Biminimal submanifolds in contact 3-manifolds

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Abstract. We study biminimal submanifolds in contact 3-manifolds. In particular, biminimal curves in homogeneous contact Riemannian 3-manifolds and biminimal Hopf cylinders in Sasakian 3-space forms are investigated.

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1 Introduction

A smooth map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is said to be *biharmonic* if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_M |\tau(\phi)|^2 dv_g,$$

where $\tau(\phi) = \operatorname{tr} \nabla d\phi$ is the tension field of ϕ. Clearly, if ϕ is harmonic, *i.e.*, $\tau(\phi) = 0$, then ϕ is biharmonic. A biharmonic map is said to be *proper* if it is not harmonic.

B. Y. Chen and S. Ishikawa [7] studied biharmonic curves and surfaces in semi-Euclidean space (see also [11]–[12]). In particular, Chen and Ishikawa proved the non-existence of proper biharmonic surfaces in Euclidean 3-space $\mathbb{R}^3$. R. Caddeo, S. Montaldo and C. Oniciuc generalized this non-existence theorem to surfaces in 3-dimensional space forms of non-positive curvature [5].

Biharmonic submanifolds in the 3-sphere S^3 are classified by Caddeo, Montaldo and Oniciuc [4]. Since, S^3 is a typical example of contact Riemannian 3-manifold, it is interesting to study biharmonic submanifolds in contact Riemannian manifolds. In our previous paper [13], we have studied biharmonic Legendre curves and Hopf cylinders in Sasakian 3-space forms. J. T. Cho, J. E. Lee and the present author [8]–[9] studied biharmonic curves in unimodular homogeneous contact Riemannian 3-manifolds.

K. Arslan, R. Ezentas, C. Murathan and T. Sasahara studied biharmonic submanifolds in 3 or 5-dimensional contact Riemannian manifolds [1], [2], [19].

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On the other hand, in [14], E. Loubeau and S. Montaldo introduced the notion of biminimal immersion.

An isometric immersion $\phi : (M, g) \to (N, h)$ is said to be *biminimal* if it is a critical point of the bienergy functional under all *normal variations*. Thus the biminimality is weaker than biharmonicity for isometric immersions, in general.

In this paper, we study biminimal submanifolds in contact 3-manifolds. In particular we study biminimality of Legendre curves and Hopf cylinders (anti-invariant surfaces) in Sasakian 3-space forms.

2 Preliminaries

2.1

Let (M^m, g) and (N^n, h) be Riemannian manifolds and $\phi : M \to N$ a smooth map. Denote by ∇^ϕ the connection of the vector bundle ϕ^*TN induced from the Levi-Civita connection ∇^h of (N, h). The *second fundamental form* $\nabla d\phi$ is defined by

$$(\nabla d\phi)(X, Y) = \nabla^\phi_X d\phi(Y) - d\phi(\nabla_X Y), \quad X, Y \in \Gamma(TM).$$

Here ∇ is the Levi-Civita connection of (M, g). The *tension field* $\tau(\phi)$ is a section of ϕ^*TN defined by

$$\tau(\phi) = \mathrm{tr}\, \nabla d\phi.$$

A smooth map ϕ is said to be *harmonic* if its tension field vanishes. It is well known that ϕ is harmonic if and only if ϕ is a critical point of the *energy*:

$$E(\phi) = \frac{1}{2} \int |d\phi|^2 dv_g$$

over every compact region of M. Now let $\phi : M \to N$ be a harmonic map. Then the Hessian $\mathcal{H}_\phi$ of E is given by

$$\mathcal{H}_\phi(V, W) = \int h(\mathcal{J}_\phi(V), W) dv_g, \quad V, W \in \Gamma(\phi^*TN).$$

Here the *Jacobi operator* $\mathcal{J}_\phi$ is defined by

$$\mathcal{J}_\phi(V) := \bar{\triangle}_\phi V - \mathcal{R}_\phi(V), \ V \in \Gamma(\phi^*TN),$$

$$\bar{\triangle}_\phi := -\sum_{i=1}^m (\nabla^\phi_{e_i} \nabla^\phi_{e_i} - \nabla^\phi_{\nabla_{e_i} e_i}), \quad \mathcal{R}_\phi(V) = \sum_{i=1}^m R^N(V, d\phi(e_i)) d\phi(e_i),$$

where R^N and $\{e_i\}$ are the Riemannian curvature of N, and a local orthonormal frame field of M, respectively. For general theory of harmonic maps, we refer to Urakawa's monograph [21].

