Stable space-like hypersurfaces with constant scalar curvature in generalized Roberston-Walker spacetimes

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Abstract. In this paper we study stable spacelike hyersurfaces with constant scalar curvature in generalized Roberston-Walker spacetime $\overline{M}^{n+1} = -I \times_\phi F^n$.

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1 Introduction

Hyersurfaces M^n with constant r-mean curvature in Riemannian manifolds or Lorentz manifolds $\overline{M}^{n+1}(c)$ with constant sectional curvature c are critical points of some area functional variations which keep constant a certain volume function. Stable hyersurfaces with constant mean curvature(CMC) (or constant r-mean curvature) in real space form are very interesting geometrical objects that were investigated by many geometricians. Barbosa and do Carmo [2] gave definition of stability of hyersurfaces with constant mean curvature in the Eucildean space R^{n+1} and proved the round spheres are the only compact stable hyersurfaces with CMC in R^{n+1}. Later, Barbosa, do Carmo and Eschenburg [3] extended ambient spaces to Riemannian manifolds and obtained the corresponding results. In [5] Barbosa and Oliker discussed stable spacelile hyersurfaces with CMC in Lorentz manifolds. At the same time, Alencar, do Carmo and Colares [1] investigated stable hyersurfaces with constant scalar curvature in Riemannian manifolds and obtained geodesic sphere is the only stable compact orientable hyersurface in Riemannain spaces. On the other hand, Barbosa and Colares [4] studied compact hyersurfaces without boundary immersed in space forms with constant r-mean curvature. Recently, Liu and Deng [9] also discussed stable spacelike hyersurfaces with constant scalar curvature in de Siter space S_1^{n+1}. Barros, Brasil and Caminha [6] classified strongly stable spacelike hypersurfaces with constant mean curvature whose warping function satisfied a certain convexity condition.

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In this paper we will study stable spacelike hypersurfaces with constant scalar curvature in generalized Roberston-Walker spacetime $\overline{M}^{n+1} = -I \times_\phi F^n$.

2 Preliminaries

Consider F^n an n-dimensional manifold, let I be a 1-dimensional manifold (either a circle or an open interval of R). We denote by $\overline{M}^{n+1} = -I \times_\phi F^n$ the $(n+1)$-dimensional product manifold $I \times F$ endowed with the Lorentzian metric

$$\overline{g} = \langle , \rangle = -dt^2 + f^2(t)\langle , \rangle_M, \tag{2.1}$$

where $f > 0$ is positive function on I, and $\langle , \rangle_M$ stands for the Riemannian metric on F^n. We refer to $-I \times_\phi F^n$ as a generalized Robertson-Walker (GRW) spacetime. In particular, when the Riemannian factor F^n has constant sectional curvature, then $-I \times_\phi F^n$ is classically called a Robertson-Walker (RW) spacetime.

A vector field V on a Lorentz manifold $\overline{M}^{n+1}$ is said to be conformal if

$$\mathcal{L}_V \overline{g} = 2\psi \overline{g}, \tag{2.2}$$

for some smooth function $\psi : \overline{M}^{n+1} \to R$, where $\mathcal{L}$ stands for the Lie derivative of Lorentz metric of $\overline{M}^{n+1}$. The function ψ is called the conformal factor of V. $V \in T\overline{M}$ is conformal if and only if

$$\langle \overline{\nabla}_X V, Y\rangle + \langle \overline{\nabla}_Y V, X\rangle = 2\psi\langle X, Y\rangle, \tag{2.3}$$

for all $X,\ Y \in T(\overline{M})$.

Any Lorentz manifold $\overline{M}^{n+1}$, possessing a globally defined, timelike conformal vector field is said to be a conformally stationary (CS) spacetime.

