Leafwise, transversal and mixed 2-jets of bundles over foliated manifolds

Adelina Manea

Abstract. In [3] we introduced the spaces $J^{l,1}\pi$and $J^{t,1}\pi$ of the leafwise and transversal first order jets of a bundle (E,π,M) where M is a foliated manifold. In this paper, taking $E=J^{l,1}\pi$, respectively $E=J^{t,1}\pi$, and using the theory from [3], we define the leafwise, transversal and mixed second order jets of the bundle π. We also prove some relations between these kinds of second order jets.

M.S.C. 2000: 55R10, 53C12, 58A20.
Key words: fiber bundles, foliated manifold, leafwise, jets.

1 Introduction

Let (E,π,M) be a fiber bundle, $\dim M=m$, $\dim E=m+n$. Weconsider that the indices $i,j,...$ take the values $1,...,m$ and the indices $\alpha,\beta,...$ take the values $1,...,n$. In [6] is defined the 1-jet of a local section as it follows:

Definition 1.1. We said that two local sections $\Phi,\Psi\in\Gamma_x(\pi)$ are *1-equivalent* at x if $\Phi(x)=\Psi(x)$ and if in some adapted coordinate system $\left(x^i,y^\alpha\right)$ around $\Phi(x)$,

$$\frac{\partial\left(y^\alpha\circ\Phi\right)}{\partial x^i}(x)=\frac{\partial\left(y^\alpha\circ\Psi\right)}{\partial x^i}(x)\,, \tag{1.1}$$

for $i=\overline{1,m}$ and $\alpha=\overline{1,n}$. The equivalence class containing Φ is called the **1**-*jet* of the section Φ at x and is denoted $j^1_x\Phi$.

The *1-jet manifold* of π is the set

$$J^1\pi=\left\{j^1_x\Phi\mid x\in M,\Phi\in\Gamma_x(\pi)\right\}.$$

Given an atlas of adapted charts (U,u) on E, where $u=\left(x^i,y^\alpha\right)$, the collection of charts $\left(U^1,u^1\right)$ is a $(m+n+mn)$-dimensional C^∞-atlas on $J^1\pi$, where $U^1=\left\{j^1_x\Phi\in J^1\pi\mid\Phi(x)\in U\right\}$ and the functions

Balkan Journal of Geometry and Its Applications, Vol.12, No.2, 2007, pp. 87-97.

$$u^1 = \left(x^i, y^\alpha, z_i^\alpha\right), \tag{1.2}$$

are defined by $x^i\left(j_x^1\Phi\right) = x^i(x)$, $y^\alpha\left(j_x^1\Phi\right) = y^\alpha(\Phi(x))$, $z_i^\alpha\left(j_x^1\Phi\right) = \frac{\partial(y^\alpha\circ\Phi)}{\partial x^i}(x)$. Moreover, $\left(J^1\pi, \pi_1, M\right)$ and $\left(J^1\pi, \pi_{1,0}, E\right)$ are bundles, where the surjection submersions $\pi_1 : J^1\pi \to M$, $\pi_{1,0} : J^1\pi \to E$ are defined by $\pi_1\left(j_x^1\Phi\right) = x$ and $\pi_{1,0}\left(j_x^1\Phi\right) = \Phi(x)$.

Now, let M be a m-dimensional foliated manifold, with dimension of foliation equal to p. In the following, we shall consider that the indices $a, b, ..$ take the values $1, ..., m-p$ and the indices $u, v, ..$ take the values $m-p+1, ..., m$.

In some adapted local coordinates (x^a, x^u) on M, the usual notation for the adapted basis on M is $\left\{\frac{\delta}{\delta x^a}, \frac{\partial}{\partial x^u}\right\}$, with

$$\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - t_a^u \frac{\partial}{\partial x^u}, \tag{1.3}$$

where t_a^u are local differentiable functions on M given by the orthogonality of structural and transversal bundles of M (see for instance [9]). Let (x^a, x^u, y^α) be the adapted coordinates on E and $\Gamma_x(\pi)$ be the space of local sections of π at $x \in M$.

In [3] there is the following definitions:

Definition 1.2. We say that two local sections $\Phi, \Psi \in \Gamma_x(\pi)$ are *leafwise 1-equivalent* at $x \in M$ if $\Phi(x) = \Psi(x)$ and if, in some adapted coordinate system (x^a, x^u, y^α) around $\Phi(x)$

$$\frac{\partial(y^\alpha \circ \Phi)}{\partial x^u}(x) = \frac{\partial(y^\alpha \circ \Psi)}{\partial x^u}(x), \tag{1.4}$$

for every $u = \overline{m-p+1, m}$. The equivalence class containing Φ is called the *leafwise 1-jet* of Φ at x and it is denoted $j_x^{l,1}\Phi$.

Definition 1.3. We say that two local sections $\Phi, \Psi \in \Gamma_x(\pi)$ are *transversal 1-equivalent* at $x \in M$ if $\Phi(x) = \Psi(x)$ and if, in some adapted coordinate system (x^a, x^u, y^α) around $\Phi(x)$

$$\frac{\delta(y^\alpha \circ \Phi)}{\delta x^a}(x) = \frac{\delta(y^\alpha \circ \Psi)}{\delta x^a}(x), \tag{1.5}$$

for every $a = \overline{1, p}$. The equivalence class containing Φ is called the *transversal 1-jet* of Φ at x and it is denoted $j_x^{t,1}\Phi$.