J. Eells and J. H. Sampson [10] suggested to study *polyharmonic maps*. In this paper, we only consider polyharmonic maps of order 2. Such maps are frequently called *biharmonic maps*.

Definition 2.1. A smooth map $\phi : (M, g) \to (N, h)$ is said to be *biharmonic* if it is a critical point of the bienergy functional:

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 \, dv_g,$$

with respect to all compactly supported variation.

The Euler-Lagrange equation of E_2 is

$$\tau_2(\phi) := -\mathcal{J}_\phi(\tau(\phi)) = 0.$$

The section $\tau_2(\phi)$ is called the *bitension field* of ϕ. If ϕ is an isometric immersion, then $\tau(\phi) = m\mathbb{H}$, where $\mathbb{H}$ is the mean curvature vector field. Hence ϕ is harmonic if and only if ϕ is a minimal immersion. As is well known, an isometric immersion $\phi : M \to N$ is minimal if and only if it is a critical point of the volume functional $\mathcal{V}$. The Euler-Lagrange equation of $\mathcal{V}$ is $\mathbb{H} = 0$.

Motivated by this coincidence, the following notion was introduced by Loubeau and Montaldo:

Definition 2.2. ([14]) An isometric immersion $\phi : (M^m, g) \to (N^n, h)$ is called a *biminimal* immersion if it is a critical point of the bienergy functional E_2 with respect to all normal variation with compact support. Here, a normal variation means a variation $\{\phi_t\}$ through $\phi = \phi_0$ such that the variational vector field $V = d\phi_t/dt|_{t=0}$ is normal to M.

The Euler-Lagrange equation of this variational problem is $\tau_2(\phi)^\perp = 0$. Here $\tau_2(\phi)^\perp$ is the normal component of $\tau_2(\phi)$. Since $\tau(\phi) = m\mathbb{H}$, the Euler-Lagrange equation is given explicitly by

$$\left\{\bar{\triangle}_\phi \mathbb{H} - \mathcal{R}_\phi(\mathbb{H})\right\}^\perp = 0 \tag{2.2.1}$$

Obviously, every biharmonic immersioin is biminimal, but the converse is not always true.

Submanifolds with harmonic mean curvature $\triangle\mathbb{H} = 0$ or normal harmonic mean curvature $\triangle^\perp\mathbb{H} = 0$ have been studied extensively. Here $\Delta^\perp$ is the Laplace-Beltrami operator of the normal bundle (and called the *normal Laplacian*). More generally, submanifolds with property $\triangle\mathbb{H} = \lambda\mathbb{H}$ or $\triangle^\perp\mathbb{H} = \lambda\mathbb{H}$ have been studied extensively by many authors (See references in [13]). Analogusly, we may generalize the notion of biminimal immersion to the following one:

Definition 2.3. An isometric immersion $\phi : M \to N$ is called a λ-*biminimal* immersion if it is a crtical point of the functional:

$$E_{2,\lambda}(\phi) = E_2(\phi) + \lambda E(\phi), \quad \lambda \in \mathbb{R}$$

The Euler-Lagrange equation for λ-biminimal immersions is

$$\tau_2(\phi)^\perp = \lambda\tau(\phi).$$

More explicitly,

$$\{\bar{\triangle}_\phi \mathbb{H} - \mathcal{R}_\phi(\mathbb{H})\}^\perp = -\lambda\mathbb{H}$$

or equivalently

$$\mathcal{J}_\phi(\mathbb{H})^\perp = -\lambda\mathbb{H}.$$

2.2

To close this section, we here collect fundamental ingredients of contact Riemannian geometry from [3] for our use.

