Clifford-Kähler manifolds

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Abstract. We consider the Clifford-Kähler manifolds defined by means of a representation of the Clifford algebra with three generators, $\mathcal{C}_3 = \mathcal{C}\ell_{03}$, on its $(1,1)$-tensor bundle, compatible with a Riemannian structure having a special group of holonomy. Such manifolds are necessarily Einstein. It is proved that its structural bundle is locally paralelizable if and only if the Ricci tensor vanishes identically.

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1 Introduction

A smooth 8n-dimensional real manifold M equipped with an action of the Clifford algebra $\mathcal{C}\ell_{03}$ on its tangent bundle is called an *almost Cliffordian manifold.*

A *Clifford Kähler manifold* is a Riemannian manifold (M^{8n}, g), whose holonomy group $Hol\,(g)$ is isomorphic to a subgroup of $Op(n)\cdot Op(1) \subset SO(8n)$. Recall that the enlarged Clifford unitary group $Op(n)\cdot Op(1)$ may be presented as the group of $\mathbb{R}$-linear transformations $T : \mathcal{O}^n \to \mathcal{O}^n$ (here $\mathcal{O} = \mathcal{C}\ell_{03}$) of the numerical n-dimensional (right) octonic space $\mathcal{O}^n$ which have the form

$$T : \xi \to \xi' = A\xi q, \quad \xi, \xi' \in \mathcal{O}^n$$

where $A \in Op(n)$ is a Clifford unitary transformation (with respect to the quasi-Hermitian product $\eta \cdot \xi = \frac{1}{2}\sum_\alpha (\overline{\eta}^\alpha \xi^\alpha + \overline{\xi}^\alpha \eta^\alpha))$ and q is a unitary octon which multiplies on the right. Note that for any element $p \in \mathcal{C}\ell_{03}$ one has $T(\xi p) = (T\xi)(p')$ with $p' = \overline{q}pq$; moreover, the Euclidean scalar product is preserved.

In this paper we shall study the Clifford Kähler manifold by using tensor calculus. In order to do this, it is rather convenient to define a Clifford-like manifold as being a manifold which admits a vector subbundle V of the bundle $End\,(TM)$ of the (1,1)-tensors, having some special properties: V is 6-dimensional as a vector bundle and admits an algebraic structure which is closely connected with Clifford algebra $\mathcal{C}\ell_{03}$.

In Section 2 we recall some notions and results on the Clifford algebra $\mathcal{C}\ell_{03}$. §3 is devoted to proving some formulae required in the last section for proving the main results of this paper.

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Manifolds, mappings and geometric objects under consideration in this paper are supposed to be of class C^∞. Further, all manifold in use will be paracompact.

2 Clifford algebra $\mathcal{C}\ell_{03}$

Recall that $\mathcal{C}\ell_{03}$ denotes the Clifford algebra with three generators $\{e_1, e_2, e_3\}$. It is a real unital associative 8-dimensional algebra for which there exists a special basis $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ such that

$$\begin{array}{ll} e_0 e_i = e_i e_0 = e_i, & i = 0, 1, ..., 7, \\ e_i^2 = -e_0, \ \ e_7^2 = e_0, & i = 1, 2, ..., 6, \\ e_i e_j + e_j e_i = 0, & i \neq j, \ \ i, j = 1, 2, ..., 6, \ \ i + j \neq 7, \\ e_i e_j = e_j e_i, & i = 0, 1, ..., 7, \ \ i \neq j, \ \ i + j = 7, \\ e_1 e_2 = e_4, \ \ e_1 e_3 = e_5, & e_2 e_3 = e_6, \ \ e_1 e_6 = e_7. \end{array}$$

For our comfort, we denote $\mathcal{O} = \mathcal{C}\ell_{03}$ and name their elements *octons*. The before introduced basis $\mathcal{B} = (e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ is called the *canonical* (or, *natural*) basis of $\mathcal{O}$. In [7], it was proved that the center of $\mathcal{O}$ is $\mathcal{C}(\mathcal{O}) = \mathbb{R}e_0 \oplus \mathbb{R}e_7$. It must be remarked that $\mathcal{C}(\mathcal{O}) \simeq \mathcal{D}$ where $\mathcal{D}$ denotes the real (associative and commutative) algebra of the so-called *double numbers*. Moreover, $\mathcal{O} \cong \mathbb{H} \otimes_{\mathbb{R}} \mathcal{D}$ because every element $a = a_0 e_0 + a_1 e_1 + ... + a_7 e_7 \in \mathcal{O}$ has the form

$$a = (a_0 e_0 + a_7 e_7) e_0 + (a_1 e_0 - a_6 e_7) e_1 + (a_2 e_0 + a_5 e_7) e_2 + (a_3 e_0 - a_4 e_7) e_3; \tag{2.1}$$

consequently, $\mathcal{O}$ is a left $\mathcal{D}$-module. The conjugation of $\mathbb{H}$ suggests us to introduce a conjugation on $\mathcal{O}$ by

$$\overline{a} = (a_0 e_0 + a_7 e_7) e_0 - (a_1 e_0 - a_6 e_7) e_1 - (a_2 e_0 + a_5 e_7) e_2 - (a_3 e_0 - a_4 e_7) e_3,$$

i.e.

$$\overline{a} = a_0 e_0 - a_1 e_1 - ... - a_6 e_6 + a_7 e_7. \tag{2.2}$$

Since

$$a \overline{a} = \left(\sum_{i=0}^{7} a_i^2 \right) e_0 + (a_0 a_7 - a_1 a_6 + a_2 a_5 - a_3 a_4) e_7 \in \mathcal{D}, \tag{2.3}$$

the following two quadratic forms $h_1, h_2 : \mathcal{O} \to \mathbb{R}$ are naturally defined by

$$h_1(a) = \sum_{i=0}^{7} a_i^2 \ , \ \ h_2(a) = a_0 a_7 - a_1 a_6 + a_2 a_5 - a_3 a_4, \quad \forall a \in \mathcal{O}.$$

The linear group preserving both these quadratic forms is isomorphic to **O(4,R)**$\times$ **O(4,R)**.

The presence of a natural conjugation on $\mathcal{O}$ suggests the possibility to define an (quasi-)inner product on it. We define now a *quasi-inner* product on $\mathcal{O}$ by

$$\langle a, b \rangle = \frac{1}{2}(a \cdot \bar{b} + b \cdot \bar{a}) \in \mathcal{D}, \quad \forall a, b \in \mathcal{O}. \tag{2.4}$$

The set $G_{\mathcal{O}} = \mathcal{O} \setminus \{\mathbf{L}_1 \cup \mathbf{L}_2\}$ is consisting only in regular elements and it is a group. The group $G_{\mathcal{O}}$ is the product of two subgroups, namely $G_{\mathcal{O}} = \mathcal{O}(1) \cdot \mathcal{D}^*$, where $\mathcal{O}(1) = \{a \in \mathcal{O} | a \cdot \bar{a} = e_0\}$ and $\mathcal{D}^*$ is the set of all invertible elements from $\mathcal{C}(\mathcal{O}) \cong \mathcal{D}$ ($\mathcal{O}(1)$ and $\mathcal{D}^*$ are normal subgroups of $G_{\mathcal{O}}$ with $\mathcal{O}(1) \cap \mathcal{D}^* = \{\pm e_0, \pm e_7\}$).

