CR-submanifolds of Kaehlerian product manifolds

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Abstract. In this paper, the geometry of F-invariant CR-submanifolds of a Kaehlerian product manifold is studied. Fundamental properties of this type submanifolds are investigated such as CR-product, $D^{\perp}$-totally geodesic and mixed geodesic submanifold. Finally, we have researched totally-umbilical F-invariant proper CR-submanifolds and CR-products in a Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$

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1 Introduction

The geometry of CR-submanifolds of a Kaehlerian is an interesting subject which was studied many geometers(see [2],[3],[9]). In particular, the geometry CR-Submanifolds of a Kaehlerian product manifold was studied in [9] by M.H. Shahid. But, he has choosed special the holomorphic distribution D and totally real distribution $D^{\perp}$ in $M = M_1 \times M_2$ such that $D \subset TM_1$ and $D^{\perp} \subset TM_2$. He demostrated CR-submanifold is a Riemannian product manifold, if it is $D^{\perp}$ totally geodesic. Moreover, He had some results which in relation to the sectional and holomorphic curvatures of CR-submanifold and CR-submanifold is D totally geodesic. Finally, necesarry and sufficient conditions are given on a minimal CR-submanifold of a Kaehlerian product manifold to be totally geodesic.

In this paper, necessary and sufficient conditions are given on F-invariant submanifolds of a Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$ to be a CR-submanifold whose distributions haven been taken such that $D \subset T(M_1 \times M_2)$ and $D^{\perp} \subset T(M_1 \times M_2)$. Moreover, we research D, $D^{\perp}$-totally geodesic and mixed-geodesic CR submanifolds in a Kaehlerian product manifold
$M = M_1(c_1) \times M_2(c_2)$. Moreover, we get the equations of Gauss, Codazzi and Ricci to F-invariant proper CR-submanifolds of a Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$. Necessary and sufficient conditions are given on F-invariant CR-submanifolds of a Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$ to be CR-product, totally geodesic and to have semi-flat normal connection.

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2 Preliminaries

Let M be a m-dimensional Riemannian manifold and N be an n-dimensional manifold isometrically immersed in M. Then N becomes a Riemannian submanifold of M with Riemannian metric induced by the Riemannian metric on M. Also we denote the Levi-Civita connections on N and M by ∇ and $\bar{\nabla}$, respectively. Then the Gauss formula is given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

for any $X, Y \in \Gamma(TN)$, where $h : \Gamma(TN) \times \Gamma(TN) \longrightarrow \Gamma(TN^{\perp})$ is the second fundamental form of N in M. Now, for any $X \in \Gamma(TN)$ and $V \in \Gamma(TN^{\perp})$, we denote the tangent part and normal part of $\bar{\nabla}_X V$ by $-A_V X$ and $\nabla^{\perp}_X V$, respectively. Then the Weingarten formula is given by

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.2}$$

where A_V is called the shape operator of N with respect to V, and $\nabla^{\perp}$ denote the operator of the normal connection in $\Gamma(TN^{\perp})$. Moreover, from (2.1) and (2.2) we have

$$g(h(X, Y), V) = g(A_V X, Y), \tag{2.3}$$

for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN^{\perp})$[4].

Definition 2.1. *Let N be a submanifold of any Riemannian manifold M. Then the mean curvature vector field H of N is defined by formula*

$$H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i),$$

where $\{e_i\}$,$1 \leq i \leq n$, is a local orthonormal basis of $\Gamma(TN)$. If the submanifold $\bar{M}$ having one of conditions

$$h = 0, \quad h(X, Y) = g(X, Y)H, \quad g(h(X, Y), H) = \lambda g(X, Y), H = 0, \lambda \in C^{\infty}(M, \mathbb{R}),$$

for any $X, Y \in \Gamma(TN)$, then it is called totally geodesic, totally umbilical, pseudo umbilical and minimal submanifold of M, respectively[4].

The covariant derivative of the second fundamental form h is defined by

$$(\bar{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y), \tag{2.4}$$

for any $X, Y, Z \in \Gamma(TN)$.

For any submanifold N of a Riemannian manifold M, the Gauss and Codazzi equations are respectively given by

$$\begin{aligned} R(X, Y)Z &= R_N(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\bar{\nabla}_X h)(Y, Z) \\ &- (\bar{\nabla}_Y h)(X, Z) \end{aligned} \tag{2.5}$$

and

$$\{R(X,Y)Z\}^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z) \tag{2.6}$$

for any $X, Y, Z \in \Gamma(TN)$, where R and R_N are the Riemannian curvature tensors of M and N, respectively. Also, $\{R(X,Y)Z\}^{\perp}$ denotes normal component of $R(X,Y)Z$.