Let $x : M^n \to \overline{M}^{n+1}$ denote an orientable spacelike hyersurface in the time-oriented Lorentz manifold $\overline{M}^{n+1}$ and N be a globally defined unit normal vector field on M^n. ∇ and $\overline{\nabla}$ denote the Levi-Civita connection of M^n and ambient space $\overline{M}^{n+1}$respectively. $\overline{\mathrm{R}}$ and $\overline{\mathrm{Ric}}$ denote the curvature tensor and Ricci curvature tensor on $\overline{M}^{n+1}$ respectively, which are defined by

$$\overline{\mathrm{R}}(X, Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z, \tag{2.4}$$

and

$$\overline{\mathrm{R}}(W, Z, X, Y) = \langle \overline{\nabla}_X\overline{\nabla}_Y Z, W\rangle - \langle \overline{\nabla}_Y\overline{\nabla}_X Y, W\rangle - \langle \overline{\nabla}_{[X,Y]}Z, W\rangle, \tag{2.5}$$

then

$$\overline{\mathrm{Ric}}(X, Y) = \sum_{k=1}^{n+1} \overline{\mathrm{R}}(e_k, X, e_k, Y), \tag{2.6}$$

where $X, Y, Z, W \in T\overline{M}$, and $\{e_k\}_{k=1}^n$ is a basis of T_pM, $e_{n+1} = N$. In particular we have

$$\overline{\mathrm{Ric}}(N,N)=\sum_{k=1}^{n}\overline{\mathrm{R}}(e_k,N,e_k,N). \tag{2.7}$$

The shape operator A associated to N of M^n, defined by

$$\mathrm{A}=-\overline{\nabla}N \quad (\texttt{i.e} \ \ \mathrm{A}e_k=-\overline{\nabla}_{e_k}N) \tag{2.8}$$

is a self-adjoint linear operator in each tangent space T_pM. Its eigenvalues are the principal curvatures of immersion and are represented by $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n$.The elementary symmetic functions S_r associated to A can be defined, using the characteristic polynomial of A, by

$$\det(t\mathrm{I}-\mathrm{A})=\sum_{k=0}^{n}(-1)^k S_k t^{n-k},$$

where $S_0=1$. If $p\in M$, and $\{e_k\}$ is a basis of T_pM formed by eigenvector of A_p, with corresponding eigenvalues λ_k, one immediately sees that

$$S_r=\sigma_r(\lambda_1,\cdots,\lambda_n),$$

where σ_r is the r-th elementary symmetric polynomial. In particular

$$\|\mathrm{A}\|^2=\sum_k\lambda_k^2=S_1^2-2S_2, \tag{2.9}$$

and

$$\sum_k\lambda_k^3=S_1^3-3S_1S_2+3S_3. \tag{2.10}$$

The r-th classical Newton transformation P_r on M is defined as following

$$\mathrm{P}_0=\mathrm{I},$$

$$\mathrm{P}_r=S_r\mathrm{I}-\mathrm{A}\mathrm{P}_{r-1},\qquad 1\le r\le n.$$

Associated to each Newton transformation P_r of immersion $x: M^n\to\overline{M}^{n+1}$, we have a second order differential operator defined by

$$\mathrm{L}_r(f)=\mathrm{trace}(\mathrm{P}_r\circ\mathrm{Hess}f). \tag{2.11}$$

When $\overline{M}^{n+1}$ has constant sectional curvature, then

$$\mathrm{L}_r(f)=\mathrm{div}(\mathrm{P}_r\nabla f), \tag{2.12}$$

where div stands for the divergence of a vector field on M, it was proved by H. Rosenberg in [12].

Remark 1.1. According (2.11) or (2.12), when $r=0$,

$$\mathrm{L}_0f=\mathrm{div}(\mathrm{P}_0\nabla f)=\triangle f$$

is Laplace operator on M^n, and if $r=1$, then

$$\begin{aligned}\mathrm{L}_1f &= \mathrm{div}[\mathrm{P}_1\circ\mathrm{hess}f]=\mathrm{div}[(S_1\mathrm{I}-\mathrm{A}\mathrm{P}_0)\circ\mathrm{hess}f]\\ &= \sum_{i,j}(S_1\delta_{ij}-h_{ij})f_{ij}\end{aligned} \tag{2.13}$$

become Cheng-Yau's operator $\square$ on M^n, where h_{ij} and f_{ij} denote the component of A and hess$f$ respectively.