Moreover, the $\mathcal{D}$-module $\mathcal{O}^n$ can be endowed with an *quasi-inner product* defined by

$$\begin{aligned} \langle p, q \rangle = \tfrac{1}{2} \textstyle\sum_{i=1}^n \left(p_i \cdot \bar{q}_i + q_i \cdot \bar{p}_i\right) \in \mathcal{D}, \\ \forall p = (p_1, p_2, ..., p_n), q = (q_1, q_2, ..., q_n) \in \mathcal{O}^n. \end{aligned} \tag{2.5}$$

As it is usual, we define the group of "isometries" $Op(n)$ as being the group consisting in all matrices $\sigma \in \mathcal{M}_n(\mathcal{O})$ such that

$$\langle \sigma p, \sigma q \rangle = \langle p, q \rangle, \quad \forall p, q \in \mathcal{O}^n.$$

It is easily to prove that $\mathcal{O}(1)$ can be identified, via an isomorphism, with $Op(1)$.

The Lie algebra $\mathcal{O}^-$ associated to the associative algebra $\mathcal{O}$ (by means of the usual bracket) is isomorphic to $su(2) \oplus su(2) \oplus \mathcal{D}^- \cong sp(1) \oplus sp(1) \oplus \mathcal{D}^-$.

It is proved in [3] that $GL_n(\mathcal{O})$ can be isomorphically identified with a subgroup of $GL(8n, \mathbb{R})$, namely

$$GL_n(\mathcal{O}) = \{\tau \in GL(8n, \mathbb{R}) |\ \tau F_i = F_i \tau,\ i = 1, 2, ..., 6\};$$

here F_i $(i = 1, 2, ..., 6)$ is the matrix of linear transformation $\mathcal{O}^n \to \mathcal{O}^n$, $q = (q_1, q_2, ..., q_n) \to q e_i = (q_1 e_i, q_2 e_i, ..., q_n e_i)$ where e_i is an element of canonical basis of $\mathcal{C}\ell_{0,3}$ in an admissible frame of $\mathcal{O}^n$. The Lie algebra $gl_n(\mathcal{O})$ of $GL_n(\mathcal{O})$ can be isomorphically identified with a subalgebra of $gl(8n, \mathbb{R})$, namely

$$gl_n(\mathcal{O}) = \{\theta \in gl(8n, \mathbb{R}) |\ \theta F_i = F_i \theta,\ i = 1, 2, ..., 6\}.$$

On the other hand, the Lie algebra g of $\mathcal{O}p_1 \cdot GL_n(\mathcal{O})$ can be isomorphically identified with a subalgebra of $gl(8n, \mathbb{R})$, namely

$$g = \left\{ \theta \in gl(8n, \mathbb{R}) \left| \begin{array}{lcrrrr} \theta F_1 - F_1 \theta & = & dF_2 & +eF_3 & -bF_4 & -cF_5, \\ \theta F_2 - F_2 \theta & = & -dF_1 & +fF_3 & +aF_4 & -cF_6, \\ \theta F_3 - F_3 \theta & = & -eF_1 & -fF_2 & +aF_5 & +bF_6, \\ \theta F_4 - F_4 \theta & = & bF_1 & -aF_2 & +fF_5 & -eF_6, \\ \theta F_5 - F_5 \theta & = & cF_1 & -aF_3 & -fF_4 & +dF_6, \\ \theta F_6 - F_6 \theta & = & cF_2 & -bF_3 & +eF_4 & -dF_5. \end{array} \right. \right\}.$$

3 Almost Cliffordian manifolds

Let M be a real smooth manifold of dimension m, and let assume that there is a 6-dimensional vector bundle V consisting of tensors of type $(1,1)$ over M such that in any coordinate neighborhood U of M, there exists a local basis $(F_1, F_2, ..., F_6)$ of V whose elements behave under the usual composition like the similar labelled elements of the natural basis of the Clifford algebra $\mathcal{C}\ell_{03}$.

Such a local basis $(F_1, F_2, ..., F_6)$ is called a *canonical basis* of the bundle V in U. Then the bundle V is called an *almost Cliffordinan structure* on M and (M, V)

is called an *almost Cliffordian manifold*. Thus, any almost Cliffordian manifold is necessarily of dimension $m = 8n$.

An almost Cliffordian structure on M is given by a reduction of the structural group of the principal frame bundle of M to $Op(n)\cdot Op(1)$. That is why the tensor fields $(F_1, F_2, ..., F_6)$ can be defined only locally. In the almost Cliffordian manifold (M, V) we take the intersecting coordinate neighborhoods U and U' and let $(F_1, F_2, ..., F_6)$ and $(F_1', F_2', ..., F_6')$ be the canonical local bases of V in U and U', respectively. Then $F_1', F_2', ..., F_6'$ are linear combinations of $F_1, F_2, ..., F_6$ on $U \cap U'$, that is

$$F_i' = \sum_{j=1}^{6} s_{ij} F_j, \quad i = 1, 2, ..., 6, \tag{3.1}$$

where s_{ij} $(i, j = 1, 2, ..., 6)$ are functions defined on $U \cap U'$. The coefficients s_{ij} appearing in (3.1) form an element $s_{UU'} = (s_{ij})$ of a proper subgroup, of dimension 6, of the special orthogonal group $SO(6)$. Consequently, any almost Cliffordian manifold is orientable.

If there exists on (M, V) a global basis $(F_1, F_2, ..., F_6)$, then (M, V) is called an *almost Clifford manifold*; the basis $(F_1, F_2, ..., F_6)$ is named a global canonical basis for V.

Example 3.1. The Clifford module $\mathcal{O}^n$ is naturally identified with $\mathbb{R}^{8n}$. It supplies the simplest example of Clifford manifold. Indeed, if we consider the Cartesian coordinate map with the coordinates $(x_1, x_2, ..., x_n, x_{n+1}, x_{n+2}, ..., x_{2n}, ..., x_{7n+1}, ..., x_{8n})$, then the standard almost Clifford structure on $\mathbb{R}^{8n}$ is defined by means of the three anticommuting operators J_1, J_2, J_3 defined by:

