspecified, the transversality condition implies $p(1) = 0$. It follows the costate $p(t) = t^1 + t^2 - 2$, the optimal control $\hat{u}_1(t) = \hat{u}_2(t) = \dfrac{1}{2}(t^1 + t^2 - 2)$ and the corresponding evolution

$$x(t) = \frac{(t^1)^2 + (t^2)^2}{4} + \frac{t^1 t^2}{2} - (t^1 + t^2).$$

Remark. In our example, the complete integrability conditions

$$\frac{\partial x}{\partial t^1} + 2u_2 \frac{\partial u_2}{\partial t^1} = \frac{\partial x}{\partial t^2} + 2u_1 \frac{\partial u_1}{\partial t^2}, \; \frac{\partial u_1}{\partial t^2} = \frac{\partial u_2}{\partial t^1}$$

are satisfied. Consequently, the curve $\Gamma_{0,1}$ is arbitrary.

3 Multitime maximum principle approach of variational calculus

It is well known that the single-time Pontryaguin's maximum principle is a generalization of the Lagrange problem in the single-time variational calculus and that these problems are equivalent when the control domain is open [7]. Does this property survive for *multitime maximum principle*? This problem has a positive answer in the case of complete integrability conditions [10]-[20], i.e., the *multitime maximum principle* motivates the *multitime Euler-Lagrange or Hamilton PDEs.*

The aim of this Section is to discuss this problem in case of the nonintegrability of the running cost 1-form and of constraint distribution. For that, suppose that the evolution system is reduced to a contact distribution system

$$dx^i(t) = u^i_\alpha(t) dt^\alpha, \; x(0) = x_0, \; t \in \Omega_{0,t_0} \subset R^m_+, \tag{CDS}$$

and the functional is a path dependent curvilinear integral

$$J(u(\cdot)) = \int_{\Gamma_{0,t_0}} X^0_\beta(x(t), u(t)) dt^\beta, \tag{J}$$

where Γ_{0,t_0} is a piecewise C^1 curve joining the points 0 and t_0, the *running cost* $\eta = X^0_\beta(x(t), u(t))dt^\beta$ is a C^1 1-form and $u = (u^i_\gamma)$.

The associated basic control problem leads necessarily to the multi-time maximum principle. Therefore, to solve it we need the control Hamiltonian 1-form

$$H(x, p_0, p, u) = (X^0_\beta(x, u) + p_i u^i_\beta) dt^\beta$$

and the adjoint distribution

$$dp_i(t) = -\frac{\partial X^0_\beta}{\partial x^i}(x(t), u(t)) dt^\beta. \tag{ADJ}$$

Suppose the multitime maximum principle is applicable:

$$\frac{\partial}{\partial u^i_\gamma} H = \frac{\partial X^0_\beta}{\partial u^i_\gamma} dt^\beta + p_i dt^\gamma = 0 \text{ or } p_i dt^\gamma = -\frac{\partial X^0_\beta}{\partial u^i_\gamma} dt^\beta. \qquad (13)$$

Suppose the functions X^0_β are dependent on x (a strong condition!). Then the Pfaff equations from (ADJ) show that

$$p_i(t) = p_i(0) - \int_{\Gamma_{0,t}} \frac{\partial X^0_\beta}{\partial x^i}(x(s), u(s)) ds^\beta, \qquad (14)$$

where $\Gamma_{0,t}$ is a piecewise C^1 curve included in the curve Γ_{0,t_0}.

3.1 Euler-Lagrange exterior system

Let $\eta = X^0_\beta(x(t), u(t)) dt^\beta$ be a C^2 1-form. From the relation (13), we find

$$dp_i \wedge dt^\gamma = -d\left(\frac{\partial X^0_\beta}{\partial u^i_\gamma}\right) \wedge dt^\beta.$$

Now, using the Pfaff equations (ADJ), we obtain the Euler-Lagrange exterior equations

$$\left(\frac{\partial X^0_\lambda}{\partial x^i}\delta^\gamma_\beta - \frac{\partial}{\partial t^\lambda}\left(\frac{\partial X^0_\beta}{\partial u^i_\gamma}\right)\right) dt^\lambda \wedge dt^\beta = 0.$$

Now we change our point of view. From the relations (13) and (14), it follows an integro-Pfaff equation

$$-\frac{\partial X^0_\beta}{\partial x^i_\gamma}(x(t), u(t)) dt^\beta = p_i(0) dt^\gamma - dt^\gamma \int_{\Gamma_{0,t}} \frac{\partial X^0_\lambda}{\partial x^i}(x(s), u(s)) ds^\lambda.$$