Let M be a 3-dimensional manifold. A one-form η is called a *contact form* on M if it satisfies $d\eta \wedge \eta \neq 0$ on M. A 3-manifold M together with a contact form η is called a *contact* 3-*manifold* (in the restricted sense). The *contact distribution* $\mathcal{D}$ of (M, η) is defined by

$$\mathcal{D} = \{X \in TM \mid \eta(X) = 0\}.$$

On a contact 3-manifold (M, η), there exist a unique vector field ξ such that

$$\eta(\xi) = 1, \ d\eta(\xi, \cdot) = 0.$$

This vector field ξ is called the *Reeb vector field* of (M, η). Moreover, there exists an endomorphism field φ and a Riemannian metric g such that

$$\varphi^2 = -I + \eta \otimes \xi, g(\xi, \cdot) = \eta,$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$d\eta(X, Y) = 2g(X, \varphi Y)$$

for all vector fields X, Y on M. A contact 3-manifold (M, η) together with structure tensors (ξ, φ, g) is called a *contact Riemannian* 3-*manifold*.

Definition 2.4. A contact Riemannian 3-manifold $(M, \eta; \xi, \varphi, g)$ is said to be a 3-dimesnional *Sasaki manifold* (or *Sasaki* 3-*maifold*) if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X$$

for all vector fields X, Y on M. Here ∇ denotes the Levi-Civita connection of (M, g).

Let $(M, \eta; \xi, \varphi, g)$ be a contact Riemannian 3-manifold. A tangent plane at a point p is said to be *holomorphic* if it is invariant under φ. The sectional curvature of a holomorphic tangent plane is called a *holomorphic sectional curvature*. If the sectional curvature function is constant on all holomorphic planes in TM, then M is said to be of *constant holomorphic sectional curvature*. In particular, complete Sasaki 3-manifolds of constant holomorphic sectional curvature are called *Sasakian* 3-*space forms*.

A contact Riemannian 3-manifold M is said to be *regular* if ξ generates a one-parameter group K of isometries on M such that the action of K on M is simply transitive. If M is regular, then φ and η are invariant under K-action. Moreover the contact Riemannian structure on M induces an almost Kähler structure $(\bar{g}, J)$ on the orbit space $\overline{M} := M/K$. The natural projection $\pi : M \to \overline{M}$ is a Riemannian submersion.

Now let M be a regular Sasaki 3-manifold. Take a regular curve $\bar{\gamma}$ parametrized by the arclength with signed curvature function $\bar{\kappa}$. Then the inverse image $S_{\bar{\gamma}} := \pi^{-1}\{\bar{\gamma}\}$ is a flat surface in M with mean curvature $H = (\bar{\kappa} \circ \pi)/2$. This flat surface is called the *Hopf cylinder* over $\bar{\gamma}$.

3 Biminimal curves

First of all we recall the following well known result (*cf.* [14]).

Lemma 3.1.

i) A curve γ in a Riemannian 2-manifold of Gaussian curvature K is biminimal if and only if its signed curvature κ satisfies:

$$\kappa'' - \kappa^3 + \kappa K = 0. \tag{3.3.1}$$

ii) A curve γ in a Riemannian 3-manifold of constant sectional curvature c is biminimal if and only if its curvature κ and torsion τ fulfill the system:

$$\begin{cases} \kappa'' - \kappa^3 - \kappa\tau^2 + \kappa c = 0 \\ \kappa^2\tau = \textit{constant} \end{cases} \tag{3.3.2}$$

Note that γ is biharmonic if and only if γ is biminimal and additionally satisfies $\kappa\kappa' = 0$. Thus a non-geodesic biharmonic curve has constant curvature κ.

Corollary 3.1. *(1) A non-geodesic curve in a Riemannian 2-manifold is biharmonic if and only if γ is a Riemannain circle of signed curvature κ. The signed curvature κ satisfies $K = \kappa^2 > 0$. Thus proper biharmonic curves can be exist only in positive curvature 2-manifolds.*

(2) There are no proper biharmonic curves in Riemannian 3-manifolds of constant nonpositive curvature.