$$\begin{array}{llllll}
J_1 \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{n+i}}, & J_2 \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{2n+i}}, & J_3 \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{3n+i}}, \\
J_1 \frac{\partial}{\partial x_{n+i}} = -\frac{\partial}{\partial x_i}, & J_2 \frac{\partial}{\partial x_{n+i}} = -\frac{\partial}{\partial x_{4n+i}}, & J_3 \frac{\partial}{\partial x_{n+i}} = -\frac{\partial}{\partial x_{5n+i}}, \\
J_1 \frac{\partial}{\partial x_{2n+i}} = \frac{\partial}{\partial x_{4n+i}}, & J_2 \frac{\partial}{\partial x_{2n+i}} = -\frac{\partial}{\partial x_i}, & J_3 \frac{\partial}{\partial x_{2n+i}} = -\frac{\partial}{\partial x_{6n+i}}, \\
J_1 \frac{\partial}{\partial x_{3n+i}} = \frac{\partial}{\partial x_{5n+i}}, & J_2 \frac{\partial}{\partial x_{3n+i}} = \frac{\partial}{\partial x_{6n+i}}, & J_3 \frac{\partial}{\partial x_{3n+i}} = -\frac{\partial}{\partial x_i}, \\
J_1 \frac{\partial}{\partial x_{4n+i}} = -\frac{\partial}{\partial x_{2n+i}}, & J_2 \frac{\partial}{\partial x_{4n+i}} = \frac{\partial}{\partial x_{n+i}}, & J_3 \frac{\partial}{\partial x_{4n+i}} = \frac{\partial}{\partial x_{n+i}}, \\
J_1 \frac{\partial}{\partial x_{5n+i}} = -\frac{\partial}{\partial x_{3n+i}}, & J_2 \frac{\partial}{\partial x_{5n+i}} = -\frac{\partial}{\partial x_{7n+i}}, & J_3 \frac{\partial}{\partial x_{5n+i}} = \frac{\partial}{\partial x_{n+i}}, \\
J_1 \frac{\partial}{\partial x_{6n+i}} = \frac{\partial}{\partial x_{7n+i}}, & J_2 \frac{\partial}{\partial x_{6n+i}} = -\frac{\partial}{\partial x_{3n+i}}, & J_3 \frac{\partial}{\partial x_{6n+i}} = \frac{\partial}{\partial x_{2n+i}}, \\
J_1 \frac{\partial}{\partial x_{7n+i}} = -\frac{\partial}{\partial x_{6n+i}}, & J_2 \frac{\partial}{\partial x_{7n+i}} = \frac{\partial}{\partial x_{5n+i}}, & J_3 \frac{\partial}{\partial x_{7n+i}} = -\frac{\partial}{\partial x_{4n+i}}.
\end{array}$$

Example 3.2. The tangent bundle of any quaternionic-like manifold endowed with a linear connection can be naturally endowed with an almost Cliffordian structure [5].

4 Connections on almost Cliffordian manifolds

An *almost Cliffordian connection* on the almost Cliffordian manifold (M, V) is a linear connection on M which preserves by parallel transport the vector bundle V. This means that if Φ is a cross-section (local or global) of the bundle V, then $\nabla_X \Phi$ is also a cross-section (local or global, respectively) of V, X being an arbitrary vector field of M. The following result was proved in [5].

Proposition 4.1. *The linear connection ∇ on the almost Cliffordian manifold (M, V) is an almost Cliffordian connection on M if and only if the covariant derivatives of the local canonical base are expressed as follows*

$$\left\{\begin{array}{l} \nabla J_1 = \quad \eta_4 \otimes J_2 + \eta_5 \otimes J_3 - \eta_2 \otimes J_4 - \eta_3 \otimes J_5 \\ \nabla J_2 = -\eta_4 \otimes J_1 + \eta_6 \otimes J_3 + \eta_1 \otimes J_4 - \eta_3 \otimes J_6 \\ \nabla J_3 = -\eta_5 \otimes J_1 - \eta_6 \otimes J_2 + \eta_1 \otimes J_5 + \eta_2 \otimes J_6 \\ \nabla J_4 = \quad \eta_2 \otimes J_1 - \eta_1 \otimes J_2 + \eta_6 \otimes J_5 - \eta_5 \otimes J_6 \\ \nabla J_5 = \quad \eta_3 \otimes J_1 - \eta_1 \otimes J_3 - \eta_6 \otimes J_4 + \eta_4 \otimes J_6 \\ \nabla J_6 = \quad \eta_3 \otimes J_2 - \eta_2 \otimes J_3 + \eta_5 \otimes J_4 - \eta_4 \otimes J_5 \end{array}\right. \tag{4.1}$$

where $\eta_1, \eta_2, ..., \eta_6$ are locally 1-forms defined on the domain of $J_1, J_2, ..., J_6$.

Let $\eta_1, \eta_2, ..., \eta_6$ be the 1-forms defined by the connection ∇ with respect to the canonical base $J_1, J_2, ..., J_6$. Then, using the relations (3.1) we get the following change formulae

$$\eta'_a = \sum_{b=1}^{6} s_{ab}\eta_b + \lambda_a, \quad a = 1, 2, ..., 6$$

where λ_a are linear combinations of s_{ab} and ds_{ab}.

Clifford Hermitian manifolds

The triple (M, g, V), where (M, V) is an almost Cliffordian manifold endowed with the Riemannian structure g, is called an *almost Cliffordian Hermitian manifold* or a *metric Cliffordian manifold* if for any canonical basis $J_1, J_2, ..., J_6$ of V in a coordinate neighborhood U, the identities

$$g(J_k X, J_k Y) = g(X, Y) \quad \forall X, Y \in \mathcal{X}(M)$$

hold. Since each J_i $(i = 1, 2, ..., 6)$ is almost Hermitian with respect to g, putting

$$\Phi_i(X, Y) = g(J_i X, Y), \quad \forall X, Y \in \mathcal{X}(M), \quad i = 1, 2, ..., 6, \tag{4.2}$$

one gets 6 local 2-forms on U. However, by means of (3.1), it results that the 4-form

$$\Omega = \Phi_1 \wedge \Phi_1 + \Phi_2 \wedge \Phi_2 + \Phi_3 \wedge \Phi_3 + \Phi_4 \wedge \Phi_4 + \Phi_5 \wedge \Phi_5 + \Phi_6 \wedge \Phi_6 \tag{4.3}$$

is globally defined on M.

By using (3.1) we easily see that

$$\Lambda = J_1 \otimes J_1 + J_2 \otimes J_2 + J_3 \otimes J_3 + J_4 \otimes J_4 + J_5 \otimes J_5 + J_6 \otimes J_6 \tag{4.4}$$

is also a global tensor field of type $(2, 2)$ on M.

If the Levi-Civita-connection $\nabla = \nabla^g$ on (M, g, V) preserves the vector bundle V by parallel transport, then (M, g, V) is called a *Clifford-Kähler manifold*. Consequently, for any Clifford-Kähler manifold, the formulae (4.1) hold (with $\nabla = \nabla^g$).

Actually, a Riemannian manifold is a Clifford-Kähler manifold if and only if its holonomy group is a subgroup of $Op(n) \cdot Op(1)$. Then, one can prove the formulae

$$\nabla \Omega = 0, \quad \nabla \Lambda = 0. \tag{4.5}$$

Conversely, if one of the equations (4.5) hold, then (M, g, V) is a Clifford-Kähler manifold. Thus we get the following result.

Theorem 4.2. *An almost Clifford Hermitian manifold is a Clifford-Kähler manifold if and only if either* $\nabla \Omega = 0$ *or* $\nabla \Lambda = 0$.