3 The variational problem in Lorentz manifolds

Let $x : M^n \to \overline{M}^{n+1}$ denotes an orientable spacelike hyersurface in the time-oriented Lorentz manifold $\overline{M}^{n+1}$ and N be a globally defined unit normal vector field on M^n. A variation of x is a smooth map $X : M^n \times (-\varepsilon, \varepsilon) \to \overline{M}^{n+1}$ satisfying the following conditions:

(1) For $t \in (-\varepsilon, \varepsilon)$, the map $X_t : M^n \to \overline{M}^{n+1}$ given by $X_t(p) = X(t,p)$ is a spaelike immersion such that $X_0 = x$.

(2) $X_t|_{\partial M} = x|_{\partial M}$, for all $t \in (-\varepsilon, \varepsilon)$.

The variational field vector associated the variation X is vector field $X_*(\frac{\partial}{\partial t}) = \frac{\partial X}{\partial t}$. Let $f = \langle \frac{\partial X}{\partial t}, N \rangle$, we have

$$\frac{\partial X}{\partial t} = (\frac{\partial X}{\partial t})^\top - fN, \tag{3.1}$$

where $\top$ denotes tangential components. The balance of volume of the variation X is the function $V : (-\varepsilon, \varepsilon) \to R$ given by

$$V(t) = \int_{M \times [0,t]} X^*(d\overline{M}), \tag{3.2}$$

where $d\overline{M}$ denotes the volume element of $\overline{M}$.

The area functional $A : (-\varepsilon, \varepsilon) \to R$ is given by

$$A(t) = \int_M S_1 dM_t, \tag{3.3}$$

where dM_t denotes the volume element of the metric induced in M by X_t. Then we have the following classical result.

Lemma 3.1. Let $\overline{M}^{n+1}$ be a time-oriented Lorentz manifold and $x : M^n \to \overline{M}^{n+1}$ a spacelike hyersurface. If $X : M^n \times (-\varepsilon, \varepsilon) \to \overline{M}^{n+1}$ is a variation of x, then

(i)

$$\frac{dV(t)}{dt}|_{t=0} = \int_M f dM; \tag{3.4}$$

(ii)

$$\frac{\partial(dM_t)}{\partial t} = (S_1 + \mathrm{div}(\frac{\partial X}{\partial t})^\top) dM_t. \tag{3.5}$$

Proof. For (i) see [3, 9], and for (ii) see [4, 11]. □

Barros, Brasil and Caminha [6] proved the following proposition:

Proposition 3.2. Let $x : M^n \to \overline{M}^{n+1}$ be a spacelike hypersurface of the time-oriented Lorentz manifold $\overline{M}^{n+1}$, and N be a globally defined unit normal vector field on M^n. If $X : M^n \times (-\varepsilon, \varepsilon) \to \overline{M}^{n+1}$ is a variation of x, then

$$\frac{dS_1}{dt} = \triangle f - (\overline{\mathrm{Ric}}(N,N) + \|\mathrm{A}\|^2) f + \langle (\frac{\partial X}{\partial t})^\top, \nabla S_1 \rangle. \tag{3.6}$$

Suppose λ is a constant, and $J : (-\varepsilon, \varepsilon) \to R$ is given by

$$J(t) = A(t) + \lambda V(t), \tag{3.7}$$

J is called the Jacobi functional associated to the variation X. Then we have the following proposition:

Proposition 3.3. Let $x : M^n \to \overline{M}^{n+1}$ be a spacelike hypersurface in the time-oriented Lorentz manifold $\overline{M}^{n+1}$, and N be a globally defined unit normal vector field on M^n. If $X : M^n \times (-\varepsilon, \varepsilon) \to \overline{M}^{n+1}$ is a variation of x, then

$$\frac{dJ(t)}{dt} = \int_M [\operatorname{div}(S_1(\frac{\partial X}{\partial t})^\top) + \Delta f - (\overline{\mathrm{Ric}}(N,N) + \|A\|^2 - S_1^2 - \lambda)f] dM_t. \tag{3.8}$$