Proper biharmonic curves in S^3 are classified in [4].

Corollary 3.2. *A non-geodesic curve γ in a Riemannian 2-manifold is λ-biminimal if and only if*

$$\kappa'' - \kappa^3 + \kappa\,(K - \lambda) = 0.$$

4 Biminimal curves in homogeneous contact 3-manifolds

4.1

A contact Riemannian 3-manifold is said to be *homogeneous* if there exists a connected Lie group G acting transitively as a group of isometries on it which preserve the contact form.

D. Perrone [18] has proven that simply connected homogeneous contact Riemannian 3-manifolds are Lie group together with a left invariant contact Riemannian structure.

Now let M be a 3-dimensional unimodular Lie group with left invariant Riemannian metric $g = \langle\cdot,\cdot\rangle$. Then M admits its compatible left-invariant contact Riemannian structure if and only if there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{m}$ such that (*cf.* [18]):

$$[e_1, e_2] = 2e_3, \ [e_2, e_3] = c_2 e_1, \ [e_3, e_1] = c_3 e_2.$$

The Reeb vector filed ξ is obtained by left translation of e_3. The contact distribution $\mathfrak{D}$ is spanned by e_1 and e_2.

By the Koszul formula, one can calculate the Levi-Civita connection ∇ in terms of the basis $\{e_1, e_2, e_3\}$ as follows:

$$\begin{aligned}
&\nabla_{e_1} e_2 = \frac{1}{2}(c_3 - c_2 + 2)e_3, \quad \nabla_{e_1} e_3 = -\frac{1}{2}(c_3 - c_2 + 2)e_2,\\
&\nabla_{e_2} e_1 = \frac{1}{2}(c_3 - c_2 - 2)e_3, \quad \nabla_{e_2} e_3 = -\frac{1}{2}(c_3 - c_2 - 2)e_1,\\
&\nabla_{e_3} e_1 = \frac{1}{2}(c_3 + c_2 - 2)e_2, \quad \nabla_{e_3} e_2 = -\frac{1}{2}(c_3 + c_2 - 2)e_1,\\
&\qquad \text{all others are zero.}
\end{aligned} \tag{4.4.1}$$

In particular, M is a Sasaki manifold if and only if $c_2 = c_3$, and it is of constant holomorphic sectional curvature $c = -3 + 2c_2$ (*cf.* [18]). The Riemannian curvature R is given by

$$\begin{aligned}
R(e_1, e_2)e_2 &= \{\frac{1}{4}(c_3 - c_2)^2 - 3 + c_3 + c_2\}e_1,\\
R(e_1, e_3)e_3 &= \{-\frac{1}{4}(c_3 - c_2)^2 - \frac{1}{2}({c_3}^2 - {c_2}^2) + 1 - c_2 + c_3\}e_1,\\
R(e_2, e_1)e_1 &= \{\frac{1}{4}(c_3 - c_2)^2 - 3 + c_3 + c_2\}e_2,\\
R(e_2, e_3)e_3 &= \{\frac{1}{4}(c_3 + c_2)^2 - {c_2}^2 + 1 + c_2 - c_3\}e_2,\\
R(e_3, e_1)e_1 &= \{-\frac{1}{4}(c_3 - c_2)^2 - \frac{1}{2}({c_3}^2 - {c_2}^2) + 1 - c_2 + c_3\}e_3,\\
R(e_3, e_2)e_2 &= \{\frac{1}{4}(c_3 + c_2)^2 - {c_2}^2 + 1 + c_2 - c_3\}e_3.
\end{aligned}$$

4.2

Now we study biharmonic curves in homogeneous contact Riemannian 3-manifold M.