We recall that N is called curvature-invariant submanifold of Riemannian manifold M, if $R(X,Y)Z \in \Gamma(TN)$, that is, $\{R(X,Y)Z\}^{\perp} = 0$ for any $X, Y, Z \in \Gamma(TN)$[6].

Now, let M be a real differentiable manifold. An almost complex structure on M is a tensor field J of type $(1,1)$ on M such that $J^2 = -I$. M is called an almost complex manifold if it has an almost complex structure.

A Hermitian metric on an almost complex manifold M is a Riemannian metric g satisfying

$$g(JX, JY) = g(X, Y)$$

for all $X, Y \in \Gamma(TM)$. Furthermore, M is called Kaehlerian manifold if the almost complex structure is parallel with respect to $\bar{\nabla}$, that is, we have
$(\bar{\nabla}_X J)Y = 0$ for any $X, Y \in \Gamma(TM)$.

For each plane γ spanned orthonormal vectors X and Y in $\Gamma(TM)$ and for each point in M, we define the sectional curvature $K(\gamma)$ by

$$K(\gamma) = K(X \wedge Y) = g(R(X,Y)Y, X).$$

If $K(\gamma)$ is a constant for all planes γ in $\Gamma(TM)$ and for all points in M, then M is called a space of constant curvature or real space form. We denote by $M(c)$ a real space form of constant sectional curvature c. Then the Riemannian curvature tensor of $M(c)$ is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}, \tag{2.7}$$

for any $X, Y, Z \in \Gamma(TM)$[4].

Now, we consider a plane γ invariant by the almost complex structure J. In this case, we can choose a basis $\{X, JX\}$ in γ, where X is a unit vector in γ. Then the sectional curvature $K(\gamma)$ is denoted by $H(X)$ and it is called holomorphic sectional curvature of M determined by the unit vector X. Then we have

$$H(X) = g(R(X, JX)JX, X).$$

If $H(X)$ is a constant for all unit vectors in $\Gamma(TM)$ and for all points in M, then M is called a space of constant holomorphic sectional curvature(or complex space form). In this case, the Riemannian curvature tensor of M is given by

$$\begin{aligned} R(X,Y)Z &= \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(Z,JY)JX - g(Z,JX)JY \\ &+ 2g(X,JY)JZ\}, \end{aligned} \tag{2.8}$$

for any $X, Y, Z \in \Gamma(TM)$, where c is the constant holomorphic sectional curvature of M[5].

3 Kaehlerian Product Manifolds

Let (M_1, J_1, g_1) and (M_2, J_2, g_2) be almost Hermitian manifolds with complex dimensional n_1 and n_2, respectively and $M_1 \times M_2$ be a Riemannian product manifold of M_1 and M_2. We denote by P and Q the projection mappings of $\Gamma(T(M_1 \times M_2))$ to $\Gamma(TM_1)$ and $\Gamma(TM_2)$, respectively. Then we have

$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0.$$

If we put $F = P - Q$, then we can easily see that $F \neq \pm I$ and $F^2 = I$, where I denotes the identity mapping of $\Gamma(T(M_1 \times M_2))$. The Riemannian metric of $M_1 \times M_2$ is given by formula

$$g(X, Y) = g_1(PX, PY) + g_2(QX, QY)$$

for any $X, Y \in \Gamma(T(M_1 \times M_2))$. From the definition of g, we get M_1 and M_2 are both totally geodesic submanifolds of Riemannian product manifold $M_1 \times M_2$. We denote the Levi-Civita connection on $M_1 \times M_2$ by $\bar{\nabla}$, then we obtain $\bar{\nabla}P = \bar{\nabla}Q = \bar{\nabla}F = 0$(for the detail, we refer to [8]).