Obviously, the both notions defined above have geometrical meaning, as we proved in [3]. We also proved that:

Proposition 1.1. *Let $\Phi, \Psi \in \Gamma_x(\pi)$ be two local sections of bundle π. Then the following assertions are equivalent:*

a) $j_x^1\Phi = j_x^1\Psi$;

b) $j_x^{l,1}\Phi = j_x^{l,1}\Psi$ and $j_x^{t,1}\Phi = j_x^{t,1}\Psi$.

The spaces

$$J^{l,1}\pi = \left\{j_x^{l,1}\Phi \mid x \in M, \Phi \in \Gamma_x(\pi)\right\},$$

$$J^{t,1}\pi = \left\{ j_x^{t,1}\Phi \mid x \in M, \Phi \in \Gamma_x(\pi) \right\},$$

of all leafwise, respectively transversal 1-jets of π have differentiable structures with local adapted coordinates defined by:

$$u^{l,1} = (x^a, x^u, y^\alpha, z_u^\alpha), \tag{1.6}$$

with $x^a\left(j_x^{l,1}\Phi\right) = x^a(x)$, $x^u\left(j_x^{l,1}\Phi\right) = x^u(x)$, $y^\alpha\left(j_x^{l,1}\Phi\right) = y^\alpha(\Phi(x))$,

$$z_u^\alpha\left(j_x^{l,1}\Phi\right) = \frac{\partial(y^\alpha \circ \Phi)}{\partial x^u}(x),$$

respectively

$$u^{t,1} = (x^a, x^u, y^\alpha, z_a^\alpha), \tag{1.7}$$

with $x^a\left(j_x^{l,1}\Phi\right) = x^a(x)$, $x^u\left(j_x^{l,1}\Phi\right) = x^u(x)$, $y^\alpha\left(j_x^{l,1}\Phi\right) = y^\alpha(\Phi(x))$,

$$z_a^\alpha\left(j_x^{t,1}\Phi\right) = \frac{\delta(y^\alpha \circ \Phi)}{\delta x^a}(x).$$

Moreover, these manifolds have fiber bundles structures given by the following maps:

$$\pi_1^l : J^{l,1}\pi \to M; \pi_1^l\left(j_x^{l,1}\Phi\right) = x,$$

for every $j_x^{l,1}\Phi \in J^{l,1}\pi$,

$$\pi_1^t : J^{t,1}\pi \to M; \pi_1^t\left(j_x^{t,1}\Phi\right) = x,$$

for $j_x^{t,1}\Phi \in J^{t,1}\pi$.

2 Second order jets

We present the notion of 2-jet of a local section of the bundle (E, π, M) following [6].

Definition 2.1. Two local sections $\Phi, \Psi \in \Gamma_x(\pi)$ are *2-equivalent* at $x \in M$ if $\Phi(x) = \Psi(x)$ and if, in some adapted coordinate system $\left(x^i, y^\alpha\right)$ around $\Phi(x)$ we have

$$\frac{\partial(y^\alpha \circ \Phi)}{\partial x^i}(x) = \frac{\partial(y^\alpha \circ \Psi)}{\partial x^i}(x), \tag{2.1}$$

$$\frac{\partial^2(y^\alpha \circ \Phi)}{\partial x^i \partial x^j}(x) = \frac{\partial^2(y^\alpha \circ \Psi)}{\partial x^i \partial x^j}(x),$$

for every $i, j = \overline{1, m}$ and $\alpha = \overline{1, n}$. The equivalence class containing Φ is called the *2-jet* of Φat x and it is denoted $j_x^2\Phi$.

The *2-jet manifold* of π is the set

$$J^2\pi = \left\{ j_x^2\Phi \mid x \in M, \Phi \in \Gamma_x(\pi) \right\}.$$

Given an atlas of adapted charts (U,u) on E, where $u=\left(x^i,y^\alpha\right)$, the collection of charts $\left(U^2,u^2\right)$ is a $\left(m+n+m\frac{(m+1)}{2}n\right)$-dimensional C^∞-atlas on $J^2\pi$, where $U^2=\left\{j_x^2\Phi\in J^2\pi \mid \Phi(x)\in U\right\}$ and the functions

$$u^2=\left(x^i,y^\alpha,z_i^\alpha,z_{ij}^\alpha\right)_{i\leq j}, \tag{2.2}$$

are defined by $x^i\left(j_x^2\Phi\right)=x^i(x)$, $y^\alpha\left(j_x^2\Phi\right)=y^\alpha(\Phi(x))$, $z_i^\alpha\left(j_x^2\Phi\right)=\frac{\partial(y^\alpha\circ\Phi)}{\partial x^i}(x)$, $z_{ij}^\alpha\left(j_x^2\Phi\right)=\frac{\partial^2(y^\alpha\circ\Phi)}{\partial x^i\partial x^j}(x)$. Moreover, $\left(J^2\pi,\pi_2,M\right)$ and $\left(J^2\pi,\pi_{2,0},E\right)$ are bundles, where the surjection submersions $\pi_2: J^2\pi\to M$, $\pi_{2,0}: J^2\pi\to E$ are defined by $\pi_2\left(j_x^2\Phi\right)=x$ and $\pi_{2,0}\left(j_x^2\Phi\right)=\Phi(x)$.