In particular, when M^n is closed and $\overline{M}^{n+1}$ has constant sectional curvature c, then

$$\frac{dJ(t)}{dt} = \int_M (2S_2 - cn + \lambda) f dM_t. \tag{3.9}$$

Proof. We can get this result from Lemma 3.1 and Proposition 3.2. In fact,

$$\begin{aligned}
\frac{dJ(t)}{dt} &= \int_M \frac{dS_1}{dt} dM_t + \int_M S_1(S_1 f + \operatorname{div}(\frac{\partial X}{\partial t})^\top) dM_t + \int_M \lambda f dM_t \\
&= \int_M [\langle (\frac{\partial X}{\partial t})^\top, \nabla S_1 \rangle + S_1 \operatorname{div}(\frac{\partial X}{\partial t})^\top + \triangle f \\
&\quad -(\overline{\mathrm{Ric}}(N,N) + \|\mathrm{A}\|^2) f + S_1^2 f + \lambda f] dM_t \\
&= \int_M [\operatorname{div}(S_1(\frac{\partial X}{\partial t})^\top) + \triangle f - (\overline{\mathrm{Ric}}(N,N) + \|\mathrm{A}\|^2 f - S_1^2 f - \lambda) f] dM_t.
\end{aligned}$$

When M^n is closed and $\overline{M}^{n+1}$ has constant sectional curvature c, then we have

$$\int_M \operatorname{div}(S_1(\frac{\partial X}{\partial t})^\top) dM_t = 0,$$

$$\int_M \triangle f dM_t = 0,$$

and $\overline{\mathrm{Ric}}(N,N) = nc$, then using (2.9), we have (3.9). □

Proposition 3.4. Let $x : M^n \to \overline{M}^{n+1}$ is a spacelike hypersurface in Lorentz space form $\overline{M}^{n+1}(c)$ with constant sectional curvature c, and $X : M^n \times (-\varepsilon, \varepsilon) \to \overline{M}^{n+1}$ is a variation of x, then

$$\frac{dS_2}{dt} = \mathrm{L}_1(f) - (S_1 S_2 - 3S_3) f - f(n-1)cS_1 + \langle (\frac{\partial X}{\partial t})^\top, \nabla S_2 \rangle. \tag{3.10}$$

In particular, if S_2 is a constant, then one has

$$\frac{dS_2}{dt} = \mathrm{L}_1(f) - (S_1 S_2 - 3S_3) f - f(n-1)cS_1. \tag{3.11}$$

Proof. According to the proof of proposition 3.2 in [6], we can get

$$\frac{dh_{kk}}{dt} = f_{kk} - cf - h_{kk}^2 f + \langle \nabla h_{kk}, (\frac{\partial X}{\partial t})^{\top} \rangle. \tag{3.12}$$

Using (2.9), we can get

$$\frac{dS_2}{dt} = S_1 \frac{dS_1}{dt} - \sum_k h_{kk} \frac{dh_{kk}}{dt}. \tag{3.13}$$

Substituting (3.6) and (3.12) into (3.13), using (2.9) and (2.10), then we have

$$\begin{aligned}
\frac{dS_2}{dt} &= S_1[\triangle f - (\overline{\mathrm{Ric}}(N,N) + \|\mathrm{A}\|^2)f + \langle (\frac{\partial X}{\partial t})^{\top}, \nabla S_1 \rangle] \\
&\quad - \sum_k h_{kk}[f_{kk} - cf - h_{kk}^2 f + \langle \nabla h_{kk}, (\frac{\partial X}{\partial t})^{\top} \rangle] \\
&= S_1 \triangle f - S_1(nc + S_1^2 - 2S_2)f + S_1 \langle (\frac{\partial X}{\partial t})^{\top}, \nabla S_1 \rangle + \sum_k (S_1 f_{kk} - h_{kk} f_{kk}) \\
&\quad - S_1 \sum_k f_{kk} + cS_1 f + f \sum_k \lambda_k^3 - \frac{1}{2} \langle \nabla (S_1^2 - 2S_2), (\frac{\partial X}{\partial t})^{\top} \rangle \\
&= L_1(f) - (S_1 S_2 - 3S_3)f - f(n-1)cS_1 + \langle (\frac{\partial X}{\partial t})^{\top}, \nabla S_2 \rangle.
\end{aligned}$$