We define a mapping by $J = J_1 P + J_2 Q$ of $\Gamma(T(M_1 \times M_2))$ to $\Gamma(T(M_1 \times M_2))$. Then, it is easily seen that $J^2 = -I$, $J_1 P = PJ$, $J_2 Q = QJ$ and $FJ = JF$. Thus J is an almost complex structure on $M_1 \times M_2$. Furthermore, if (M_1, J_1, g_1) and (M_2, J_2, g_2) are both almost Hermitian manifolds, then we have

$$\begin{aligned} g(JX, JY) &= g_1(PJX, PJY) + g_2(QJX, QJY) \\ &= g_1(J_1 PX, J_1 PY) + g_2(J_2 QX, J_2 QY) \\ &= g_1(PX, PY) + g_2(QX, QY) \\ &= g(X, Y) \end{aligned}$$

for any $X, Y \in \Gamma(T(M_1 \times M_2))$. Thus, $(M_1 \times M_2, J, g)$ is an almost Hermitian manifold. By direct calculations, we obtain

$$(3.1)\quad (\bar{\nabla}_X J)Y = (\bar{\nabla}_{PX} J_1)PY + (\bar{\nabla}_{QX} J_2)QY + (\bar{\nabla}_{QX} J_1)PY + (\bar{\nabla}_{PX} J_2)QY.$$

If $(M_1 \times M_2, J, g)$ is a Kaehlerian manifold, then we have

$$(3.2)\quad (\bar{\nabla}_{PX} J_1)PY + (\bar{\nabla}_{QX} J_2)QY + (\bar{\nabla}_{QX} J_1)PY + (\bar{\nabla}_{PX} J_2)QY = 0,$$

for any $X, Y \in \Gamma(T(M_1 \times M_2))$. We take FX instead of X in (3.2), then we obtain

$$(3.3)\quad (\bar{\nabla}_{PX} J_1)PY + (\bar{\nabla}_{QX} J_2)QY - (\bar{\nabla}_{QX} J_1)PY - (\bar{\nabla}_{PX} J_2)QY = 0.$$

Thus together with (3.2) and (3.3) give $(\bar{\nabla}_{PX} J_1)PY = (\bar{\nabla}_{QX} J_2)QY = 0$, that is, (M_1, J_1, g_1) and (M_2, J_2, g_2) are Kaehlerian manifolds. We denote Kaehlerian product manifold by $(M_1 \times M_2, J, g)$ througthout this paper.

If M_1 and M_2 are complex space forms with constant holomorphic sectional curvatures c_1, c_2 and we denote them by $M_1(c_1)$ and $M_2(c_2)$, respectively, then the

Riemannian curvature tensor R of Kaehlerian product manifold $M_1(c_1) \times M_2(c_2)$ is given by formula

$$\begin{aligned} R(X,Y)Z &= \frac{1}{16}(c_1+c_2)\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY \\ &+ 2g(X,JY)JZ + 2g(FY,Z)FX - g(FX,Z)FY + g(FJY,Z)FJX \\ &- g(FJX,Z)FJY + 2g(FX,JY)FJZ\} \\ &+ \frac{1}{16}(c_1-c_2)\{g(FY,Z)X - g(FX,Z)Y + g(Y,Z)FX - g(X,Z)FY \\ &+ g(FJY,Z)JX - g(FJX,Z)JY + g(JY,Z)FJX - g(JX,Z)FJY \\ &+ 2g(FX,JY)JZ + 2g(X,JY)JFZ\} \end{aligned} \tag{3.4}$$

for all $X,Y,Z \in \Gamma(T(M_1 \times M_2))$[6].

We suppose that $K(X \wedge Y)$ be the sectional curvature of $M_1 \times M_2$ determined by orthonormal vectors X and Y. Then by using (3.4), we obtain

$$\begin{aligned} K(X \wedge Y) &= \frac{1}{16}(c_1+c_2)\{1 + 3g(X,JY)^2 + 2g(FY,Y)g(FX,X) - g(FX,Y)^2 \\ &+ 3g(X,JFY)^2\} + \frac{1}{16}(c_1-c_2)\{g(FY,Y) + g(FX,X) \\ &+ 6g(FJX,Y)g(JX,Y)\}. \end{aligned} \tag{3.5}$$

Similarly, if $H(X)$ is the holomorphic sectional curvature of Kaehlerian product manifold $M_1 \times M_2$ determined by the unit vectors X and JX, then by using (3.4), we derive

$$\begin{aligned} H(X) = K(X,JX,JX,X) &= \frac{1}{16}(c_1+c_2)\{4 + 5g(FX,X)^2\} \\ &+ \frac{1}{2}(c_1-c_2)\{g(FX,X)\} \end{aligned} \tag{3.6}$$