We can consider also the repeted 1-jet of a local section Φ at a point x: $j_x^1\left(j^1\Phi\right)$. For this one, we have to remark that every section $\Phi\in\Gamma_x(\pi)$ define a section $j^1\Phi\in\Gamma_x(\pi_1)$. It is known from [6] that:

Proposition 2.1. *The map $i_{1,1}: J^2\pi\to J^1\pi$ defined by*

$$i_{1,1}\left(j_x^2\Phi\right)=j_x^1\left(j^1\Phi\right),$$

is an embedding.

Let be $\Phi,\Psi\in\Gamma_x(\pi)$. Taking into account proposition 2.1, from the injectivity of $i_{1,1}$, results:

Remark 2.1. Two local sections $\Phi,\Psi\in\Gamma_x(\pi)$ are *2-equivalent* at $x\in M$ iff the sections $j^1\Phi$, $j^1\Psi\in\Gamma_x(\pi_1)$ are 1-equivalent at x.

The idea of repeted jets will be used in the following sections, where we shall introduce the leafwise, transversal, respectively mixed second order jets as some repeted first order jets.

3 Leafwise second order jets

In this section we consider the following bundles over the foliated manifold M: $\left(J^{l,1}\pi,\pi_1^l,M\right)$, $\left(J^{t,1}\pi,\pi_1^t,M\right)$, $\left(J^1\pi,\pi_1,M\right)$ and we study the leafwise first order jets of some of their sections. We give the following definition:

Definition 3.1. We say that two local sections $\Phi,\Psi\in\Gamma_x(\pi)$ are *leafwise 2-equivalent* at $x\in M$ if the sections $j^{l,1}\Phi$, $j^{l,1}\Psi\in\Gamma_x\left(\pi_1^l\right)$ are leafwise 1-equivalent at x. We denote $j_x^{l,1}\left(j^{l,1}\Phi\right)$ by $j_x^{l,2}\Phi$ and we call it the *leafwise 2-jet* of Φ at x.

Using the definition 1.2 , the above definition could be expresed by:

$$j_x^{l,2}\Phi=j_x^{l,2}\Psi\Leftrightarrow j_x^{l,1}\left(j^{l,1}\Phi\right)=j_x^{l,1}\left(j^{l,1}\Psi\right)\Leftrightarrow$$

$$\Leftrightarrow j_x^{l,1}\Phi=j_x^{l,1}\Psi,\ \frac{\partial\left(y^\alpha\circ j^{l,1}\Phi\right)}{\partial x^u}(x)=\frac{\partial\left(y^\alpha\circ j^{l,1}\Psi\right)}{\partial x^u}(x),$$

$$\frac{\partial\left(z_u^\alpha\circ j^{l,1}\Phi\right)}{\partial x^u}(x)=\frac{\partial\left(z_u^\alpha\circ j^{l,1}\Psi\right)}{\partial x^u}(x),$$

and from the definitions of the adapted coordinates (6), $y^\alpha\circ j^{l,1}\Phi=y^\alpha\circ\Phi$, $z_u^\alpha\circ j^{l,1}\Phi=\frac{\partial(y^\alpha\circ\Phi)}{\partial x^u}$, it results:

Proposition 3.1. *Two local sections $\Phi,\Psi\in\Gamma_x(\pi)$ determine the same leafwise 2-jet at $x\in M$ iff*

$$j_x^{l,1}\Phi=j_x^{l,1}\Psi,\ \frac{\partial^2(y^\alpha\circ\Phi)}{\partial x^u\partial x^v}(x)=\frac{\partial^2(y^\alpha\circ\Psi)}{\partial x^u\partial x^v}(x), \tag{3.1}$$

for all $u,v=\overline{m-p+1,m}$, $u\le v$.

The *leafwise 2-jet manifold* of π is the set

$$J^{l,2}\pi=\left\{j_x^{l,2}\Phi\mid x\in M,\Phi\in\Gamma_x(\pi)\right\}.$$

Given an atlas of adapted charts (U,u) on E, where $u=(x^a,x^u,y^\alpha)$, the collection of charts $\left(U^{l,2},u^{l,2}\right)$ is a $\left(m+n+p\frac{(p+3)}{2}n\right)$-dimensional C^∞-atlas on $J^{l,2}\pi$, where $U^{l,2}=\left\{j_x^{l,2}\Phi\in J^{l,2}\pi\mid\Phi(x)\in U\right\}$ and the functions

$$u^{l,2}=\left(x^a,x^u,y^\alpha,z_u^\alpha,z_{uv}^\alpha\right)_{u\le v}, \tag{3.2}$$

are defined by $x^i\left(j_x^{l,2}\Phi\right)=x^i(x)$, $y^\alpha\left(j_x^{l,2}\Phi\right)=y^\alpha(\Phi(x))$, $z_u^\alpha\left(j_x^{l,2}\Phi\right)=\frac{\partial(y^\alpha\circ\Phi)}{\partial x^u}(x)$,

$$z_{uv}^\alpha\left(j_x^{l,2}\Phi\right)=\frac{\partial^2(y^\alpha\circ\Phi)}{\partial x^u\partial x^v}(x).$$

Moreover, $\left(J^{l,2}\pi,\pi_2^l,M\right)$ and $\left(J^{l,2}\pi,\pi_{2,0}^l,E\right)$ are bundles, where the surjection submersions $\pi_2^l:J^{l,2}\pi\to M$, $\pi_{2,0}^l:J^{l,2}\pi\to E$ are defined by $\pi_2^l\left(j_x^{l,2}\Phi\right)=x$ and $\pi_{2,0}^l\left(j_x^{l,2}\Phi\right)=\Phi(x)$, respectively.