5 Some formulae

Let (M, V, g) be a Clifford-Kähler manifold with $dim\ M = 8n$. In a coordinate neighborhood (U, x^h) of M we denote by g_{ij} the components of g and by $\overset{k}{J}{}_i^j$ the components of $\overset{k}{J}$, with $k = 1, 2, ..., 6$ (here and in what follows we shall put the label of any element of a local basis in V above it, i.e. $(\overset{1}{J}, \overset{2}{J}, ..., \overset{6}{J})$ is a canonical local basis of V in U). Then formulae (4.1) become

$$\begin{cases}
\nabla \overset{1}{J}{}_i{}^h = \overset{4}{\eta}_j \overset{2}{J}{}_i{}^h + \overset{5}{\eta}_j \overset{3}{J}{}_i{}^h - \overset{2}{\eta}_j \overset{4}{J}{}_i{}^h - \overset{3}{\eta}_j \overset{5}{J}{}_i{}^h \\
\nabla \overset{2}{J}{}_i{}^h = -\overset{4}{\eta}_j \overset{1}{J}{}_i{}^h + \overset{6}{\eta}_j \overset{3}{J}{}_i{}^h + \overset{1}{\eta}_j \overset{4}{J}{}_i{}^h - \overset{3}{\eta}_j \overset{6}{J}{}_i{}^h \\
\nabla \overset{3}{J}{}_i{}^h = -\overset{5}{\eta}_j \overset{1}{J}{}_i{}^h - \overset{6}{\eta}_j \overset{2}{J}{}_i{}^h + \overset{1}{\eta}_j \overset{5}{J}{}_i{}^h + \overset{2}{\eta}_j \overset{6}{J}{}_i{}^h \\
\nabla \overset{4}{J}{}_i{}^h = \overset{2}{\eta}_j \overset{1}{J}{}_i{}^h - \overset{1}{\eta}_j \overset{2}{J}{}_i{}^h + \overset{6}{\eta}_j \overset{5}{J}{}_i{}^h - \overset{5}{\eta}_j \overset{6}{J}{}_i{}^h \\
\nabla \overset{5}{J}{}_i{}^h = \overset{3}{\eta}_j \overset{1}{J}{}_i{}^h - \overset{1}{\eta}_j \overset{3}{J}{}_i^h - \overset{6}{\eta}_j \overset{4}{J}{}_i{}^h + \overset{4}{\eta}_j \overset{6}{J}{}_i{}^h \\
\nabla \overset{6}{J}{}_i{}^h = \overset{3}{\eta}_j \overset{2}{J}{}_i{}^h - \overset{2}{\eta}_j \overset{3}{J}{}_i{}^h + \overset{4}{\eta}_j \overset{4}{J}{}_i{}^h - \overset{4}{\eta}_j \overset{5}{J}{}_i{}^h ,
\end{cases} \tag{5.1}$$

where $\overset{i}{\eta}_j$ are the components of $\overset{i}{\eta}$ $(i = 1, 2, ..., 6)$ in (U, x^h).

Using Ricci formula, from (5.1) one gets:

$$\begin{cases}
K_{kjs}{}^h \overset{1}{J}{}_i{}^s - K_{kji}{}^s \overset{1}{J}{}_s{}^h = \overset{4}{\omega}_{kj} \overset{2}{J}{}_i{}^h + \overset{5}{\omega}_{kj} \overset{3}{J}{}_i{}^h - \overset{2}{\omega}_{kj} \overset{4}{J}{}_i{}^h - \overset{3}{\omega}_{kj} \overset{5}{J}{}_i{}^h \\
K_{kjs}{}^h \overset{2}{J}{}_i{}^s - K_{kji}{}^s \overset{2}{J}{}_s{}^h = -\overset{4}{\omega}_{kj} \overset{1}{J}{}_i{}^h + \overset{6}{\omega}_{kj} \overset{3}{J}{}_i{}^h + \overset{1}{\omega}_{kj} \overset{4}{J}{}_i{}^h - \overset{3}{\omega}_{kj} \overset{6}{J}{}_i{}^h \\
K_{kjs}{}^h \overset{3}{J}{}_i{}^s - K_{kji}{}^s \overset{3}{J}{}_s{}^h = -\overset{5}{\omega}_{kj} \overset{1}{J}{}_i{}^h - \overset{6}{\omega}_{kj} \overset{2}{J}{}_i{}^h + \overset{1}{\omega}_{kj} \overset{5}{J}{}_i{}^h + \overset{2}{\omega}_{kj} \overset{6}{J}{}_i{}^h \\
K_{kjs}{}^h \overset{4}{J}{}_i{}^s - K_{kji}{}^s \overset{4}{J}{}_s{}^h = \overset{2}{\omega}_{kj} \overset{1}{J}{}_i{}^h - \overset{1}{\omega}_{kj} \overset{2}{J}{}_i{}^h + \overset{6}{\omega}_{kj} \overset{5}{J}{}_i{}^h - \overset{5}{\omega}_{kj} \overset{6}{J}{}_i{}^h \\
K_{kjs}{}^h \overset{5}{J}{}_i{}^s - K_{kji}{}^s \overset{5}{J}{}_s{}^h = \overset{3}{\omega}_{kj} \overset{1}{J}{}_i{}^h - \overset{1}{\omega}_{kj} \overset{3}{J}{}_i{}^h - \overset{6}{\omega}_{kj} \overset{4}{J}{}_i{}^h + \overset{4}{\omega}_{kj} \overset{6}{J}{}_i{}^h \\
K_{kjs}{}^h \overset{6}{J}{}_i{}^s - K_{kji}{}^s \overset{6}{J}{}_s{}^h = \overset{3}{\omega}_{kj} \overset{2}{J}{}_i{}^h - \overset{2}{\omega}_{kj} \overset{3}{J}{}_i{}^h + \overset{4}{\omega}_{kj} \overset{4}{J}{}_i{}^h - \overset{4}{\omega}_{kj} \overset{5}{J}{}_i{}^h ,
\end{cases} \tag{5.2}$$

where $K_{kjs}{}^h$ are the components of the curvature tensor K of the Clifford-Kähler manifold (M, V, g) and $\overset{1}{\omega}, \overset{2}{\omega}, ..., \overset{6}{\omega}$ are defined by

$$(5.3)\qquad \begin{cases} \overset{1}{\omega} = d\overset{1}{\eta} + \overset{2}{\eta} \wedge \overset{6}{\eta} + \overset{3}{\eta} \wedge \overset{5}{\eta} \\ \overset{2}{\omega} = d\overset{2}{\eta} + \overset{4}{\eta} \wedge \overset{1}{\eta} + \overset{6}{\eta} \wedge \overset{3}{\eta} \\ \overset{3}{\omega} = d\overset{3}{\eta} + \overset{5}{\eta} \wedge \overset{1}{\eta} + \overset{6}{\eta} \wedge \overset{2}{\eta} \\ \overset{4}{\omega} = d\overset{4}{\eta} + \overset{1}{\eta} \wedge \overset{2}{\eta} + \overset{5}{\eta} \wedge \overset{6}{\eta} \\ \overset{5}{\omega} = d\overset{5}{\eta} + \overset{1}{\eta} \wedge \overset{3}{\eta} + \overset{6}{\eta} \wedge \overset{4}{\eta} \\ \overset{6}{\omega} = d\overset{6}{\eta} + \overset{2}{\eta} \wedge \overset{3}{\eta} + \overset{4}{\eta} \wedge \overset{5}{\eta}, \end{cases}$$

and

$$(5.4)\qquad \overset{k}{\omega}_{ij} = -\overset{k}{\omega}_{ji} \quad \overset{k}{\omega} = \frac{1}{2}\overset{k}{\omega}_{ij} dx^i \wedge dx^j, \quad k = 1,2,...,6.$$

Thus $\overset{i}{\omega}$, $i = 1,2,...,6$, are local 2-forms defined on U.