4 CR-Submanifolds of a Kaehlerian Product Manifold

Definition 4.1. *Let N be an isometrically immersed submanifold of a Kaehlerian manifold M with complex structure J. N is said to be a CR-submanifold of M if there exist a differentiable distribution*

$$D : x \longrightarrow D_x \subset T_xN$$

on N satisfying the following conditions.
i) D is holomorphic(invariant), i.e., $J(D_x) = D_x$, for each $x \in N$.
ii) The orthogonal complementary distribution

$$D^\perp : x \longrightarrow D_x^\perp \subset T_xN$$

is totally-real(anti-invariant), i.e., $J(D_x^\perp) \subset T_xN^\perp$, for each $x \in N$[2].

We denote by p and q the dimensional of the distributions D and $D^{\perp}$, respectively. In particular, $q = 0$(resp. $p = 0$) for each $x \in N$, then the CR-submanifold N is called holomorphic submanifold(resp. totally real submanifold) of M. A proper CR-submanifold is a CR submanifold which is neither a holomorphic submanifold nor a totally real submanifold.

Let N be a CR-submanifold of any Kaehlerian manifold M with complex structure J. For any vector field X tangent to N, we put

$$JX = fX + \omega X, \tag{4.1}$$

where fX and ωX are the tangential and normal parts of JX, respectively. Similarly, for any vector field V normal to N, we put

$$JV = BV + CV, \tag{4.2}$$

where BV and CH are the tangential and normal parts of JV, respectively.

Theorem 4.1. *Let N be a F-invariant submanifold of a Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$ with $c_1.c_2 \neq 0$. Then N is a CR-submanifold if and only if the maximal holomorphic subspaces*

$$D_x = T_xN \cap J(T_xN), \quad x \in N$$

define a nontrivial differentiable distribution D on N such that

$$K(D, D, D^{\perp}, D^{\perp}) = 0, \tag{4.3}$$

where $D^{\perp}$ denotes the orthogonal complementary distribution of D in TN.

Proof. We suppose that N be a CR-submanifold of Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$. Then by using (3.4), we obtain

$$\begin{aligned} R(X,Y)Z &= \frac{1}{8}(c_1 + c_2)\{g(X, JY)JZ + g(FX, JY)JFZ\} \\ &+ \frac{1}{8}(c_1 + c_2)\{g(FX, JY)JZ + g(X, JY)FJZ\}, \end{aligned}$$

for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. Thus we have

$$K(X, Y, Z, W) = g(R(X, Y)Z, W) = 0,$$

for any $W \in \Gamma(D^{\perp})$, since JZ is normal to N for any $Z \in \Gamma(D^{\perp})$.

Conversely, if the maximal holomorphic subspaces D_x for each $x \in N$, define a nontrivial distribution D such that (4.3) holds, then (3.4) implies that

$$\begin{aligned} K(X, JX, Z, W) = &- \frac{1}{8}(c_1 + c_2)\{g(X, X)g(JZ, W) + g(FX, X)g(FJZ, W)\} \\ &- \frac{1}{8}(c_1 - c_2)\{g(X, FX)g(JZ, W) + g(X, X)g(FJZ, W)\} = 0, \end{aligned}$$

for any $X \in \Gamma(D)$ and $Z, W \in \Gamma(D^\perp)$. From above the equation, we obtain

$$\begin{aligned} & g(X,X)\{g(JZ,W)(c_1+c_2)+g(JZ,FW)(c_1-c_2)\} \\ + \quad & g(FX,X)\{g(JZ,FW)(c_1+c_2)+g(JZ,W)(c_1-c_2)\}=0. \end{aligned}$$

Thus we have

$$\{g(JZ,W)(c_1+c_2)+g(JZ,FW)(c_1-c_2)\}=0$$

and

$$\{g(JZ,FW)(c_1+c_2)+g(JZ,W)(c_1-c_2)\}=0,$$

because vector fields X and FX are independent. It follow that $g(JZ,W) = g(JZ,FW) = 0$, that is, $J(D_x^\perp)$ is perpendicular to $D_x^\perp$ for each $x \in N$. Since D is invariant by J, $J(D_x^\perp)$ is also perpendicular to D_x. Therefore, $J(D_x^\perp) \subset T_xN^\perp$ and N is a CR-submanifold of a Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$. This completes the proof of the theorem.
□

The aim of this paragraph is to obtain some results on sectional curvature of F-invariant CR-submanifolds of a Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$.