Example 3.1. If E is the trivial bundle $M\times R$, the proposition 3.1. gives exactly the definition [2] of the leafwise 2-jet of a differentiable function on M at a point x.

Remark 3.1. From the definition 3.1, the map

$$i^l:J^{l,2}\pi\to J^{l,1}\pi_1^l,\ i^l\left(j_x^{l,2}\Phi\right)=j_x^{l,1}\left(j^{l,1}\Phi\right),$$

is injective . It is not surjective because not every section of π_1^l is a prolongation of a section of π.

Definition 3.2. Two local sections $\Phi,\Psi\in\Gamma_x(\pi)$ are *l-t 2-equivalent* at $x\in M$ if the sections $j^{t,1}\Phi$, $j^{t,1}\Psi\in\Gamma_x\left(\pi_1^t\right)$ are leafwise 1-equivalent at x. We denote $j_x^{l,1}\left(j^{t,1}\Phi\right)$ by $j_x^{l,t}\Phi$ and we call it the *l-t 2-jet* of Φ at x.

Using the definition 1.2, the above definition could be expresed by:

$$j_x^{l,t}\Phi=j_x^{l,t}\Psi\Leftrightarrow j_x^{l,1}\left(j^{t,1}\Phi\right)=j_x^{l,1}\left(j^{t,1}\Psi\right)\Leftrightarrow$$

$$\Leftrightarrow j_x^{t,1}\Phi = j_x^{t,1}\Psi, \frac{\partial\left(y^\alpha\circ j^{t,1}\Phi\right)}{\partial x^u}(x) = \frac{\partial\left(y^\alpha\circ j^{t,1}\Psi\right)}{\partial x^u}(x),$$

$$\frac{\partial\left(z_a^\alpha\circ j^{t,1}\Phi\right)}{\partial x^u}(x) = \frac{\partial\left(z_a^\alpha\circ j^{t,1}\Psi\right)}{\partial x^u}(x),$$

and from the definitions of the adapted coordinates (7), $y^\alpha\circ j^{t,1}\Phi = y^\alpha\circ\Phi$, $z_a^\alpha\circ j^{t,1}\Phi = \frac{\delta(y^\alpha\circ\Phi)}{\delta x^a}$, it results:

Proposition 3.2. *Two sections $\Phi,\Psi\in\Gamma_x(\pi)$ determine the same l-t 2-jet at $x\in M$ iff*

$$j_x^{t,1}\Phi = j_x^{t,1}\Psi, j_x^{l,1}\Phi = j_x^{l,1}\Psi, \frac{\partial}{\partial x^u}\left(\frac{\delta\left(y^\alpha\circ\Phi\right)}{\delta x^a}\right)(x) = \frac{\partial}{\partial x^u}\left(\frac{\delta\left(y^\alpha\circ\Psi\right)}{\delta x^a}\right)(x), \tag{3.3}$$

for all $u = \overline{m-p+1,m}$ and $a = \overline{1,m-p}$.

From the propositions 1.1 and 3.2, we can say that:

Remark 3.2. Two sections $\Phi,\Psi\in\Gamma_x(\pi)$ determine the same l-t 2-jet at $x\in M$ iff they determine the same 1-jet at x and they satisfy the conditions

$$\frac{\partial}{\partial x^u}\left(\frac{\delta\left(y^\alpha\circ\Phi\right)}{\delta x^a}\right)(x) = \frac{\partial}{\partial x^u}\left(\frac{\delta\left(y^\alpha\circ\Psi\right)}{\delta x^a}\right)(x). \tag{3.4}$$

Let be $\Phi,\Psi\in\Gamma_x(\pi)$. They induce the local sections $j^1\Phi$, $j^1\Psi\in\Gamma_x(\pi_1)$. From the definition 1.2, these local sections determine the same leafwise 1-jet at x iff the following conditions are satisfying:

$$j_x^1\Phi = j_x^1\Psi, \frac{\partial\left(y^\alpha\circ j^1\Phi\right)}{\partial x^u}(x) = \frac{\partial\left(y^\alpha\circ j^1\Psi\right)}{\partial x^u}(x), \frac{\partial\left(z_i^\alpha\circ j^1\Phi\right)}{\partial x^u}(x) = \frac{\partial\left(z_i^\alpha\circ j^1\Psi\right)}{\partial x^u}(x).$$

Taking into account the definitions of local coordinates (2), the above conditions are equivalent with the following ones:

$$j_x^1\Phi = j_x^1\Psi, \frac{\partial\left(y^\alpha\circ\Phi\right)}{\partial x^u}(x) = \frac{\partial\left(y^\alpha\circ\Psi\right)}{\partial x^u}(x), \frac{\partial^2\left(y^\alpha\circ\Phi\right)}{\partial x^u\partial x^i}(x) = \frac{\partial^2\left(y^\alpha\circ\Psi\right)}{\partial x^u\partial x^i}(x), \tag{3.5}$$

for all $u = \overline{m-p+1,m}$ and $i = \overline{1,m}$.