If S_2 is constant, then the last term in the above is equal to zero, so we have (3.11). □

If M has constant normalized scalar curvature R, and we choose

$$\lambda = 2S_2 - nc = n(n-1)(c - R) - nc, \tag{3.14}$$

then λ is a constant too, so we have

Proposition 3.5. Let $x : M^n \to \overline{M}^{n+1}(c)$ is a spacelike hypersurface in the time-oriented Lorentz manifold $\overline{M}^{n+1}(c)$, and $X : M^n \times (-\varepsilon, \varepsilon) \to \overline{M}^{n+1}$ is a variation of x, and S_2 is constant, then

$$\frac{d^2 J(0)}{dt^2}(f) = 2 \int_M [\mathrm{L}_1(f) - (S_1 S_2 - 3S_3)f - f(n-1)cS_1] f dM. \tag{3.15}$$

Proof. Since $\lambda = 2S_2 - nc = n(n-1)(c-R) - nc$, using (3.9) and (3.11), we can get

$$\frac{d^2 J(0)}{dt^2}(f) = 2 \int_M \frac{dS_2(0)}{dt} f dM = 2 \int_M [\mathrm{L}_1(f) - (S_1 S_2 - 3S_3)f - f(n-1)cS_1] f dM.$$

□

Definition 3.6. Suppose $x : M^n \to \overline{M}^{n+1}(c)$ has constant scalar curvature. The immersion x is stable if

$$\frac{d^2 J(0)}{dt^2}(f) = 2 \int_M [\mathrm{L}_1(f) - (S_1 S_2 - 3S_3)f - f(n-1)cS_1] f dM \le 0, \tag{3.16}$$

for all volume-presering variations of x. If M^n is noncompact, x is stable if for every conpact submanifolds $M' \subset M^n$ with boundary, the restriction $x\,|_{M'}$ is stable.

For conformally stationary spacetimes, we have the following proposition.

Proposition 3.7. Let $\overline{M}^{n+1}$ be a conformally stationary Lorentz manifold, with conformal vector V having conformal factor $\psi : \overline{M}^{n+1} \to R$. Suppose $x : M^n \to \overline{M}^{n+1}$ is a spacelike hypersurface in $\overline{M}^{n+1} = I \times_\phi F^n$ with constant sectional curvature c, and N a future-pointing, unit normal vector field globally defined on M^n, $f = \langle V, N\rangle$, then

$$\mathrm{L}_1(f) = (S_1S_2 - 3S_3)f + f(n-1)cS_1 - (n-1)S_1N(\psi) - 2S_2\psi - \langle V^\top, \nabla S_2\rangle. \tag{3.17}$$

In particular, if R is constant, then S_2 is a constant too, so

$$\Box f = \mathrm{L}_1(f) = (S_1S_2 - 3S_3)f + f(n-1)cS_1 - (n-1)S_1N(\psi) - 2S_2\psi. \tag{3.18}$$

Proof. We can choose $\{e_k\}$ as a moving frame on neighborhood $U \subset M$ of p, geodesic at p, and diagonalizing the shape operator A of M at p, with $\mathrm{A}e_k = \lambda_k e_k$, for $1 \le k \le n$. Extend N and e_k $(1 \le k \le n)$ to a neighborhood of p in $\overline{M}$, such that

$$\langle N, e_k\rangle = 0 \quad \text{and} \quad (\overline{\nabla}_N e_k)(p) = 0.$$

Let

$$V = \sum_l \alpha_l e_l - fN,$$

so we have

$$e_k(f) = \langle \overline{\nabla}_{e_k} N, V\rangle + \langle N, \overline{\nabla}_{e_k} V\rangle = -\langle \mathrm{A}e_k, V\rangle + \langle N, \overline{\nabla}_{e_k} V\rangle.$$