Theorem 4.2. *Let N be a F-invariant proper CR-submanifold of a Kaehlerian product manifold $M = M_1 \times M_2$. Then there exist no F-invariant totally umbilical proper CR-submanifold in a Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$ with $c_1 + c_2 \neq 0$.*

Proof. We suppose that N be a F-invariant proper totally umbilical CR-submanifold in a Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$. From (3.5) we obtain

$$\begin{aligned} K(X,Y,X,Y) &= \frac{1}{16}(c_1+c_2)\{-1+2g(FX,Y)^2-g(FX,X)g(FY,Y) \\ &- 3g(FX,JY)^2\}-\frac{1}{16}(c_1-c_2)\{g(FX,X) \\ &+ g(FY,Y)\}, \end{aligned} \tag{4.4}$$

for any orthonormal vector fields $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$. Since vector fields X and FX are independent, they can be choosen orthogonal to each other. Then from (4.4), we have

$$K(X \wedge Y) = -\frac{1}{16}(c_1+c_2). \tag{4.5}$$

On the other hand, since N is totally umbilical proper CR-submanifold from (2.4), we have

$$(\bar{\nabla}_X h)(Y,Z)-(\bar{\nabla}_Y h)(X,Z)=g(Y,Z)\nabla_X^\perp H-g(X,Z)\nabla_Y^\perp H,$$

for any $X, Y, Z \in \Gamma(TN)$. Furthermore, taking account of (4.5) we obtain

$$K(X,Y,Z,V)=g(Y,Z)g(\nabla_X^\perp H,V)-g(X,Z)g(\nabla_Y^\perp H,V), \tag{4.6}$$

for any $V \in \Gamma(TN^{\perp})$. By putting $X = Z \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$ in (4.6), then we get $JX \in \Gamma(D)$ and $JY \in \Gamma(D^{\perp})$. Thus from (2.6), we infer

$$K(X, Y, JX, JY) = g(Y, JX)g(\nabla^{\perp}_X H, JY) - g(X, JX)g(\nabla^{\perp}_Y H, JY) = 0.$$

Since M is a Kaehlerian product manifold, we have

$$K(X, Y, JX, JY) = K(X, Y, X, Y) = 0,$$

which proves our assertion.
□

Theorem 4.3. *Let N be a F-invariant proper CR-submanifold of a Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$. If N is $D^{\perp}$-totally geodesic submanifold, then $N = N_1(\frac{1}{4}c_1) \times N_2(\frac{1}{4}c_2)$, where $N_1(\frac{1}{4}c_1)$ is a real space form of constant curvature $\frac{1}{4}c_1$ and $N_2(\frac{1}{4}c_2)$ is a real space form of constant curvature $\frac{1}{4}c_2$.*

Proof. If N is $D^{\perp}$-totally geodesic, then by using (2.5) and (3.4), we obtain

$$\begin{aligned} R_N(X, Y)Z &= \frac{1}{8}c_1\{g(Y, Z)PX - g(X, Z)PY - g(FX, Z)PY + g(FY, Z)PX\} \\ &+ \frac{1}{8}c_2\{g(Y, Z)QX - g(X, Z)QY - g(FY, Z)QX + g(FX, Z)QY\} \\ &= \frac{1}{4}c_1\{g(PY, PZ)PX - g(PX, PZ)PY\} \\ &+ \frac{1}{4}c_2\{g(QY, QZ)QX - g(QX, QZ)QY\}, \end{aligned}$$

for any $X, Y, Z, W \in \Gamma(D^{\perp})$, where R_N is the Riemannian curvature tensor of N. This completes the proof of the theorem.
□

Now, we calculate holomorphic bisectional curvature $H_B(X, Y)$ for any unit vector fields $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$. From (3.4), by a direct calculation, we derive

$$\begin{aligned} H_B(X, Y) = g(R(X, JX)JY, Y) &= \frac{1}{8}(c_1 + c_2)\{1 + g(FX, X)g(FY, Y)\} \\ &+ \frac{1}{8}(c_1 - c_2)\{g(FX, X) + g(FY, Y)\}. \end{aligned}$$