Considering the particular cases $i = v = \overline{m-p+1,m}$, respectively $i = a = \overline{1,m-p}$ in relations (14), we obtain:

$$\frac{\partial^2\left(y^\alpha\circ\Phi\right)}{\partial x^u\partial x^v}(x) = \frac{\partial^2\left(y^\alpha\circ\Psi\right)}{\partial x^u\partial x^v}(x), \tag{3.6}$$

$$\frac{\partial^2\left(y^\alpha\circ\Phi\right)}{\partial x^u\partial x^a}(x) = \frac{\partial^2\left(y^\alpha\circ\Psi\right)}{\partial x^u\partial x^a}(x). \tag{3.7}$$

Using relation (3) $\frac{\partial}{\partial x^a} = \frac{\delta}{\delta x^a} + t_a^u\frac{\partial}{\partial x^u}$, in (16) and taking into account relations (15), result exactly relations (13). So, we can conclude that the conditions (14) assure the conditions (10), (12) from propositions 3.1 and 3.2. The reverse is also true, hence we have:

Theorem 3.1. *Two local sections* $\Phi, \Psi \in \Gamma_x(\pi)$ *are satisfying* $j_x^{l,1}\left(j^1\Phi\right) = j_x^{l,1}\left(j^1\Psi\right)$ *if and only if they determine the same leafwise 2-jet and the same l-t 2-jet at* x.

4 Transversal second order jets

In this section we study the transversal 1-jets of the sections of the bundles $\left(J^{l,1}\pi, \pi_1^l, M\right)$, $\left(J^{t,1}\pi, \pi_1^t, M\right)$, $\left(J^1\pi, \pi_1, M\right)$. We give the following definition:

Definition 4.1. We say that two local sections $\Phi, \Psi \in \Gamma_x(\pi)$ are *transversal 2-equivalent* at $x \in M$ if the sections $j^{t,1}\Phi$, $j^{t,1}\Psi \in \Gamma_x(\pi_1^t)$ are transversal 1-equivalent at x. We denote $j_x^{t,1}\left(j^{t,1}\Phi\right)$ by $j_x^{t,2}\Phi$ and we call it the *transversal 2-jet* of Φ at x.

Using the definition 1.3 , the above definition could be expresed by:

$$j_x^{t,2}\Phi = j_x^{t,2}\Psi \Leftrightarrow j_x^{t,1}\left(j^{t,1}\Phi\right) = j_x^{t,1}\left(j^{t,1}\Psi\right) \Leftrightarrow$$

$$\Leftrightarrow j_x^{t,1}\Phi = j_x^{t,1}\Psi, \frac{\delta\left(y^\alpha \circ j^{t,1}\Phi\right)}{\delta x^a}(x) = \frac{\delta\left(y^\alpha \circ j^{t,1}\Psi\right)}{\delta x^a}(x),$$

$$\frac{\delta\left(z_b^\alpha \circ j^{t,1}\Phi\right)}{\delta x^a}(x) = \frac{\delta\left(z_b^\alpha \circ j^{t,1}\Psi\right)}{\delta x^a}(x),$$

and from the definitions of the adapted coordinates (7), $y^\alpha \circ j^{t,1}\Phi = y^\alpha \circ \Phi$, $z_a^\alpha \circ j^{t,1}\Phi = \frac{\delta(y^\alpha \circ \Phi)}{\delta x^a}$, it results:

Proposition 4.1. *Two local sections* $\Phi, \Psi \in \Gamma_x(\pi)$ *determine the same transversal 2-jet at* $x \in M$ *iff*

$$j_x^{t,1}\Phi = j_x^{t,1}\Psi, \frac{\delta^2\left(y^\alpha \circ \Phi\right)}{\delta x^a \delta x^b}(x) = \frac{\delta^2\left(y^\alpha \circ \Psi\right)}{\delta x^a \delta x^b}(x), \tag{4.1}$$

for all $a, b = \overline{1, m-p}$.

Remark 4.1. The conditions $\frac{\delta^2(y^\alpha \circ \Phi)}{\delta x^a \delta x^b}(x) = \frac{\delta^2(y^\alpha \circ \Psi)}{\delta x^a \delta x^b}(x)$ does not assure $\frac{\delta^2(y^\alpha \circ \Phi)}{\delta x^b \delta x^a}(x) = \frac{\delta^2(y^\alpha \circ \Psi)}{\delta x^b \delta x^a}(x)$, so it is not sufficiently to demande $a \leq b$. Indeed, we have

$$\left[\frac{\delta}{\delta x^a}, \frac{\delta}{\delta x^b}\right] = \left(\frac{\delta t_b^u}{\delta x^a} - \frac{\delta t_a^u}{\delta x^b}\right)\frac{\partial}{\partial x^u},$$

and we shall need $\frac{\partial(y^\alpha \circ \Phi)}{\partial x^u}(x) = \frac{\partial(y^\alpha \circ \Psi)}{\partial x^u}(x)$, so the condition $j_x^{l,1}\Phi = j_x^{l,1}\Psi$, to obtain in relations (15) $a \leq b$.

