# On two natural Riemannian metrics on a tube

M.-L. Labbi

**Abstract.** During an operation of surgery on a Riemannian manifold and along a given embedded submanifold, (see [1, 2, 3]), one needs to replace the (old) metric induced by the exponential map on a tubular neighborhood of the submanifold by the Sasakian metric. So a good understanding of the behavior of these two metrics is important, this is our main goal in this paper. In particular, we prove that these two metrics are tangent up to the order one if and only if the submanifold is totally geodesic. In the case where the ambient space is an Euclidean space, we prove that the difference of these two metrics is quadratic in the radius of the tube and depends only on the second fundamental form of the submanifold. Also the case of spherical and hyperbolic space forms are studied.

**M.S.C. 2000**: 53A07, 53B20.
**Key words**: tube, Sasakian metric.

## 1 Statement of the results

Let $(X, g)$ be a smooth Riemannian manifold of dimension $n + p$ and let $M$ be an embedded (compact) $n$-submanifold of $X$. Let

$$T_\epsilon = \{(x, v) : x \in M, v \in N_x M \quad \text{and} \quad g(v, v) < \epsilon^2\}$$

be a tube of radius $\epsilon$ around M, where $N_x M$ denotes the normal space to $M$ at $x$. It is well known that there exists $\epsilon_0 > 0$ such that the exponential map, $\exp : T_\epsilon \to X$, is a diffeomorphism onto its image for all $\epsilon \leq \epsilon_0$. We shall denote by $\exp^* g$ the pull back to $T_\epsilon$ of the metric $g$ on $X$.

The normal sub-bundle $T_\epsilon$ can also be endowed with a second natural metric, namely, the Sasakian metric. It is defined to be the metric $h$ compatible with the normal connection of the normal (sub)bundle such that the natural projection $\pi : (T_\epsilon, h) \to (M, g)$ is a Riemannian submersion.

In this paper we investigate the behavior of these two metrics near the zero section of the normal bundle.

Balkan Journal of Geometry and Its Applications, Vol.12, No.2, 2007, pp. 81-86.

Let $(p, rn)$ be an arbitrary point in $T_\epsilon$, where $r < \epsilon$ and $n$ is a unit normal vector to $M$ at $p$. We shall denote by $A_n$ the shape operator of the submanifold $M$ in the direction of $n$.

**Theorem A.** *Let $R$ denote the Riemann curvature $(0,4)$-tensor of $(X, g)$. Then for $u_1, u_2 \in T_{(p,rn)}T_\epsilon$, we have*

$$\begin{array}{ll} \exp^* g(u_1,u_2) & = h(u_1,u_2) - 2g(A_n\pi_*u_1, \pi_*u_2)r \\ + & \big\{g(A_n\pi_*u_1, A_n\pi_*u_2) + R(\pi_*u_1, n, \pi_*u_2, n) + \frac{2}{3}R(\pi_*u_1, n, Ku_2, n) \\ + & \frac{2}{3}R(\pi_*u_2, n, Ku_1, n) + \frac{1}{3}R(Ku_1, n, Ku_2, n)\big\}r^2 + O(r^3) \end{array}$$

*In particular,*

$$\frac{d}{dr}_{|r=0}\exp^* g = \frac{d}{dr}_{|r=0} h - 2\pi^*(II_n).$$

*Where $II_n(u,v) = g(A_n\pi_*u, v)$ is the second fundamental form of $M$.*

Remark. Note that in [1], at the beginning of the proof of Lemma 2 page 430, it is claimed that the metrics $\exp^* g$ and $h$ are sufficiently close in the $C^2$-topology as $r \to 0$. The same error is also in [2]. But this does not affect the corresponding conclusions in both papers (after minor changes), see [3].
An alternative short way to notice this fact is as follows:
With respect to the metric $h$, the zero section $M \hookrightarrow T_\epsilon$ is totally geodesic (since for a Riemannian submersion the horizontal lift of a geodesic is a geodesic). But on the other side, the zero section $M \hookrightarrow T_\epsilon$ is totally geodesic for the metric $\exp^* g$ if and only if $M$ is totally geodesic in $(X, g)$.