Moreover, if N is a CR-product, then we have

$$H_B(X, Y) = 2\|h(X, Y)\|^2,$$

for any unit vector fields $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$[2]. Thus if N is a CR-product, then we obtain

$$\begin{aligned} \|h(X, Y)\|^2 &= \frac{1}{4}(c_1 + c_2)\{1 + g(FX, X)g(FY, Y)\} \\ &+ \frac{1}{4}(c_1 - c_2)\{g(FX, X) + g(FY, Y)\}, \end{aligned} \tag{4.7}$$

for any unit vector fields $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$, where taking X and FX are orthogonal vector fields in (4.7), then we have

$$\|h(X,Y)\|^2 = \frac{1}{4}(c_1 + c_2), \tag{4.8}$$

for any vector fields $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$. Thus we have the following theorems.

Theorem 4.4. *Let N be a F-invariant proper CR-submanifold in a Kaehlerian product manifold $M = M_1 \times M_2$. Then there exist no F-invariant totally geodesic proper CR-products N in any Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$ with $c_1 + c_2 \neq 0$.*

Theorem 4.5. *Let N be a F-invariant proper CR-submanifold in a Kaehlerian product manifold $M = M_1 \times M_2$. Then there exist no F-invariant mixed-geodesic proper CR-products N in any Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$ with $c_1 + c_2 \neq 0$.*

Theorem 4.6. *Let N be a proper CR-submanifold of a Kaehlerian product manifold $M = M_1 \times M_2$. Then N is a CR-product manifold if and only if*

$$A_{JZ}X = 0,$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

Proof. Let us suppose that N be a CR-product. Then we have $\nabla_X Y \in \Gamma(D)$ and $\nabla_W Z \in \Gamma(D^{\perp})$ for any $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^{\perp})$. By using (2.1), (2.2) and (2.3) we infer

$$\begin{aligned} g(A_{JZ}X, Y) &= -g(\bar{\nabla}_X JZ, Y) = g(\bar{\nabla}_X Z, JY) \\ &= -g(\bar{\nabla}_X JY, Z) = -g(\nabla_X JY, Z) \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} g(A_{JZ}X, W) &= g(A_{JZ}W, X) = -g(\bar{\nabla}_W JZ, X) \\ &= g(\bar{\nabla}_W Z, JX) = g(\nabla_W Z, JX), \end{aligned} \tag{4.10}$$

for any $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^{\perp})$. From equations (4.9) and (4.10), we obtain that the distribution D and $D^{\perp}$ are integrable and their leaves are totally geodesic submanifolds in N if and only if $A_{JZ}X \in \Gamma(D)$ and $A_{JZ}X \in \Gamma(D^{\perp})$, which proves our assertion.
□

Making use of the equations (2.5) and (3.4), we have special forms for the structure equations of Gauss, Codazzi and Ricci for the CR-submanifold N in Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$.

$$\begin{array}{rcl} R_N(X,Y)Z &=& \frac{1}{16}(c_1+c_2)\{g(Y,Z)X-g(X,Z)Y+g(JY,Z)JX \\ &-& g(JX,Z)JY+2g(X,JY)JZ+2g(FY,Z)FX-g(FX,Z)FY \\ &+& g(FJY,Z)FJX-g(FJX,Z)FJY+2g(FX,JY)FJZ\} \\ &+& \frac{1}{16}(c_1-c_2)\{g(FY,Z)X-g(FX,Z)Y+g(Y,Z)FX \\ &-& g(X,Z)FY+g(FJY,Z)JX-g(FJX,Z)JY \\ &+& g(JY,Z)FJX-g(JX,Z)FJY+2g(FX,JY)JZ \\ &+& 2g(X,JY)FJZ\}+A_{h(Y,Z)}X-A_{h(X,Z)}Y \\ &-& (\bar{\nabla}_X h)(Y,Z)+(\bar{\nabla}_Y h)(X,Z), \end{array} \tag{4.11, 4.12}$$

for any vector fields X, Y, Z tangent to N. Taking account of (4.1) and (4.2), then the equation of Gauss becomes