The *transversal 2-jet manifold* of π is the set

$$J^{t,2}\pi = \left\{j_x^{t,2}\Phi \mid x \in M, \Phi \in \Gamma_x(\pi)\right\}.$$

Given an atlas of adapted charts (U, u) on E, where $u = (x^a, x^u, y^\alpha)$, the collection of charts $\left(U^{t,2}, u^{t,2}\right)$ is a $(m+n+(m-p+1)(m-p)n)$-dimensional C^∞-atlas on $J^{t,2}\pi$, where $U^{t,2} = \left\{j_x^{t,2}\Phi \in J^{t,2}\pi \mid \Phi(x) \in U\right\}$ and the functions

$$u^{t,2} = \left(x^a, x^u, y^\alpha, z_a^\alpha, z_{ab}^\alpha\right), \tag{4.2}$$

are defined by $x^i\left(j_x^{t,2}\Phi\right) = x^i(x)$, $y^\alpha\left(j_x^{t,2}\Phi\right) = y^\alpha(\Phi(x))$, $z_a^\alpha\left(j_x^{t,2}\Phi\right) = \frac{\delta(y^\alpha\circ\Phi)}{\delta x^u}(x)$, $z_{ab}^\alpha\left(j_x^{t,2}\Phi\right) = \frac{\delta^2(y^\alpha\circ\Phi)}{\delta x^a\delta x^b}(x)$. Moreover, $\left(J^{t,2}\pi, \pi_2^t, M\right)$ and $\left(J^{t,2}\pi, \pi_{2,0}^t, E\right)$ are bundles, where the surjection submersions $\pi_2^t : J^{t,2}\pi \to M$, $\pi_{2,0}^t : J^{t,2}\pi \to E$ are defined by $\pi_2^t\left(j_x^{t,2}\Phi\right) = x$ and $\pi_{2,0}^t\left(j_x^{t,2}\Phi\right) = \Phi(x)$.

Remark 4.2. From the definition 4.1, the map

$$i^t : J^{t,2}\pi \to J^{t,1}\pi_1^t, i^t\left(j_x^{t,2}\Phi\right) = j_x^{t,1}\left(j^{t,1}\Phi\right),$$

is injective. It is not surjective because not every section of π_1^t is a prolongation of a section of π.

Definition 4.2. We say that two local sections $\Phi, \Psi \in \Gamma_x(\pi)$ are *t-l 2-equivalent* at $x \in M$ if the sections $j^{l,1}\Phi$, $j^{l,1}\Psi \in \Gamma_x\left(\pi_1^l\right)$ are transversal 1-equivalent at x. We denote $j_x^{t,1}\left(j^{l,1}\Phi\right)$ by $j_x^{t,l}\Phi$ and we call it the *t-l 2-jet* of Φ at x.

Using the definition 1.3, the above definition could be expresed by:

$$j_x^{t,l}\Phi = j_x^{t,l}\Psi \Leftrightarrow j_x^{t,1}\left(j^{l,1}\Phi\right) = j_x^{t,1}\left(j^{l,1}\Psi\right) \Leftrightarrow$$

$$\Leftrightarrow j_x^{l,1}\Phi = j_x^{l,1}\Psi, \frac{\delta\left(y^\alpha \circ j^{l,1}\Phi\right)}{\delta x^a}(x) = \frac{\delta\left(y^\alpha \circ j^{l,1}\Psi\right)}{\delta x^a}(x),$$

$$\frac{\delta\left(z_u^\alpha \circ j^{l,1}\Phi\right)}{\delta x^a}(x) = \frac{\delta\left(z_u^\alpha \circ j^{l,1}\Psi\right)}{\delta x^a}(x),$$

and from the definitions of the adapted coordinates (6), $y^\alpha \circ j^{l,1}\Phi = y^\alpha \circ \Phi$, $z_u^\alpha \circ j^{l,1}\Phi = \frac{\partial(y^\alpha\circ\Phi)}{\partial x^u}$, it results:

Proposition 4.2. *Two sections $\Phi, \Psi \in \Gamma_x(\pi)$ determine the same t-l 2-jet at $x \in M$ iff*

$$j_x^{t,1}\Phi = j_x^{t,1}\Psi, j_x^{l,1}\Phi = j_x^{l,1}\Psi, \frac{\delta}{\delta x^a}\left(\frac{\partial\left(y^\alpha \circ \Phi\right)}{\partial x^u}\right)(x) = \frac{\delta}{\delta x^a}\left(\frac{\partial\left(y^\alpha \circ \Psi\right)}{\partial x^u}\right)(x), \tag{4.3}$$

for all $u = \overline{m-p+1, m}$ and $a = \overline{1, m-p}$.