In the case when the ambient space $(X, g)$ is the Euclidean space $R^n$, we prove the following simple formula relating the metrics $\exp^* g$ and $h$:

**Theorem B.** *Let $M$ be an embedded submanifold in the Euclidean space $R^n$, then for $u_1, u_2 \in T_{(p,rn)}T_\epsilon$, we have*

$$\exp^* g(u_1,u_2) = h(u_1,u_2) - 2g(A_n\pi_*u_1, \pi_*u_2)r + g(A_n\pi_*u_1, A_n\pi_*u_2)r^2.$$

A similar result is proved for any space form, as follows:

**Theorem C.** *Let $M$ be an embedded submanifold in a space form $(X, g)$ with curvature $k$, then for $u_1, u_2 \in T_{(p,rn)}T_\epsilon$, we have*

$$\begin{array}{ll} \exp^* g(u_1,u_2) & = \frac{\sin_k^2(r)}{r^2}h(u_1,u_2) - 2\sin_k(r)\cos_k(r)g(A_n\pi_*u_1, u_2) \\ + & \sin_k^2(r)g(A_n\pi_*u_1, A_n\pi_*u_2) + \{\cos_k^2(r) - \frac{\sin_k^2(r)}{r^2}\}g(\pi_*u_1, \pi_*u_2) \\ + & \{\frac{\sin_k(r)}{r} - 1\}^2 g(Ku_1, n)g(Ku_2, n). \end{array}$$

# 2 Preliminaries

## 2.1 The Sasakian Metric on the Normal Bundle

Let $M$ and $(X, g)$ be as above and let $\pi : \nu(M) \to M$ be the normal bundle of the embedding. Using the normal connection $\nabla$ of $\nu(M)$, the tangent bundle $T(\nu(M))$ splits naturally to

$$T(\nu(M)) = \mathcal{V} \oplus \mathcal{H}.$$

Where $\mathcal{V}$ and $\mathcal{H}$ are respectively the vertical and horizontal bundles. Recall that at a given point $(p, v) \in \nu(M)$, we have $\mathcal{V}_{(p,v)} = T_{(p,v)}\pi^{-1}(p)$, that is the tangent to the fiber over $p$. Hence, using parallel displacement in the fiber, we can canonically identify the vertical space at $(p, v)$ with the fiber $\nu_p(M)$. Thus we get a map , called the connection map,

$$K : T(\nu(M)) \to \nu(M)$$

It is the composition of the projection onto the vertical space followed by a parallel displacement in the fiber as above. In particular, we have

$$K(\mathcal{V}_{(p,v)}) = \nu_p(M) \quad \text{and} \quad K(\mathcal{H}_{(p,v)}) = \{0\}.$$

More explicitly, if $u$ is a tangent vector at $t = 0$ to a curve $(p(t), v(t))$ in $\nu(M)$, then

$$Ku = \nabla_{\dot{p}} v(0). \tag{2.1}$$

On the other hand, a tangent vector $u$, as above, is horizontal if and only if $v(t)$ is $\nabla$-parallel along $p(t)$.
The Sasakian metric on $\nu(M)$ is defined by

$$h(u_1, u_2) = g(Ku_1, Ku_2) + g(\pi_* u_1, \pi_* u_2). \tag{2.2}$$

Note that clearly $\pi : (\nu(M), h) \to (M, g)$ is then a Riemannian submersion.

## 2.2 Jacobi Fields and the Exponential Map

Let $r$ be positive and let $n$ be a unit normal vector at $p \in M$. Let $u \in T_{(p,rn)}\nu(M)$, then $u = \frac{d}{dt}_{|t=0}(p(t), rn(t))$. Consider $U(s) = \frac{d}{dt}_{|t=0}(p(t), sn(t))$. It is a vector field along the curve $c(s) = (p, sn)$ such that $U(r) = u$. Next, set

$$Y(s) = \exp_* U(s).$$

Fact: *The vector field $Y(s)$ is a Jacobi field in $(X, g)$ such that*

$$\begin{aligned} Y(o) &= \dot{p}(0) = \pi_*(u) \in T_pM, \\ \tfrac{D}{ds} Y(0) &= \tfrac{1}{r} Ku - A_n(\pi_*(u)). \end{aligned} \tag{2.3}$$

*Where $D$ and $A_n$ denote respectively the Riemannian connection of $(X, g)$ and the shape operator of $M$.*

*Proof:* Let $\xi(s) = \exp_\nu c(s) = \exp_\nu(p, sn)$ be the unit speed geodesic in $(X, g)$ normal to $M$ with $\xi(0) = p \in M$ and $n = \dot{\xi}(0) \in \nu_p(M)$. Remark that

$$Y(s) = \exp_* U(s) = \frac{d}{dt}_{|t=0} \exp_\nu(p(t), sn(t)).$$

The vector field $Y(s)$ is then generated from a variation of geodesics, in $(X, g)$, normal to $M$, namely, $c(t, s) = \exp_\nu(p(t), sn(t))$. Then $Y(s)$ is a Jacobi field along $\xi(s)$.