$$\begin{array}{rcl} R_N(X,Y)Z &=& \frac{1}{16}(c_1+c_2)\{g(Y,Z)X-g(X,Z)Y+g(fY,Z)fX \\ &-& g(fX,Z)fY+2g(X,fY)fZ+2g(FY,Z)FX-g(FX,Z)FY \\ &+& g(FfY,Z)FfX-g(FfX,Z)FfY+2g(FX,fY)FfZ\} \\ &+& \frac{1}{16}(c_1-c_2)\{g(FY,Z)X-g(FX,Z)Y+g(Y,Z)FX \\ &-& g(X,Z)FY+g(FfY,Z)fX-g(FfX,Z)fY \\ &+& g(fY,Z)FfX-g(fX,Z)FfY+2g(FX,fY)fZ \\ &+& 2g(X,fY)FfZ\}+A_{h(Y,Z)}X-A_{h(X,Z)}Y \end{array} \tag{4.13}$$

and the equation of Codazzi is given by

$$\begin{array}{rcl} (\bar{\nabla}_X h)(Y,Z)-(\bar{\nabla}_Y h)(X,Z) &=& \frac{1}{16}(c_1+c_2)\{g(fY,Z)\omega X-g(fX,Z)\omega Y \\ &+& 2g(X,fY)\omega Z+g(fY,FZ)F\omega X \\ &-& g(fX,FZ)F\omega Y+2g(FX,fY)F\omega Z\} \\ &+& \frac{1}{16}(c_1-c_2)\{g(FZ,fY)\omega X-g(FZ,fX)\omega Y \\ &+& g(fY,Z)F\omega X-g(fX,Z)F\omega Y \\ &+& 2g(FX,fY)\omega Z+2g(X,fY)F\omega Z\}, \end{array} \tag{4.14}$$

for any vector fields X, Y, Z tangent to N. Finally, the equation of Ricci is given by

$$\begin{array}{rcl} K(X,Y,V,W) &=& g(R^{\perp}(X,Y)V,W)+g([A_W,A_V]X,Y) \\ &=& \frac{1}{16}(c_1+c_2)\{g(\omega Y,V)g(\omega X,W)-g(\omega X,V)g(\omega Y,W) \\ &+& 2g(X,fY)g(CV,W)+g(\omega Y,FV)g(\omega X,FW) \\ &-& g(\omega X,FV)g(\omega Y,FW)+2g(FX,fY)g(CV,FW)\} \\ &+& \frac{1}{16}(c_1-c_2)\{g(\omega Y,FV)g(\omega X,W)-g(\omega X,FV)g(\omega Y,W) \\ &+& g(\omega Y,V)g(\omega X,FW)-g(\omega X,V)g(\omega Y,FW) \\ &+& 2g(FX,fY)g(CV,W)+2g(X,fY)g(CV,FW)\}, \end{array} \tag{4.15}$$

for any vector fields $X, Y,$ tangent to N and V, W normal to N, where $R^{\perp}$ is the curvature tensor of the normal connection of $\Gamma(TN^{\perp})$. Thus we have the following Theorem.

Theorem 4.7. *Let N be a F invariant proper CR-submanifold of a Kaehlerian product manifold $M = M_1 \times M_2$. Then there exist no curvature-invariant proper CR-submanifolds in Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$.*

Proof. Let us suppose that N be a curvature-invariant proper CR-submanifold in a Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$. Then from (4.14), we obtain

$$\begin{aligned} 0 &= \frac{1}{16}(c_1 + c_2)\{g(fY, Z)\omega X - g(fX, Z)\omega Y + 2g(X, fY)\omega Z \\ &+ g(fY, FZ)F\omega X - g(fX, FZ)F\omega Y + 2g(FX, fY)F\omega Z\} \\ &+ \frac{1}{16}(c_1 - c_2)\{g(FZ, fY)\omega X - g(FZ, fX)\omega Y \\ &+ g(fY, Z)F\omega X - g(fX, Z)F\omega Y + 2g(FX, fY)\omega Z \\ &+ 2g(X, fY)F\omega Z\}, \end{aligned} \tag{4.16}$$

for any vector fields X, Y, Z tangent to N. Taking $Y \in (D^{\perp})$ in the equation (4.16), then we infer

$$\begin{aligned} 0 &= \omega Y\{(c_1 + c_2)g(fX, Z) + (c_1 - c_2)g(fX, FZ)\} \\ &+ F\omega Y\{(c_1 + c_2)g(fX, FZ) + (c_1 - c_2)g(fX, Z)\}, \end{aligned}$$

that is,

$$\begin{aligned} & g(fX, Z)\{(c_1 + c_2)\omega Y + (c_1 - c_2)F\omega Y\} \\ + \ & g(fX, FZ)\{(c_+ c_2)F\omega Y + (c_1 - c_2)\omega Y\} = 0, \end{aligned}$$

which implies that

$$(c_1 + c_2)g(fX, Z) + (c_1 - c_2)g(fX, FZ) = 0$$

and

$$(c_1 + c_2)g(fX, FZ) + (c_1 - c_2)g(fX, Z) = 0,$$

for any $X, Z \in \Gamma(TN)$. Thus we have $g(fX, Z) = 0$. This is impossible. The proof is complete.
□