From (5.2) we get

$$(5.5)\qquad \begin{cases} [K(X,Y),\overset{1}{J}] = \overset{4}{\omega}(X,Y)\overset{2}{J} + \overset{5}{\omega}(X,Y)\overset{3}{J} - \overset{2}{\omega}(X,Y)\overset{4}{J} - \overset{3}{\omega}(X,Y)\overset{5}{J} \\ [K(X,Y),\overset{2}{J}] = -\overset{4}{\omega}(X,Y)\overset{1}{J} + \overset{6}{\omega}(X,Y)\overset{3}{J} + \overset{1}{\omega}(X,Y)\overset{4}{J} - \overset{3}{\omega}(X,Y)\overset{6}{J} \\ [K(X,Y),\overset{3}{J}] = -\overset{5}{\omega}(X,Y)\overset{1}{J} - \overset{6}{\omega}(X,Y)\overset{2}{J} + \overset{1}{\omega}(X,Y)\overset{5}{J} + \overset{2}{\omega}(X,Y)\overset{6}{J} \\ [K(X,Y),\overset{4}{J}] = \overset{2}{\omega}(X,Y)\overset{1}{J} - \overset{1}{\omega}(X,Y)\overset{2}{J} + \overset{6}{\omega}(X,Y)\overset{5}{J} - \overset{5}{\omega}(X,Y)\overset{6}{J} \\ [K(X,Y),\overset{5}{J}] = \overset{3}{\omega}(X,Y)\overset{1}{J} - \overset{1}{\omega}(X,Y)\overset{3}{J} - \overset{6}{\omega}(X,Y)\overset{4}{J} + \overset{4}{\omega}(X,Y)\overset{6}{J} \\ [K(X,Y),\overset{6}{J}] = \overset{3}{\omega}(X,Y)\overset{2}{J} - \overset{2}{\omega}(X,Y)\overset{3}{J} + \overset{5}{\omega}(X,Y)\overset{4}{J} - \overset{4}{\omega}(X,Y)\overset{5}{J}, \end{cases}$$

in a coordinate neighborhood (U, x^h), X and Y being arbitrary vector fields in M. In another coordinate neighborhood (U', x'^h) we get

$$(5.6)\qquad \begin{cases} [K'(X,Y),\overset{1}{J'}] = \overset{4}{\omega'}(X,Y)\overset{2}{J'} + \overset{5}{\omega'}(X,Y)\overset{3}{J'} - \overset{2}{\omega'}(X,Y)\overset{4}{J'} - \overset{3}{\omega'}(X,Y)\overset{5}{J'} \\ [K'(X,Y),\overset{2}{J'}] = -\overset{4}{\omega'}(X,Y)\overset{1}{J'} + \overset{6}{\omega'}(X,Y)\overset{3}{J'} + \overset{1}{\omega'}(X,Y)\overset{4}{J'} - \overset{3}{\omega'}(X,Y)\overset{6}{J'} \\ [K'(X,Y),\overset{3}{J'}] = -\overset{5}{\omega'}(X,Y)\overset{1}{J'} - \overset{6}{\omega'}(X,Y)\overset{2}{J'} + \overset{1}{\omega'}(X,Y)\overset{5}{J'} + \overset{2}{\omega'}(X,Y)\overset{6}{J'} \\ [K'(X,Y),\overset{4}{J'}] = \overset{2}{\omega'}(X,Y)\overset{1}{J'} - \overset{1}{\omega'}(X,Y)\overset{2}{J'} + \overset{6}{\omega'}(X,Y)\overset{5}{J'} - \overset{5}{\omega'}(X,Y)\overset{6}{J'} \\ [K'(X,Y),\overset{5}{J'}] = \overset{3}{\omega'}(X,Y)\overset{1}{J'} - \overset{1}{\omega'}(X,Y)\overset{3}{J'} - \overset{6}{\omega'}(X,Y)\overset{4}{J'} + \overset{4}{\omega'}(X,Y)\overset{6}{J'} \\ [K'(X,Y),\overset{6}{J'}] = \overset{3}{\omega'}(X,Y)\overset{2}{J'} - \overset{2}{\omega'}(X,Y)\overset{3}{J'} + \overset{5}{\omega'}(X,Y)\overset{4}{J'} - \overset{4}{\omega'}(X,Y)\overset{5}{J'}, \end{cases}$$

where $(\overset{1}{J'}, \overset{2}{J'}, ..., \overset{6}{J'})$ form a canonical local basis of V in U'. Since $S_{U,U'} = (s_{ij}) \in SO(6,\mathbb{R})$, by means of (3.1) we find in $U \cap U'$

$$(5.7)\qquad \overset{i}{\omega'} = s_{i1}\overset{1}{\omega} + s_{i2}\overset{2}{\omega} + ... + s_{i6}\overset{6}{\omega}, \qquad i = 1,2,...,6.$$

Using (5.7) we see that the local 4-form

$$(5.8)\qquad \Sigma = \overset{1}{\omega} \wedge \overset{1}{\omega} + \overset{2}{\omega} \wedge \overset{2}{\omega} + \overset{3}{\omega} \wedge \overset{3}{\omega} + \overset{4}{\omega} \wedge \overset{4}{\omega} + \overset{5}{\omega} \wedge \overset{5}{\omega} + \overset{6}{\omega} \wedge \overset{6}{\omega}$$

determines in M a global 4-form, which is denoted also by Σ. This Σ is, in some sense, the curvature tensor of a linear connection defined in the bundle V by means of (4.1). Now, using (5.3) we can prove

Lemma 5.1. *Let* (M,V,g) *be a Clifford-Kähler 8n-dimensional real manifold. A necessary and sufficient condition for the 4-form* Σ *to vanish on M, is that in each coordinate neighborhood U to exist a canonical local basis* $(\overset{1}{J},\overset{2}{J},...,\overset{6}{J})$ *of V satisfying*

$$\nabla \overset{i}{J}=0, \quad i=1,2,...,6$$

i.e., that the bundle V be locally paralelizable.

Assuming that a Clifford-Kähler manifold satisfies the conditions stated in Lemma 5.1, we see that the functions s_{ij} appearing in (3.1) are constant in a connected component of $U \cap U'$, U and U' being coordinate neighborhoods, if we take $(\overset{1}{J},\overset{2}{J},...,\overset{6}{J})$ such that $\nabla \overset{i}{J}=0$, $i=1,2,...,6$ in each U. In a Clifford-Kähler manifold with M a simply connected manifold and the bundle V is locally paralelizable, then V has a canonical global basis.