Let $\gamma : I \to M$ be a curve parametrized by arc-length with Frenet frame $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$. Expand $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}$ as $\boldsymbol{t} = T_1 e_1 + T_2 e_2 + T_3 e_3$, $\boldsymbol{n} = N_1 e_1 + N_2 e_2 + N_3 e_3$, $\boldsymbol{b} = B_1 e_1 + B_2 e_2 + B_3 e_3$ with respect to the basis $\{e_1, e_2, e_3\}$. Since $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ is positively oriented,

$$B_1 = T_2 N_3 - T_3 N_2, B_2 = T_3 N_1 - T_1 N_3, B_3 = T_1 N_2 - T_2 N_1.$$

Direct computation shows

$$\begin{aligned}R(\boldsymbol{n},\boldsymbol{t})\boldsymbol{t} &= \big[{B_1}^2\{\frac{1}{4}(c_3+c_2)^2-{c_2}^2+1+c_2-c_3\}\\ &\quad -{B_2}^2\{\frac{1}{4}(c_3-c_2)^2+\frac{1}{2}({c_3}^2-{c_2}^2)-1+c_2-c_3\}\\ &\quad +{B_3}^2\{\frac{1}{4}(c_3-c_2)^2-3+c_2+c_3\}\big]\boldsymbol{n}\\ &\quad +\big[-B_1N_1\{\frac{1}{4}(c_3+c_2)^2-{c_2}^2+1+c_2-c_3\}\\ &\quad +B_2N_2\{\frac{1}{4}(c_3-c_2)^2+\frac{1}{2}({c_3}^2-{c_2}^2)-1+c_2-c_3\}\\ &\quad -B_3N_3\{\frac{1}{4}(c_3-c_2)^2-3+c_2+c_3\}\big]\boldsymbol{b}.\end{aligned}$$

The bitension field $\tau_2(\gamma)$ is given by $\tau_2(\gamma)=\nabla_{\boldsymbol{t}}\nabla_{\boldsymbol{t}}\nabla_{\boldsymbol{t}}\boldsymbol{t}+R(\kappa\boldsymbol{n},\boldsymbol{t})\boldsymbol{t}$.
Hence we have [8]:

$$\begin{aligned}\tau_2(\gamma)^{\perp} &= \Big[(\kappa''-\kappa^3-\kappa\tau^2)+\kappa\big\{{B_1}^2(\frac{1}{4}(c_3+c_2)^2-{c_2}^2+1+c_2-c_3)\\ &\quad -{B_2}^2(\frac{1}{4}(c_3-c_2)^2+\frac{1}{2}({c_3}^2-{c_2}^2)-1+c_2-c_3)\\ &\quad +{B_3}^2(\frac{1}{4}(c_3-c_2)^2-3+c_2+c_3)\big\}\Big]\boldsymbol{n}\\ &\quad +\Big[(2\tau\kappa'+\kappa\tau')-\kappa\big\{-B_1N_1(\frac{1}{4}(c_3+c_2)^2-{c_2}^2+1+c_2-c_3)\\ &\quad +B_2N_2(\frac{1}{4}((c_3-c_2)^2+\frac{1}{2}({c_3}^2-{c_2}^2)-1+c_2-c_3)\\ &\quad -B_3N_3(\frac{1}{4}(c_3-c_2)^2-3+c_2+c_3)\big\}\Big]\boldsymbol{b}.\end{aligned}$$

Now we assume that γ is a *Legendre curve*, that is, γ tangents to the contact distribution. Then $B_1=B_2=T_3=N_3=0$ and $B_3=1$.

$$\begin{aligned}\tau_2(\gamma)^{\perp} &= \Big[\nabla_{\boldsymbol{t}}\nabla_{\boldsymbol{t}}\nabla_{\boldsymbol{t}}\boldsymbol{t}+R(\kappa\boldsymbol{n},\boldsymbol{t})\boldsymbol{t}\Big]^{\perp}\\ &= \Big[(\kappa''-\kappa^3-\kappa\tau^2)+\kappa\big\{(\frac{1}{4}(c_3-c_2)^2-3+c_2+c_3)\big\}\Big]\boldsymbol{n}\\ &\quad +\Big[(2\tau\kappa'+\kappa\tau')\Big]\xi.\end{aligned}$$