From the propositions 1.1 and 4.2, we can say that:

Remark 4.3. Two sections $\Phi, \Psi \in \Gamma_x(\pi)$ determine the same t-l 2-jet at $x \in M$ iff they determine the same 1-jet at x and they satisfy the conditions

$$\frac{\delta}{\delta x^a}\left(\frac{\partial\left(y^\alpha \circ \Phi\right)}{\partial x^u}\right)(x) = \frac{\delta}{\delta x^a}\left(\frac{\partial\left(y^\alpha \circ \Psi\right)}{\partial x^u}\right)(x). \tag{4.4}$$

Proposition 4.3. *Let be $\Phi, \Psi \in \Gamma_x(\pi)$. We have the equvalence*

$$j_x^{t,l}\Phi = j_x^{t,l}\Psi \Leftrightarrow j_x^{l,t}\Phi = j_x^{l,t}\Psi.$$

Proof. The equality $j_x^{t,l}\Phi = j_x^{t,l}\Psi$ is equivalent with the relations (19) from proposition 4.2 and for the equality $j_x^{l,t}\Phi = j_x^{l,t}\Psi$ we have the equivalent conditions (12). We have to prove the equivalence between the relations (13) and (20). Taking into account the relation (3), it is known that

$$\left[\frac{\delta}{\delta x^a}, \frac{\partial}{\partial x^u}\right] = \frac{\partial t_a^v}{\partial x^u}\frac{\partial}{\partial x^v},$$

and $j_x^{l,1}\Phi = j_x^{l,1}\Psi$ assures $\frac{\partial(y^\alpha\circ\Phi)}{\partial x^v}(x) = \frac{\partial(y^\alpha\circ\Psi)}{\partial x^v}(x)$, for all $v = \overline{m-p+1,m}$, which ends the proof.

Remark 4.4. The previous proposition says that the l-t 2-jet of Φ at x contains exactly the same sections like the t-l 2-jet of Φat x. That means that the l-t 2-jet and the t-l 2-jet are equivalent notions, and in the following we shall call it the *mixed 2-jet* of a section. We denote by $j_x^{l,t}\Phi$ the mixed 2-jet of Φ at x.

The *mixed 2-jet manifold* of π is the set

$$J^{l,t}\pi = \left\{j_x^{l,t}\Phi \mid x\in M, \Phi\in\Gamma_x(\pi)\right\}.$$

Given an atlas of adapted charts (U,u) on E, where $u = (x^a, x^u, y^\alpha)$, the collection of charts $\left(U^{l,t}, u^{l,t}\right)$ is a $(m+n+mn+p(m-p)n)$-dimensional C^∞-atlas on $J^{l,t}\pi$, where $U^{l,t} = \left\{j_x^{l,t}\Phi\in J^{l,t}\pi \mid \Phi(x)\in U\right\}$ and the functions

$$u^{l,t} = \left(x^a, x^u, y^\alpha, z_a^\alpha, z_u^\alpha, z_{au}^\alpha\right), \tag{4.5}$$

are defined by $x^i\left(j_x^{l,t}\Phi\right) = x^i(x)$, $y^\alpha\left(j_x^{l,t}\Phi\right) = y^\alpha(\Phi(x))$, $z_a^\alpha\left(j_x^{l,t}\Phi\right) = \frac{\delta(y^\alpha\circ\Phi)}{\delta x^u}(x)$, $z_u^\alpha\left(j_x^{l,t}\Phi\right) = \frac{\partial(y^\alpha\circ\Phi)}{\partial x^u}(x)$ $z_{au}^\alpha\left(j_x^{l,t}\Phi\right) = \frac{\delta}{\delta x^a}\left(\frac{\partial(y^\alpha\circ\Phi)}{\partial x^u}\right)(x)$. Moreover, $\left(J^{l,t}\pi, \pi_2^{l,t}, M\right)$ and $\left(J^{l,t}\pi, \pi_{2,0}^{l,t}, E\right)$ are bundles, where the surjection submersions $\pi_2^{l,t} : J^{l,t}\pi\to M$, $\pi_{2,0}^{l,t} : J^{l,t}\pi\to E$ are defined by $\pi_2^{l,t}\left(j_x^{l,t}\Phi\right) = x$ and $\pi_{2,0}^{l,t}\left(j_x^{l,t}\Phi\right) = \Phi(x)$.

Let be $\Phi,\Psi\in\Gamma_x(\pi)$. They induce the local sections $j^1\Phi$, $j^1\Psi\in\Gamma_x(\pi_1)$. From the definition 1.3, these local sections determine the same transversal 1-jet at x iff the following conditions are satisfying:

$$j_x^1\Phi = j_x^1\Psi, \frac{\delta\left(y^\alpha\circ j^1\Phi\right)}{\delta x^a}(x) = \frac{\delta\left(y^\alpha\circ j^1\Psi\right)}{\delta x^a}(x), \frac{\delta\left(z_i^\alpha\circ j^1\Phi\right)}{\delta x^a}(x) = \frac{\delta\left(z_i^\alpha\circ j^1\Psi\right)}{\delta x^a}(x).$$

Taking into account the definitions of local coordinates (2), the above conditions are equivalent with the following ones:

$$j_x^1\Phi = j_x^1\Psi, \frac{\delta\left(y^\alpha\circ\Phi\right)}{\delta x^a}(x) = \frac{\delta\left(y^\alpha\circ\Psi\right)}{\delta x^a}(x), \frac{\delta}{\delta x^a}\left(\frac{\partial\left(y^\alpha\circ\Phi\right)}{\partial x^i}\right)(x) = \frac{\delta}{\delta x^a}\left(\frac{\partial\left(y^\alpha\circ\Psi\right)}{\partial x^i}\right)(x), \tag{4.6}$$

for all $a = \overline{1,m-p}$ and $i = \overline{1,m}$.