Transvecting the 6 equations of (5.2) by $\overset{i}{J}_{hu}=\overset{i}{J}{}^t_h g_{tu}$ $(i=1,2,...,6)$ and changing indices, we find respectively

$$(5.9)\qquad \begin{cases} -K_{kjts}\overset{1}{J}{}^t_i\overset{1}{J}{}^s_h+K_{kjih}= & \overset{4}{\omega}_{kj}\overset{4}{J}_{ih}+\overset{5}{\omega}_{kj}\overset{5}{J}_{ih}+\overset{2}{\omega}_{kj}\overset{2}{J}_{ih}+\overset{3}{\omega}_{kj}\overset{3}{J}_{ih}\\ -K_{kjts}\overset{2}{J}{}^t_i\overset{2}{J}{}^s_h+K_{kjih}= & \overset{4}{\omega}_{kj}\overset{4}{J}_{ih}+\overset{6}{\omega}_{kj}\overset{6}{J}_{ih}+\overset{1}{\omega}_{kj}\overset{1}{J}_{ih}+\overset{3}{\omega}_{kj}\overset{3}{J}_{ih}\\ -K_{kjts}\overset{3}{J}{}^t_i\overset{3}{J}{}^s_h+K_{kjih}= & \overset{5}{\omega}_{kj}\overset{5}{J}_{ih}+\overset{6}{\omega}_{kj}\overset{6}{J}_{ih}+\overset{1}{\omega}_{kj}\overset{1}{J}_{ih}+\overset{2}{\omega}_{kj}\overset{2}{J}_{ih}\\ -K_{kjts}\overset{4}{J}{}^t_i\overset{4}{J}{}^s_h+K_{kjih}= & \overset{2}{\omega}_{kj}\overset{2}{J}_{ih}+\overset{1}{\omega}_{kj}\overset{1}{J}_{ih}+\overset{6}{\omega}_{kj}\overset{6}{J}_{ih}+\overset{5}{\omega}_{kj}\overset{5}{J}_{ih}\\ -K_{kjts}\overset{5}{J}{}^t_i\overset{5}{J}{}^s_h+K_{kjih}= & \overset{3}{\omega}_{kj}\overset{3}{J}_{ih}+\overset{1}{\omega}_{kj}\overset{1}{J}_{ih}+\overset{6}{\omega}_{kj}\overset{6}{J}_{ih}+\overset{4}{\omega}_{kj}\overset{4}{J}_{ih}\\ -K_{kjts}\overset{6}{J}{}^t_i\overset{6}{J}{}^s_h+K_{kjih}= & \overset{3}{\omega}_{kj}\overset{3}{J}_{ih}+\overset{2}{\omega}_{kj}\overset{2}{J}_{ih}+\overset{5}{\omega}_{kj}\overset{5}{J}_{ih}+\overset{4}{\omega}_{kj}\overset{4}{J}_{ih}, \end{cases}$$

where $K_{kjih}=K^s_{kji}g_{sh}$ and $\overset{k}{J}_{ih}=\overset{k}{J}{}^s_i g_{sh}$ $(k=1,2,...,6)$ are the components of $\overset{k}{\Phi}$ defined by (4.2).

Transvecting the second equation (5.9) with $\overset{1}{J}{}^{ih}=g^{ip}\overset{1}{J}{}^h_p$ we get

$$-K_{kjts}\overset{2}{J}{}^t_i\overset{2}{J}{}^s_h\overset{1}{J}{}^{ih}+K_{kjih}\overset{1}{J}{}^{ih}=\overset{4}{\omega}_{kj}\overset{4}{J}_{ih}\overset{1}{J}{}^{ih}+\overset{5}{\omega}_{kj}\overset{5}{J}_{ih}\overset{1}{J}{}^{ih}+\overset{2}{\omega}_{kj}\overset{2}{J}_{ih}\overset{1}{J}{}^{ih}+\overset{3}{\omega}_{kj}\overset{3}{J}_{ih}\overset{1}{J}{}^{ih}.$$

But

$$-K_{kjts}\overset{2}{J}{}^t_i\overset{2}{J}{}^s_h g^{ip}\overset{1}{J}{}^h_p=K_{kjts}\overset{2}{J}{}^t_i g^{ip}\overset{4}{J}{}^s_p=-K_{kjts}\overset{2}{J}{}^t_i g^{sp}\overset{4}{J}{}^i_p=-K_{kjts}\overset{1}{J}{}^t_p g^{sp}=$$

$$=K_{kjts}\overset{1}{J}{}^s_p g^{tp}=K_{kjts}\overset{1}{J}{}^{ts}$$

so that

$$2K_{kjih}\overset{1}{J}^{ih}=8m\overset{1}{\omega}_{kj}\qquad\Longleftrightarrow\qquad\overset{1}{\omega}_{kj}=\frac{1}{4m}K_{kjih}\overset{1}{J}^{ih}.$$

Similarly we obtain

$$\overset{s}{\omega}_{kj}=\frac{1}{4n}K_{kjih}\overset{s}{J}^{ih}\qquad s=1,2,...,6. \tag{5.10}$$

Using (5.10) and identity $K_{kjth}+K_{jtkh}+K_{tkjh}=0$, one gets

$$K_{ktsh}\overset{i}{J}^{ts}=-n\overset{i}{\omega}_{kh}\qquad i=1,2,...,6. \tag{5.11}$$

On the other hand, taking into account of (5.11) and transvecting succesively (5.9) with g^{ij} it results:

$$\begin{cases} K_{kh}=-2n\overset{1}{\omega}_{ks}\overset{1}{J}{}^s_h-\overset{2}{\omega}_{ks}\overset{2}{J}{}^s_h-\overset{3}{\omega}_{ks}\overset{3}{J}{}^s_h-\overset{4}{\omega}_{ks}\overset{4}{J}{}^s_h-\overset{5}{\omega}_{ks}\overset{5}{J}{}^s_h\\ K_{kh}=-\overset{1}{\omega}_{ks}\overset{1}{J}{}^s_h-2n\overset{2}{\omega}_{ks}\overset{2}{J}{}^s_h-\overset{3}{\omega}_{ks}\overset{3}{J}{}^s_h-\overset{4}{\omega}_{ks}\overset{4}{J}{}^s_h-\overset{6}{\omega}_{ks}\overset{6}{J}{}^s_h\\ K_{kh}=-\overset{1}{\omega}_{ks}\overset{1}{J}{}^s_h-\overset{2}{\omega}_{ks}\overset{2}{J}{}^s_h-2n\overset{3}{\omega}_{ks}\overset{3}{J}{}^s_h-\overset{5}{\omega}_{ks}\overset{5}{J}{}^s_h-\overset{6}{\omega}_{ks}\overset{6}{J}{}^s_h\\ K_{kh}=-\overset{1}{\omega}_{ks}\overset{1}{J}{}^s_h-\overset{2}{\omega}_{ks}\overset{2}{J}{}^s_h-2n\overset{4}{\omega}_{ks}\overset{4}{J}{}^s_h-\overset{5}{\omega}_{ks}\overset{5}{J}{}^s_h-\overset{6}{\omega}_{ks}\overset{6}{J}{}^s_h\\ K_{kh}=-\overset{1}{\omega}_{ks}\overset{1}{J}{}^s_h-\overset{3}{\omega}_{ks}\overset{3}{J}{}^s_h-\overset{4}{\omega}_{ks}\overset{4}{J}{}^s_h-2n\overset{5}{\omega}_{ks}\overset{5}{J}{}^s_h-\overset{6}{\omega}_{ks}\overset{6}{J}{}^s_h\\ K_{kh}=-\overset{2}{\omega}_{ks}\overset{2}{J}{}^s_h-\overset{3}{\omega}_{ks}\overset{3}{J}{}^s_h-\overset{4}{\omega}_{ks}\overset{4}{J}{}^s_h-\overset{5}{\omega}_{ks}\overset{5}{J}{}^s_h-2n\overset{6}{\omega}_{ks}\overset{6}{J}{}^s_h. \end{cases} \tag{5.12}$$