Furthermore, we have

$$Y(o) = \dot{p}(0) = \pi_*(u) \in T_pM.$$

Also, using (2.1) we get

$$\begin{array}{ll} \frac{D}{ds}Y(0) = & \frac{D}{ds}\big(\frac{d}{dt}{}_{|t=0}\exp_\nu(p(t), sn(t))\big)(0) = \frac{D}{dt}\big(\frac{d}{ds}{}_{|s=0}\exp_\nu(p(t), sn(t))\big)(0) \\ = & \frac{D}{dt}n(t)(0) = \frac{1}{r}\frac{D^\perp}{dt}rn(t)(0) + \frac{D^T}{dt}n(t)(0) \\ = & \frac{1}{r}Ku - A_n(\pi_*(u)). \end{array}$$

This completes the proof. ■

Finally, note the following remarks:

$$\begin{array}{l} \pi_*(U(s)) = Y(0) = \pi_*(u) = \dot{p}(0), \\ K(U(s)) = \nabla_{\dot{p}} sn(t) = \frac{s}{r}\nabla_{\dot{p}} rn(t) = \frac{s}{r}Ku. \end{array}$$

# 3 Proof of Theorems

Let $u_1, u_2 \in T_{(p,rn)}\nu(M)$, then

$$\exp^* g(u_1, u_2) = g(\exp_* u_1, \exp_* u_2) = g(\exp_* U_1(r), \exp_* U_2(r)) = g(Y_1(r), Y_2(r)),$$

where, for $j = 1, 2$, $Y_j(s) = \exp_* U_j(s)$ and $U_j$ is the vector field associated to the vector $u_j$ as above.

## 3.1 Proof of Theorem A

It results from the Jacobi equation that for every $k \geq 0$ and for $j = 1$ or 2, we have

$$D^{k+2}_{\dot{\xi}} Y_j(o) = -\sum_{i=0}^{k} C^k_i (D^{k-i}_{\dot{\xi}} R)(D^i_{\dot{\xi}} Y_j(0), n)n$$

In particular, using (2.3), we have

$$\begin{array}{ll} D^2_{\dot{\xi}}Y_j(o) & = -R(\pi_* u_j, n)n, \\ D^3_{\dot{\xi}}Y_j(o) & = -D_{\dot{\xi}}R(\pi_* u_j, n)n - \frac{1}{r}R(Ku_j, n)n + R(\pi_* u_j, n)n \\ D^4_{\dot{\xi}}Y_j(o) & = -D^2_{\dot{\xi}}R(\pi_* u_j, n)n - \frac{2}{r}D_{\dot{\xi}}R(\pi_* u_j, n)n + 2D_{\dot{\xi}}R(A_n\pi_* u_j, n)n \\ & +R(R(\pi_* u_j, n)n, n)n \end{array}$$

Next, the Taylor expansion of $g(Y_1(r), Y_2(r))$ shows that

$$\begin{array}{ll} \exp^* g(u_1, u_2) & = g(Y_1(r), Y_2(r)) \\ & = g(\pi_* u_1, \pi_* u_2) - 2g(A_n\pi_* u_1, \pi_* u_2)r \\ & +\big\{2R(\pi_* u_1, n, \pi_* u_2, n) + \frac{2}{r^2}g(Ku_1, Ku_2) + 2g(A_n\pi_* u_1, A_n\pi_* u_2)\big\}\frac{r^2}{2!} \\ & +\big\{\frac{4}{r}R(\pi_* u_1, n, Ku_2, n) + \frac{4}{r}R(Ku_1, n, \pi_* u_2, n) + O(1)\big\}\frac{r^3}{3!} \\ + & \big\{\frac{8}{r^2}R(Ku_1, n, Ku_2, n) + O(\frac{1}{r})\big\}\frac{r^4}{4!} + ... \end{array}$$