Then

$$\begin{aligned} e_ke_k(f) &= -e_k\langle \mathrm{A}e_k, V\rangle + e_k\langle N, \overline{\nabla}_{e_k} V\rangle \\ &= -\langle \overline{\nabla}_{e_k}(\mathrm{A}e_k), V\rangle - 2\langle \mathrm{A}e_k, \overline{\nabla}_{e_k} V\rangle + \langle N, \overline{\nabla}_{e_k}\overline{\nabla}_{e_k} V\rangle. \end{aligned} \tag{3.19}$$

For the first term in (3.19), we have

$$\begin{aligned} \langle \overline{\nabla}_{e_k}(\mathrm{A}e_k), V\rangle &= \langle \overline{\nabla}_{e_k}(\mathrm{A}e_k), \sum_l \alpha_l e_l - fN\rangle \\ &= \sum_j e_k(h_{kj})\langle e_j, \sum_l \alpha_l e_l\rangle + \sum_j h_{kj}\langle \overline{\nabla}_{e_k} e_j, -fN\rangle \\ &= \sum_l \alpha_l e_k(h_{kl}) - \sum_l h_{kl}^2 f. \end{aligned} \tag{3.20}$$

For the second term in (3.19), we have

$$\langle \mathrm{A}e_k, \overline{\nabla}_{e_k} V\rangle = \sum_j h_{kj}\langle e_j, \overline{\nabla}_{e_k} V\rangle = \lambda_k\langle e_k, \overline{\nabla}_{e_k} V\rangle = \lambda_k\psi, \tag{3.21}$$

where in the last equality we use the fact that V is conformal vector having conformal factor ψ, and we have

$$\langle N, \overline{\nabla}_{e_k} V\rangle + \langle e_k, \overline{\nabla}_N V\rangle = 2\psi\langle e_k, N\rangle = 0,$$

then we can get

$$\langle\overline{\nabla}_{e_k} N, \overline{\nabla}_{e_k} V\rangle + \langle N, \overline{\nabla}_{e_k}\overline{\nabla}_{e_k} V\rangle + \langle\overline{\nabla}_{e_k} e_k, \overline{\nabla}_N V\rangle + \langle e_k, \overline{\nabla}_{e_k}\overline{\nabla}_N V\rangle = 0. \tag{3.22}$$

Since

$$\langle\overline{\nabla}_{e_k} N, \overline{\nabla}_{e_k} V\rangle = -\langle \mathrm{A} e_k, \overline{\nabla}_{e_k} V\rangle = -\lambda_k\psi,$$

and

$$\begin{aligned}\langle\overline{\nabla}_N V, \overline{\nabla}_{e_k} e_k\rangle &= -\langle\overline{\nabla}_N V, \langle\overline{\nabla}_{e_k} e_k, N\rangle N\rangle \\ &= -\langle\overline{\nabla}_N V, h_{kk} N\rangle = -h_{kk}\psi\langle N, N\rangle = \lambda_k\psi,\end{aligned}$$

then

$$\langle N, \overline{\nabla}_{e_k}\overline{\nabla}_{e_k} V\rangle = -\langle e_k, \overline{\nabla}_{e_k}\overline{\nabla}_N V\rangle. \tag{3.23}$$

On the other hand, noting that

$$[N, e_k](p) = -\overline{\nabla}_N e_k(p) - \overline{\nabla}_{e_k} N(p) = -\lambda_k e_k(p),$$

so we have

$$\begin{aligned}\langle\overline{\mathrm{R}}(N, e_k)V, e_k\rangle &= \langle\overline{\nabla}_N\overline{\nabla}_{e_k} V - \overline{\nabla}_{e_k}\overline{\nabla}_N V - \overline{\nabla}_{[N,e_k]} V, e_k\rangle_p \\ &= N\langle\overline{\nabla}_{e_k} V, e_k\rangle + \langle N, \overline{\nabla}_{e_k}\overline{\nabla}_{e_k} V\rangle - \langle\overline{\nabla}_{\lambda_k e_k} V, e_k\rangle \\ &= \langle N, \overline{\nabla}_{e_k}\overline{\nabla}_{e_k} V\rangle + N(\psi) - \lambda_k\psi.\end{aligned}$$