Here, $K_{kh}=K_{kjih}g^{ji}$ are the components of the Ricci tensor S of (M,V,g).

From these equations it follows that

$$K_{kh}=-2(n+2)\overset{i}{\omega}_{ks}\overset{i}{J}{}^s_h\qquad i=1,2,...,6. \tag{5.13}$$

Formulae (5.13) give

$$\overset{i}{\omega}_{kh}=\frac{1}{2(n+2)}K_{ks}\overset{i}{J}{}^s_h\qquad i=1,2,...,6. \tag{5.14}$$

Substituting (5.14) in (5.9) we get

$$\begin{cases} -K_{kjts}\overset{1}{J}{}^t_i\overset{1}{J}{}^s_h+K_{kjih}=\frac{1}{2(m+2)}\left(\overset{2}{J}{}^t_j\overset{2}{J}_{ih}+\overset{3}{J}{}^t_j\overset{3}{J}_{ih}+\overset{4}{J}{}^t_j\overset{4}{J}_{ih}+\overset{5}{J}{}^t_j\overset{5}{J}_{ih}\right)\\ -K_{kjts}\overset{2}{J}{}^t_i\overset{2}{J}{}^s_h+K_{kjih}=\frac{1}{2(m+2)}\left(\overset{1}{J}{}^t_j\overset{1}{J}_{ih}+\overset{3}{J}{}^t_j\overset{3}{J}_{ih}+\overset{4}{J}{}^t_j\overset{4}{J}_{ih}+\overset{6}{J}{}^t_j\overset{6}{J}_{ih}\right)\\ -K_{kjts}\overset{3}{J}{}^t_i\overset{3}{J}{}^s_h+K_{kjih}=\frac{1}{2(m+2)}\left(\overset{1}{J}{}^t_j\overset{1}{J}_{ih}+\overset{2}{J}{}^t_j\overset{2}{J}_{ih}+\overset{5}{J}{}^t_j\overset{5}{J}_{ih}+\overset{6}{J}{}^t_j\overset{6}{J}_{ih}\right)\\ -K_{kjts}\overset{4}{J}{}^t_i\overset{4}{J}{}^s_h+K_{kjih}=\frac{1}{2(m+2)}\left(\overset{1}{J}{}^t_j\overset{1}{J}_{ih}+\overset{2}{J}{}^t_j\overset{2}{J}_{ih}+\overset{5}{J}{}^t_j\overset{5}{J}_{ih}+\overset{6}{J}{}^t_j\overset{6}{J}_{ih}\right)\\ -K_{kjts}\overset{5}{J}{}^t_i\overset{5}{J}{}^s_h+K_{kjih}=\frac{1}{2(m+2)}\left(\overset{1}{J}{}^t_j\overset{1}{J}_{ih}+\overset{3}{J}{}^t_j\overset{3}{J}_{ih}+\overset{4}{J}{}^t_j\overset{4}{J}_{ih}+\overset{6}{J}{}^t_j\overset{6}{J}_{ih}\right)\\ -K_{kjts}\overset{6}{J}{}^t_i\overset{6}{J}{}^s_h+K_{kjih}=\frac{1}{2(m+2)}\left(\overset{2}{J}{}^t_j\overset{2}{J}_{ih}+\overset{3}{J}{}^t_j\overset{3}{J}_{ih}+\overset{4}{J}{}^t_j\overset{4}{J}_{ih}+\overset{5}{J}{}^t_j\overset{5}{J}_{ih}\right). \end{cases} \tag{5.15}$$

Since $\overset{i}{\omega}_{ks}$ $(i = 1, 2, ..., 6)$ are all skew-symmetric, using (5.15) we find

$$K_{ts}\overset{i}{J}{}^t_k\overset{i}{J}{}^s_j = K_{kj} \qquad i = 1, 2, ..., 6. \tag{5.16}$$

Using (5.3) we get the identities

$$\begin{cases} d\overset{1}{\omega} = & \overset{4}{\eta}\wedge\overset{2}{\omega} + \overset{5}{\eta}\wedge\overset{3}{\omega} - \overset{2}{\eta}\wedge\overset{4}{\omega} - \overset{3}{\eta}\wedge\overset{5}{\omega} \\ d\overset{2}{\omega} = & -\overset{4}{\eta}\wedge\overset{1}{\omega} + \overset{6}{\eta}\wedge\overset{3}{\omega} + \overset{1}{\eta}\wedge\overset{4}{\omega} - \overset{3}{\eta}\wedge\overset{6}{\omega} \\ d\overset{3}{\omega} = & -\overset{5}{\eta}\wedge\overset{1}{\omega} - \overset{6}{\eta}\wedge\overset{2}{\omega} + \overset{1}{\eta}\wedge\overset{5}{\omega} + \overset{2}{\eta}\wedge\overset{6}{\omega} \\ d\overset{4}{\omega} = & \overset{2}{\eta}\wedge\overset{1}{\omega}1\overset{1}{\eta}\wedge\overset{2}{\omega} + \overset{6}{\eta}\wedge\overset{5}{\omega} - \overset{5}{\eta}\wedge\overset{6}{\omega} \\ d\overset{5}{\omega} = & \overset{5}{\eta}\wedge\overset{1}{\omega} - \overset{1}{\eta}\wedge\overset{3}{\omega} - \overset{6}{\eta}\wedge\overset{4}{\omega} + \overset{4}{\eta}\wedge\overset{6}{\omega} \\ d\overset{6}{\omega} = & \overset{3}{\eta}\wedge\overset{2}{\omega} - \overset{2}{\eta}\wedge\overset{3}{\omega} + \overset{5}{\eta}\wedge\overset{4}{\omega} - \overset{4}{\eta}\wedge\overset{5}{\omega}. \end{cases} \tag{5.17}$$