Considering the particular cases $i = u = \overline{m-p+1,m}$, respectively $i = b = \overline{1,m-p}$ in relation (22), we obtain:

$$\text{(4.7)} \qquad \frac{\delta}{\delta x^a}\left(\frac{\partial\left(y^\alpha\circ\Phi\right)}{\partial x^u}\right)(x)=\frac{\delta}{\delta x^a}\left(\frac{\partial\left(y^\alpha\circ\Psi\right)}{\partial x^u}\right)(x)\,,$$

$$\text{(4.8)} \qquad \frac{\delta}{\delta x^a}\left(\frac{\partial\left(y^\alpha\circ\Phi\right)}{\partial x^b}\right)(x)=\frac{\delta}{\delta x^a}\left(\frac{\partial\left(y^\alpha\circ\Psi\right)}{\partial x^b}\right)(x)\,.$$

Using relation (3) $\frac{\partial}{\partial x^b}=\frac{\delta}{\delta x^b}+t_b^u\frac{\partial}{\partial x^u}$, in (24) and taking into account relations (23), result exactly relations (17). So, we can conclude that the conditions (22) assure the conditions(17), (19) from propositions 4.1 and 4.2. The reverse is also true, hence we have:

Theorem 4.1. *Two local sections $\Phi,\Psi\in\Gamma_x(\pi)$ are satisfying $j_x^{t,1}\left(j^1\Phi\right)=j_x^{t,1}\left(j^1\Psi\right)$ if and only if they determine the same transversal 2-jet and the same mixed 2-jet at x.*

Now we can give the main theorem of this paper:

Theorem 4.2. *Two local sections $\Phi,\Psi\in\Gamma_x(\pi)$ determine the same 2-jet at x if and only if they determine the same leafwise, transversal and mixed 2-jet at x.*

Proof. From the remark 2.1, we have $j_x^2\Phi=j_x^2\Psi\Leftrightarrow j_x^1\left(j^1\Phi\right)=j_x^1\left(j^1\Psi\right)$.
¿From the proposition 1.1, $j_x^1\left(j^1\Phi\right)=j_x^1\left(j^1\Psi\right)\Leftrightarrow j_x^{l,1}\left(j^1\Phi\right)=j_x^{l,1}\left(j^1\Psi\right)$and $j_x^{t,1}\left(j^1\Phi\right)=j_x^{t,1}\left(j^1\Psi\right)$. Now, the theorems 3.1 and 4.1 assure:

$$j_x^{l,1}\left(j^1\Phi\right)=j_x^{l,1}\left(j^1\Psi\right)\Leftrightarrow j_x^{l,2}\Phi=j_x^{l,2}\Psi, j_x^{l,t}\Phi=j_x^{l,t}\Psi$$

$$j_x^{t,1}\left(j^1\Phi\right)=j_x^{t,1}\left(j^1\Psi\right)\Leftrightarrow j_x^{t,2}\Phi=j_x^{t,2}\Psi, j_x^{l,t}\Phi=j_x^{l,t}\Psi$$

and proposition 4.3 ends the proof.
As a consequence, we have:

Theorem 4.3. *The following map:*

$$\varphi: J^2\pi\to J^{l,2}\pi\times J^{l,t}\pi\times J^{t,2}\pi,$$

defined by

$$\varphi\left(j_x^2\Phi\right)=\left(j_x^{l,2}\Phi, j_x^{l,t}\Phi, j_x^{t,2}\Phi\right),$$

for all $j_x^2\Phi\in J^2\pi$, is injective.

References

[1] I. Kolar and P.W. Michor and J. Slovak, *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin Heidelberg, 1993.

[2] A. Manea, *Cohomology of fields of 2-jets on a foliated manifold,* Libertas Mathematica 24 (2004), 95-102.

[3] A. Manea, *First order jets of bundles over a foliated manifold,* Proc. Int. Workshop Diff. Geom., Deva 2005, Modern Trends in Geometry and Topology, 273-281.

[4] R. Miron, *The Geometry of Higher Order Lagrange spaces. Applications to Mechanics and Physics*, Kluwer Acad. Publ., 1996.

[5] P. Molino, *Riemannian Foliations,* Progr.Math., Birkhäuser, Vol 73, 1988.

[6] D.J. Saunders, *The Geometry of Jet Bundles*, London Math. Soc. Lecture Note Series, 142, Cambridge U.P., 1989.

[7] G. Sardanashvily, *Differential geometry of fibre bundles over foliated manifolds*, arXiv: math-ph/010820, 2001.

[8] Ph. Tondeur, *Foliations on Riemannian Manifolds,* Universitext, Springer-Verlag, New-York, 1988.

[9] I. Vaisman, *Varietes riemanniennes feuilletees,* Czechosl. Math. J. 21 (1971), 46-75.

[10] I. Vaisman, *Cohomology and Differential Forms,* Marcel Dekker Inc., New York, 1973.

Author's address:

Adelina Manea
Transilvania University of Brasov, Romania.
e-mail: amanea28@yahoo.com