For the third term in (3.19), we have

$$\langle N, \overline{\nabla}_{e_k}\overline{\nabla}_{e_k} V\rangle = \langle\overline{\mathrm{R}}(N, e_k)V, e_k\rangle - N(\psi) + \lambda_k\psi. \tag{3.24}$$

Also we have

$$\begin{aligned}\langle\overline{\mathrm{R}}(N, e_k)V, e_k\rangle &= \langle\overline{\mathrm{R}}(N, e_k)(\sum_l \alpha_l e_l - fN), e_k\rangle \\ &= \sum_l \alpha_l\overline{\mathrm{R}}(N, e_k)e_l, e_k\rangle - f\langle\overline{\mathrm{R}}(N, e_k)N, e_k\rangle,\end{aligned}$$

and

$$\langle\overline{\mathrm{R}}(N, e_k)e_l, e_k\rangle = \overline{\mathrm{R}}(e_k, e_l, N, e_k) = e_k(h_{kl}) - e_l(h_{kk}),$$

so (3.24) become

$$\langle N, \overline{\nabla}_{e_k}\overline{\nabla}_{e_k} V\rangle = \sum_l [\alpha_l e_k(h_{kl}) - e_l(h_{kk})] + f\overline{\mathrm{R}}(N, e_k, N, e_k) - N(\psi) + \lambda_k\psi. \tag{3.25}$$

Substituting (3.20), (3.21) and (3.25) into (3.19) we can get

$$
\begin{aligned}
f_{kk} &= e_k e_k(f) = -\sum_l \alpha_l e_k(h_{kl}) + \sum_l h_{kl}^2 f - 2\lambda_k\psi + \\
&\quad \sum_l (\alpha_l e_k(h_{kl}) - \alpha_l e_l(h_{kk})) + f\overline{\mathrm{R}}(N, e_k, N, e_k) - N(\psi) + \lambda_k\psi \\
&= (\sum_l h_{kl}^2 + \overline{\mathrm{R}}(N, e_k, N, e_k))f - \langle V^\top, \nabla h_{kk}\rangle - N(\psi) - \lambda_k\psi. \qquad (3.26)
\end{aligned}
$$

So we have

$$
\triangle f = (\|\mathrm{A}\|^2 + \overline{\mathrm{Ric}}(N,N))f - \langle V^\top, \nabla S_1\rangle - nN(\psi) - S_1\psi, \qquad (3.27)
$$

and

$$
\begin{aligned}
\sum_k \lambda_k f_{kk} &= -2\sum_k \lambda_k^2\psi - \sum_{k,l}\lambda_k e_k(h_{kl}) + \sum_k \lambda_k h_{kl}^2 f + \sum_k \lambda_k\alpha_l(e_k(h_{kl}) \\
&\quad -e_l(h_{kk})) + \sum_k f\lambda_k\overline{\mathrm{R}}(N, e_k, N, e_k) - \sum_k \lambda_k N(\psi) + \sum_k \lambda_k^2\psi \\
&= -\sum_k \lambda_k^2\psi + \sum_k \lambda_k^3 f - \sum_{k,l}\alpha_l\lambda_k e_l(\lambda_k) + fcS_1 - S_1N(\psi). \qquad (3.28)
\end{aligned}
$$

Note that (2.9) and (2.10)

$$
\sum_k \lambda_k^2 = S_1^2 - 2S_2, \qquad \sum_k \lambda_k^3 = S_1^3 - 3S_1S_2 + 3S_3,
$$

substituting (3.27) and (3.28) into (2.13), so we can get

$$
\begin{aligned}
\mathrm{L}_1(f) &= \sum_{i,j}(S_1\delta_{ij} - h_{ij})f_{ij} = \sum_i S_1 f_{ii} - \sum_i h_{ii}f_{ii} \\
&= (S_1S_2 - 3S_3)f + f(n-1)cS_1 - (n-1)S_1N(\psi) - 2S_2\psi - \langle V^\top, \nabla S_2\rangle.
\end{aligned}
$$