(5.1) gives

$$\nabla_k\left(K_{js}\overset{1}{J}{}^s_i\right) = (\nabla_k K_{js})\,\overset{1}{J}{}^s_i + K_{js}\left(\overset{4}{\eta}_k\overset{2}{J}{}^s_i + \overset{5}{\eta}_k\overset{3}{J}{}^s_i - \overset{2}{\eta}_k\overset{4}{J}{}^s_i - \overset{3}{\eta}_k\overset{5}{J}{}^s_i\right);$$

taking into account that $(\nabla_k K_{js})\,\overset{1}{J}{}^s_i + (\nabla_k K_{is})\,\overset{1}{J}{}^s_j = 0$, one gets

$$\nabla_k K_{ij} = (\nabla_k K_{ts})\,\overset{1}{J}{}^t_i\overset{1}{J}{}^s_j.$$

The following identity holds:

$$\nabla_k K_{ij} = (\nabla_k K_{ts})\,\overset{p}{J}{}^t_i\overset{p}{J}{}^s_j. \qquad p = 1, 2, ..., 6. \tag{5.18}$$

6 Some Theorems

Lemma 6.1. *For any Clifford-Kähler manifold (M,V,g) the Ricci tensor is parallel.*

Proof. By means of formulae (4.1) and (5.14) and the first identity (5.17) it follows

$$(\nabla_k K_{js})\overset{p}{J}{}^s_i + (\nabla_j K_{is})\overset{p}{J}{}^s_k + (\nabla_i K_{ks})\overset{p}{J}{}^s_j = 0, \;\; p = 1, 2, ..., 6. \tag{6.1}$$

Transvecting (4.1) with $\overset{1}{J}{}^i_h$ one gets

$$(\nabla_k K_{js})\overset{1}{J}{}^s_i\overset{1}{J}{}^i_h + (\nabla_j K_{is})\overset{1}{J}{}^s_k\overset{1}{J}{}^i_h + (\nabla_i K_{ks})\overset{1}{J}{}^s_j\overset{1}{J}{}^i_h = 0,$$

i.e.

$$-\nabla_k K_{jh} + (\nabla_j K_{ts})\overset{1}{J}{}^s_k\overset{1}{J}{}^t_h + (\nabla_t K_{ks})\overset{1}{J}{}^t_h\overset{1}{J}{}^s_j = 0.$$

Substituting in this equation $(\nabla_j K_{ts})\overset{1}{J}{}^t_h\overset{1}{J}{}^s_k = \nabla_j K_{kh}$ (which is a consequence of (5.18)), one obtains

$$-\nabla_k K_{jh} + \nabla_j K_{kh} = -(\nabla_t K_{ks})\overset{1}{J}{}^{t}_{h}\overset{1}{J}{}^{s}_{j}\ .$$

If we substitute in this equation $\nabla_t K_{ks} = (\nabla_t K_{ba})\overset{2}{J}{}^{b}_{k}\overset{2}{J}{}^{a}_{s}$ (which is obtained in a similar way as (5.18)), then we find

$$\nabla_j K_{kh} - \nabla_k K_{jh} = (\nabla_c K_{ba})\overset{1}{J}{}^{c}_{h}\overset{2}{J}{}^{b}_{k}\overset{4}{J}{}^{a}_{j}.$$

Similarly, we get

$$\nabla_j K_{kh} - \nabla_k K_{jh} = -(\nabla_c K_{ba})\overset{2}{J}{}^{c}_{h}\overset{4}{J}{}^{b}_{k}\overset{1}{J}{}^{a}_{j} = -(\nabla_c K_{ba})\overset{4}{J}{}^{c}_{h}\overset{1}{J}{}^{b}_{k}\overset{2}{J}{}^{a}_{j}.$$

Combining the last two equations gives

$$(\nabla_c K_{ba})\overset{1}{J}{}^{c}_{k}\overset{2}{J}{}^{b}_{j}\overset{4}{J}{}^{a}_{i} = (\nabla_c K_{ba})\overset{2}{J}{}^{c}_{k}\overset{4}{J}{}^{b}_{j}\overset{1}{J}{}^{a}_{i} = (\nabla_c K_{ba})\overset{4}{J}{}^{c}_{k}\overset{1}{J}{}^{b}_{j}\overset{2}{J}{}^{a}_{i}\,. \tag{6.2}$$

In particular, one gets

$$(\nabla_c K_{ba})\overset{1}{J}{}^{c}_{k}\overset{2}{J}{}^{b}_{j}\overset{4}{J}{}^{a}_{i} = (\nabla_c K_{ba})\overset{2}{J}{}^{c}_{k}\overset{4}{J}{}^{b}_{j}\overset{1}{J}{}^{a}_{i}$$

from which, by transvecting with $\overset{4}{J}{}^{k}_{r}\overset{1}{J}{}^{j}_{q}\overset{2}{J}{}^{i}_{p}$ it follows

$$-(\nabla_c K_{ba})\overset{2}{J}{}^{c}_{r}\overset{4}{J}{}^{b}_{q}\overset{1}{J}{}^{a}_{p} = (\nabla_c K_{ba})\overset{1}{J}{}^{c}_{r}\overset{2}{J}{}^{b}_{q}\overset{4}{J}{}^{a}_{p}. \tag{6.3}$$

Thus, by combining (6.2) and (6.3) it follows

$$(\nabla_c K_{ba})\overset{1}{J}{}^{c}_{k}\overset{2}{J}{}^{b}_{j}\overset{4}{J}{}^{a}_{i} = 0,$$

which implies

$$\nabla_c K_{ba} = 0. \tag{6.4}$$

□

Lemma 6.1 allow us to prove:

Theorem 6.2. *Any Clifford-Kähler manifold is an Einstein space.*

Theorem 6.3. *The restricted holonomy group of a Clifford-Kähler 8m-dimensional manifold is a subgroup of $Op(m)$ if and only if the Ricci tensor vanishes identically.*

Proof. From (5.10) and (5.14) we get

$$K_{kjih}\overset{p}{J}{}^{ih} = \frac{4m}{2(m+2)}K_{ks}\overset{p}{J}{}^{s}_{j}, \quad p = 1, 2, ..., 6. \tag{6.5}$$

If Ricci tensor vanish identically, then we obtain for successive covariant derivatives of the curvature tensor the identities

$$
(6.6) \qquad \begin{aligned}
&K_{kjih}\overset{p}{J}{}^{ih} = 0, \quad p = 1, 2, ..., 6,\\
&(\nabla_\ell K_{kjih})\overset{p}{J}{}^{ih} = 0, \quad p = 1, 2, ..., 6,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&(\nabla_s ... \nabla_\ell K_{kjih})\overset{p}{J}{}^{ih} = 0, \quad p = 1, 2, ..., 6,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

Therefore, by Ambrose-Singer theorem, the restricted holonomy group of (M, g, V) is a subgroup of $Op(m)$. Conversely, if the restricted holonomy group is a subgroup of $Op(m)$, then (6.6) hold and hence $K_{ij} = 0$ (by taking account of (6.4)). □

Taking into account of Lemma 5.1, we have:

Theorem 6.4. *For a Clifford-Kähler manifold (M,V,g) the bundle V is locally paralelizable if and only if the Ricci tensor vanishes identically.*

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