Consequently, using (2.2) we get

$$\begin{array}{ll} \exp^* g(u_1,u_2) & = h(u_1,u_2) - 2g(A_n\pi_*u_1,\pi_*u_2)r + \big\{g(A_n\pi_*u_1,A_n\pi_*u_2) \\ + & R(\pi_*u_1,n,\pi_*u_2,n) + \frac{2}{3}R(\pi_*u_1,n,Ku_2,n) + \frac{2}{3}R(\pi_*u_2,n,Ku_1,n) \\ + & \frac{1}{3}R(Ku_1,n,Ku_2,n)\big\}r^2 + O(r^3). \end{array}$$

This completes the proof. ∎

## 3.2 Proof of Theorems B and C

Here, we suppose the manifold $(X,g)$ is with constant sectional curvature $k$, then the Jacobi equation for $Y_j$, $j=1$ or 2, becomes

$$Y_j''(s) + kY_j(s) - kg\big(Y_j(s),\dot\xi(s)\big)\dot\xi(s) = 0. \tag{3.1}$$

Next, note that

$$Y_j(s) = g\big(Y_j(s),\dot\xi(s)\big)\dot\xi(s) + Y_j^\perp(s),$$

and it is easy to check that

$$g(Y_j(s),\dot\xi(s)) = g(Y_j(0),\dot\xi(0)) + g(Y_j'(0),\dot\xi(0))s.$$

On the other hand, we also have

$$(Y_j^\perp)''(s) + R\big(Y_j^\perp(s),\dot\xi(s)\big)\dot\xi(s) = 0,$$

then the vector field $Y_j^\perp(s)$ satisfies

$$\begin{array}{ll} (Y_j^\perp)''(s) & +kY_j^\perp(s) = 0, \\ Y_j^\perp(0) & = Y_j(0), \\ (Y_j^\perp)'(0) & = Y_j'(0) - g(Y_j'(0),n)n. \end{array}$$

The solutions of this differential equation are in terms of parallel translation $\tau_s$ along $\xi(s)$ as follows

$$Y_j^\perp(s) = \cos_k(s)\tau_s(Y_j^\perp(0)) + \sin_k(s)\tau_s\big((Y_j^\perp)'(0)\big). \tag{3.2}$$

Where, $\cos_k(s) = \frac{d}{ds}\sin_k(s)$ and

$$\sin_k(s) = \begin{cases} \frac{\sin\sqrt{k}s}{\sqrt{k}} & \text{if } k>0 \\ s & \text{if } k=0 \\ \frac{\sinh\sqrt{|k|}s}{\sqrt{|k|}} & \text{if } k<0 \end{cases} \tag{3.3}$$

Consequently, after using formula (2.3), the Jacobi fields $Y_j$ are explicitly given by

$$\begin{array}{ll} Y_j(s) = \frac{s}{r}g(Ku_j,n)\dot\xi(s) + & \cos_k(s)\tau_s(\pi_*u_j) \\ + & \sin_k(s)\tau_s\big\{\frac{1}{r}Ku_j - A_n\pi_*u_j - g(\frac{1}{r}Ku_j,n)n\big\} \end{array}$$

Finally, a direct computation shows that

$$\begin{array}{ll} \exp^* g(u_1,u_2) & = g(Y_1(r),Y_2(r)) \\ = & \frac{\sin_k^2(r)}{r^2}h(u_1,u_2) - 2\sin_k(r)\cos_k(r)g(A_n\pi_*u_1,u_2) \\ + & \sin_k^2(r)g(A_n\pi_*u_1,A_n\pi_*u_2) + \{\cos_k^2(r) - \frac{\sin_k^2(r)}{r^2}\}g(\pi_*u_1,\pi_*u_2) \\ + & \{\frac{\sin_k(r)}{r} - 1\}^2 g(Ku_1,n)g(Ku_2,n). \end{array}$$

This completes the proof. ∎

## References

[1] M. Gromov, H.B. Lawson, *The classification of simply connected manifolds of positive scalar curvature*, Annals of Math. 111 (1980), 423-434.

[2] M.-L. Labbi, *Stability of the p-curvature positivity under surgeries and manifolds with positive Einstein tensor,* Annals of Global Analysis and Geometry 15 (1997) 299-312.

[3] M.-L. Labbi, *Manifolds with positive second Gauss-Bonnet curvature*, Pacific Journal of Mathematics 227-2 (2006), 295-310.

*Author's address:*

M.-L. Labbi
Department of Mathematics, College of Science,
University of Bahrain, 32038 Isa Town, Kingdom of Bahrain.
e-mail: labbi@sci.uob.bh