□

4 Stable hyersurfaces with constant scalar curvature in GRW

In the following, we will consider the generalized Roberston-Walker spaces $\overline{M}^{n+1} = -I \times_\phi F^n$, let

$$
\pi_I : \overline{M}^{n+1} \to I
$$

denote the canonical projection onto the I. Then the vector field

$$
V = (\phi \circ \pi_I)\frac{\partial}{\partial t}
$$

is conformal, timelike and closed (in the sense that its metrically equavalent 1-form is closed), with conformal factor $\psi = \phi'$. Now we have the follow corollary

Corollary 4.1. If M^n is a closed spacelike hypersurface having constant normalized scalar curvature R in generalized Roberston-Walker spaces $\overline{M}^{n+1} = -I \times_\phi F^n$ with constant sectional curvature c. Let N be a future-pointing unit normal vector field globally defined on M^n. If $V = (\phi \circ \pi_I)\frac{\partial}{\partial t}$ and $f = \langle V, N\rangle$, then

$$\mathrm{L}_1(f) = (S_1S_2 - 3S_3)f + f(n-1)cS_1 + (n-1)S_1\phi''\langle N, \frac{\partial}{\partial t}\rangle - 2S_2\psi. \tag{4.1}$$

Proof. Since we have

$$\overline{\nabla}\phi' = -\langle\overline{\nabla}\phi', \frac{\partial}{\partial t}\rangle\frac{\partial}{\partial t} = -\phi''\frac{\partial}{\partial t}, \tag{4.2}$$

then

$$N(\phi') = \langle N, \overline{\nabla}\phi'\rangle = -\phi''\langle\frac{\partial}{\partial t}, N\rangle. \tag{4.3}$$

Substituting (4.3) into (3.18), we can get (4.1). □

Now we can state and prove our main result:

Theorem 4.2. If M^n a is closed hypersurface, having constant normalized scalar curvature R in generalized Roberston-Walker spaces $\overline{M}^{n+1} = -I \times_\phi F^n$ with constant sectional curvature c. If the warping function ϕ is not constant and satisfies $H\phi'' \geq \max\{(R-c)\phi', 0\}$, and M^n is stable, then
(I) $R = c$ on M, or
(II) M is spacelike slice $M_{t_0} = t_0 \times F^n$, for some $t_0 \in I$, satisfying

$$H\phi'' = (R-c)\phi'.$$

Proof. Using Proposition 3.5 and Corollary 4.1, we can get

$$\begin{aligned} J''(0)(f) &= 2\int_M [(n-1)S_1\phi''\langle N, \frac{\partial}{\partial t}\rangle - 2S_2\phi']f dM \\ &= 2\int_M [(n-1)S_1\phi''\langle N, \frac{\partial}{\partial t}\rangle - n(n-1)(c-R)\phi']\phi\langle N, \frac{\partial}{\partial t}\rangle dM. \end{aligned}$$

Let $\langle N, \frac{\partial}{\partial t}\rangle = -\cosh\theta$, where θ denotes the hyperbolic angle between the timelike verctor fields N and $\frac{\partial}{\partial t}$. Since M^n is stable, so

$$\begin{aligned} 0 &\geq 2 \int_M [(n-1)S_1\phi''\langle N, \frac{\partial}{\partial t}\rangle - n(n-1)(c-R)\phi']\phi\langle N, \frac{\partial}{\partial t}\rangle dM \\ &\geq 2 \int_M (n-1)S_1\phi\phi''\cosh\theta(\cosh\theta - 1)dM \geq 0, \end{aligned}$$

and hence

$$H\phi''(\cosh\theta - 1) = 0 \quad \text{and} \quad H\phi'' = (R-c)\phi' \tag{4.4}$$

holds on M^n. If $R \neq c$ and $\phi' \neq 0$ then $H\phi'' \neq 0$, and $\cosh\theta = 1$, so M is an umbilical leaf satisfying $H\phi'' = (R-c)\phi'$. □

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