Giving the parametrization $\Gamma_{0,t_0} : s^\alpha = s^\alpha(\tau)$, we rewrite

$$-\frac{\partial X^0_\beta}{\partial x^i_\gamma}(x(t(\tau)), u(t(\tau)))$$

$$= \delta^\gamma_\beta \left(p_i(0) - \int_0^\tau \frac{\partial X^0_\lambda}{\partial x^i}(x(s(\tau)), u(s(\tau))) \frac{ds^\lambda}{d\tau}(\tau) d\tau\right) + \mu^\gamma_\beta(\tau),$$

where

$$\mu^\gamma_\beta(\tau) \frac{dt^\beta}{d\tau}(\tau) = 0.$$

Also

$$u^i_\alpha(t(\tau)) = x^i_\alpha(t(\tau)) + \nu^i_\alpha(\tau),$$

with

$$\nu^i_\alpha(\tau) \frac{dt^\alpha}{d\tau}(\tau) = 0.$$

Suppose that X^0_β are functions of class C^2. Then we can apply the operator $\dfrac{d}{d\tau}$, transforming the previous equality into an Euler-Lagrange like ODE system.

Remark. The curve Γ_{0,t_0} can be included in the control. For related theory, see [5].

3.2 Conversion to multi-time Hamilton-Pfaff equations (canonical variables)

Let $u(\cdot)$ be an optimal control, $x(\cdot)$ the optimal evolution, and $p(\cdot)$ be the solution of (ADJ) which corresponds to $u(\cdot)$ and $x(\cdot)$. The control Hamiltonian 1-form $H = X^0_\beta dt^\beta + p_j u^j_\beta dt^\beta$ must satisfy

$$\frac{\partial H}{\partial u^i_\gamma} = p_i dt^\gamma + \frac{\partial X^0_\beta}{\partial u^i_\gamma} dt^\beta = 0.$$

This relation defines the costate p as a *moment* along the curve $\Gamma_{0,t}$. Suppose that the critical point condition admits a unique solution that satisfies $u^i_\gamma(t)dt^\gamma = u^i_\gamma(x(t),p(t))dt^\gamma = dx^i(t)$. Then, using a path dependent curvilinear integral, we can write

$$x^i(t) = x^i(0) + \int_{\Gamma_{0,t}} u^i_\gamma(x(s),p(s))ds^\gamma,$$

where $\Gamma_{0,t}$ is a piecewise C^1 curve included in the curve Γ_{0,t_0}. On the other hand

$$\frac{\partial H}{\partial p_i} = \frac{\partial X^0_\beta}{\partial u^j_\gamma} dt^\beta \frac{\partial u^j_\gamma}{\partial p_i} + u^i_\beta dt^\beta + p_j \frac{\partial u^j_\beta}{\partial p_i} dt^\beta = u^i_\beta dt^\beta$$

or

$$dx^i(t) = \frac{\partial H}{\partial p_i}(x(t),p(t),u(t)).$$

Now, the relation

$$-\frac{\partial H}{\partial x^i} = -\left(\frac{\partial X^0_\beta}{\partial x^i} dt^\beta + \frac{\partial X^0_\beta}{\partial u^j_\gamma} dt^\beta \frac{\partial u^j_\gamma}{\partial x^i}\right) - p_j \frac{\partial u^j_\beta}{\partial x^i} dt^\beta$$

and (ADJ) shows

$$dp_i(t) = -\frac{\partial H}{\partial x^i}(x(t),p(t),u(t)).$$

In this way we find the canonical variables x, p and the *multitime Hamilton-Pfaff* equations

$$dx^i(t) = \frac{\partial H}{\partial p_i}(x(t),p(t)),\ dp_i(t) = -\frac{\partial H}{\partial x^i}(x(t),p(t)),$$

along the curve $\Gamma_{0,t}$.

Acknowledgements: Partially supported by Grant CNCSIS 86/ 2008, by 15-th Italian-Romanian Executive Programme of S&T Co-operation for 2006-2008, University Politehnica of Bucharest, and by Academy of Romanian Scientists, Splaiul Independentei 54, 050094 Bucharest, Romania.

We thank Professors Franco Giannessi, Valeriu Prepeliţă, Marius Rădulescu, Ionel Ţevy and Assistant Cristian Ghiu for their suggestions improving the paper.

References

[1] J. Baillieul, P. E. Crouch, J. E. Marsden, *Nonholonomic Mechanics and Control*, Springer-Verlag NY, 2003.

[2] V. Barbu, I. Lasiecka, D. Tiba, C. Varsan, *Analysis and Optimization of Differential Systems*, IFIP TC7/WG7.2 International Working Conference on Analysis and Optimization of Differential Systems, September 10-14, 2002, Constanta, Romania, Kluwer Academic Publishers, 2003.

[3] G. Barles, A. Tourin, *Commutation properties of semigroups for first-order Hamilton-Jacobi equations and application to multi-time equations*, Indiana University Mathematics Journal, 50, 4 (2001), 1523-1544.