Let N be a F-invariant proper CR-submanifold of a Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$. Then we say that N has semi-flat normal connection if its normal curvature $K^{\perp}$ satisfies

$$\begin{aligned} K^{\perp}(X, Y, V, W) &= g(R^{\perp}(X, Y)V, W) = \frac{1}{8}\{g(X, fY)g(JV, W) \\ &+ g(FX, fY)g(JV, FW)\} \\ &+ \frac{1}{8}(c_1 - c_2)\{g(X, fY)g(JV, FW) + g(FX, fY)g(JV, W)\}, \end{aligned}$$

for any $X, Y \in \Gamma(TN)$ and $V, W \in \Gamma(TN^{\perp})$. Making use of the equation (4.15), we obtain that a F-invariant proper CR-submanifold N of a Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$ has semi-flat normal connection if and only if

$$\begin{aligned} g([A_W, A_V]X, Y) &= \frac{1}{16}(c_1 + c_2)\{g(\omega Y, V)g(\omega X, W) + g(\omega X, V)g(\omega Y, V) \\ &+ g(\omega Y, FV)g(\omega X, FW) - g(\omega X, FV)g(\omega Y, FW)\} \\ &+ \frac{1}{16}(c_1 - c_2)\{g(\omega Y, FV)g(\omega X, W) - g(\omega X, FV)g(\omega Y, W) \\ &+ g(\omega Y, V)g(\omega X, FW) - g(\omega X, V)g(\omega Y, FW)\}. \end{aligned}$$

Theorem 4.8. *Let N be a F-invariant proper CR-submanifold of a Kaehlerian product manifold $M = M_1 \times M_2$. Then there exist no F-invariant totally umbilical proper CR-submanifolds of a Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$ such that c_1 and c_2 don't vanish.*

Proof. Choosing $Y \in \Gamma(D^{\perp})$ in the equation (4.14), we have

$$\begin{aligned} (\bar{\nabla}_X h)(Y, JX) - (\bar{\nabla}_Y h)(X, JX) &= -\frac{1}{16}(c_1 + c_2)\{g(X, X)\omega Y + g(FX, X)F\omega Y\} \\ &- \frac{1}{16}(c_1 - c_2)\{g(X, X)F\omega Y + g(X, FX)\omega Y\} \end{aligned}$$

for any $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$.

On the other hand, since N is totally umbilical proper CR-submanifold, we obtain

$$\begin{aligned} g(Y, JX)\nabla^{\perp}_X H - g(X, JX)\nabla^{\perp}_Y H = &- \frac{1}{16}(c_1 + c_2)\{g(X, X)\omega Y + g(FX, X)F\omega Y\} \\ &- \frac{1}{16}(c_1 - c_2)\{g(X, X)F\omega Y + g(X, FX)\omega Y\}, \end{aligned}$$

that is,

$$\begin{aligned} &(c_1 + c_2)\{g(X, X)\omega Y + g(FX, X)F\omega Y\} + \\ &\quad (c_1 - c_2)\{g(X, X)F\omega Y + g(X, FX)\omega Y\} = 0, \end{aligned}$$

which implies that

$$\omega Y\{(c_1 + c_2)g(X, X) + (c_1 - c_2)g(X, FX)\} = 0$$

and

$$F\omega Y\{(c_1 + c_2)g(X, FX) + (c_1 - c_2)g(X, X)\} = 0.$$

It follow that $4c_1c_2\omega Y = 0$. This is a contradiction. Thus the proof is complete. □

Since a totally geodesic submanifold is always curvature-invariant, we have the following theorem from the theorem 4.8.

Theorem 4.9. *Let N be a F-invariant proper CR-submanifold of a Kaehlerian product manifold $M = M_1 \times M_2$. Then there exist no F-invariant totally geodesic proper CR-submanifolds of a Kaehlerian product manifold $M = M_1(c_1) \times M_2(c_2)$ such that c_1 and c_2 don't vanish.*

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