[4] F. Cardin, C. Viterbo, *Commuting Hamiltonians and Hamilto-Jacobi multi-time equations*, INTERNET, 2007.

[5] M. Motta, F. Rampazo, *Nonsmooth multi-time Hamilton-Jacobi systems*, Indiana University Mathematics Journal, 55, 5 (2006), 1573-1614.

[6] S. Pickenhain, M. Wagner, *Piecewise continuous controls in Dieudonne-Rashevsky type problems*, JOTA, 127 (2005), pp.145-163.

[7] L. Pontriaguine, V. Boltianski, R. Gamkrelidze, E. Michtchenko, *Théorie Mathématique des Processus Optimaux*, Edition MIR, Moscou, 1974.

[8] D. F. M. Torres, *On the Noether Theorem for Optimal Control*, European Journal of Control (EJC), 8, 1 (2002), 56-63.

[9] C. Udrişte, A. M. Teleman, *Hamiltonian approaches of field theory*, IJMMS, 57 (2004), 3045-3056.

[10] C. Udrişte, *Multi-time maximum principle*, Short Communication at International Congress of Mathematicians, Madrid, August 22-30, 2006; Plenary Lecture at 6-th WSEAS International Conference on Circuits, Systems, Electronics, Control&Signal Processing (CSECS'07) and 12-th WSEAS International Conference on Applied Mathematics, Cairo, Egypt, December 29-31, 2007.

[11] C. Udrişte, I. Ţevy, *Multi-time Euler-Lagrange dynamics*, Proceedings of the 7th WSEAS International Conference on Systems Theory and Scientific Computation (ISTASC'07), Vouliagmeni Beach, Athens, Greece, August 24-26 (2007), 66-71.

[12] C. Udrişte, I. Ţevy, *Multi-Time Euler-Lagrange-Hamilton Theory*, WSEAS Transactions on Mathematics, 6, 6 (2007), 701-709.

[13] C. Udrişte, *Controllability and observability of multitime linear PDE systems*, Proceedings of The Sixth Congress of Romanian Mathematicians, Bucharest, Romania, June 28 - July 4, 2007, vol. 1, 313-319.

[14] C. Udrişte, *Multi-time stochastic control theory*, Selected Topics on Circuits, Systems, Electronics, Control&Signal Processing, Proceedings of the 6-th WSEAS International Conference on Circuits, Systems, Electronics, Control&Signal Processing (CSECS'07), Cairo, Egypt, December 29-31, 2007, 171-176.

[15] C. Udrişte, *Finsler optimal control and Geometric Dynamics*, Mathematics and Computers In Science and Engineering, Proceedings of the American Conference on Applied Mathematics, Cambridge, Massachusetts, 2008, 33-38.

[16] C. Udrişte, *Lagrangians constructed from Hamiltonian systems*, Mathematics a Computers in Business and Economics, Proceedings of the 9th WSEAS International Conference on Mathematics a Computers in Business and Economics(MCBE-08), Bucharest, Romania, June 24-26, 2008, 30-33.

[17] C. Udrişte, *Multitime controllability, observability and bang-bang principle*, Journal of Optimization Theory and Applications, 139, 1(2008), 141-157; DOI 10.1007/s10957-008-9430-2.

[18] C. Udrişte, L. Matei, *Lagrange-Hamilton theories* (in Romanian), Monographs and Textbooks 8, Geometry Balkan Press, Bucharest, 2008.

[19] C. Udrişte, *Simplified multitime maximum principle*, Balkan Journal of Geometry and Its Applications, 14, 1 (2009), 102-119.

[20] A. Pitea, C. Udrişte, S. Mititelu, *PDI&PDE-constrained optimization problems with curvilinear functional quotients as objective vectors*, Balkan Journal of Geometry and Its Applications, 14, 2 (2009), 75-88.

[21] C. Udrişte, *Tzitzeica theory - opportunity for reflection in Mathematics*, Balkan Journal of Geometry and Its Applications, 10, 1 (2005), 110-120.

[22] C. Udrişte, T. Oprea, *H-convex Riemannian submanifolds*, Balkan Journal of Geometry and Its Applications, 13, 2 (2008), 110-120.

[23] M. T. Calapso, C. Udrişte, *Isothermic surfaces as solutions of Calapso PDE*, Balkan Journal of Geometry and Its Applications, 13, 1 (2008), 20-26.

Author's address:

Constantin Udriste
University Politehnica of Bucharest, Faculty of Applied Sciences,
Department of Mathematics-Informatics, Splaiul Independentei 313,
Bucharest 060042, Romania.
E-mail: udriste@mathem.pub.ro, anet.udri@yahoo.com