Proposition 4.1. *Let γ be a Legendre curve in a unimodular homogeneous contact Riemannian 3-manifold. Then γ is biminimal if and only if*

$$(\kappa''-\kappa^3-\kappa\tau^2)+\kappa\left\{\frac{1}{4}(c_3-c_2)^2-3+c_2+c_3\right\}=0$$

and

$$2\tau\kappa'+\kappa\tau'=0.$$

Corollary 4.1. *Let γ be a non-geodesic Legendre curve in a unimodular homogeneous contact Riemannian 3-manifold. Then γ is biharmonic if and only if γ is a helix such that*

$$\kappa^2 + \tau^2 = \frac{1}{4}(c_3 - c_2)^2 - 3 + c_2 + c_3.$$

In particular, there are no proper biharmonic Legendre curves in homogeneous contact 3*-manifold with* $\frac{1}{4}(c_3 - c_2)^2 - 3 + c_2 + c_3 \leq 0$.

Note that Corollary 4.1 is a special case of [2]. In fact, every homogeneous contact Riemannian 3-manifold is a (κ, μ)-space.

Example 4.1. (Solvable Lie groups) Choose $c_3 = 0$ and $c_2 > 0$. Then M is the Euclidean motion group $E(2)$. Hence, if $c_2 > 2$, then $M = E(2)$ admits proper biharmonic Legendre helices. On the other hand, if $c_3 = 0$ and $c_2 < 0$, then M is the Minkowski motion group $E(1,1)$. In this case, $M = E(1,1)$ admits proper biharmonic Legendre helices if and only if $c_2 < -6$. Note that $E(1,1)$ with left invariant metric $c_2 < -6$ is not isomorphic to the model space Sol ($c_2 = -2$) of the solvegeometry in the sense of W. Thurston. Hence Sol admits no proper biharmonic Legendre curves.

5 Biminimal submanifolds in Sasakian 3-space forms

5.1

Let us denote by $\mathcal{M}^3(c)$ a Sasakian 3-space form of constant holomorphic sectional curvature c. Then $\mathcal{M}^3(c)$ is regular and its orbit space $\overline{\mathcal{M}^2}$ is a complex space form of constant curvature $(c{+}3)$. Take a curve $\bar{\gamma}(s)$ in the orbit space $\overline{\mathcal{M}^2}(c{+}3)$ parametrized by arclength s. Denote by $\{\bar{\boldsymbol{t}}, \overline{\boldsymbol{n}}\}$ the Frenet frame of $\bar{\gamma}$. The arclength parameter s is also an arclength parameter of the horizontal lift $\bar{\gamma}^*$ in $\mathcal{M}^3(c)$. Thus the Frenet frame of $\bar{\gamma}^*$ is given by $(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3)$, where

$$\boldsymbol{p}_1 = \boldsymbol{t} = \bar{\boldsymbol{t}}^*, \ \boldsymbol{p}_2 = \boldsymbol{n} = \overline{\boldsymbol{n}}^* = \varphi\, \boldsymbol{t}, \ \boldsymbol{p}_3 = \pm\xi.$$

Without loss of generality, we may assume that $\boldsymbol{p}_3 = \xi$.

5.2

Let $\gamma : I \to \mathcal{M}^3(c)$ be a curve in a Sasakian 3-space form which is not a geodesic. Then the bitension field of γ is computed as ([13], p. 175):

$$\tau_2(\gamma) = -3\kappa\kappa'\boldsymbol{p}_1 + (\kappa'' - \kappa^3 - \kappa\tau^2)\boldsymbol{p}_2 + (2\kappa'\tau + \kappa\tau')\boldsymbol{p}_3 + \kappa R(\boldsymbol{p}_2, \boldsymbol{p}_1)\boldsymbol{p}_1.$$

Now assume that γ is Legendre. Then $R(\boldsymbol{p}_2, \boldsymbol{p}_1)\boldsymbol{p}_1 = c\boldsymbol{p}_2$. Hence

$$\tau_2(\gamma)^\perp = (\kappa'' - \kappa^3 - \kappa + c\kappa)\varphi\boldsymbol{t} + 2\kappa'\xi$$

Thus γ is λ-biminimal if and only if

$$(\kappa'' - \kappa^3 - \kappa + c\kappa)\varphi\boldsymbol{t} + 2\kappa'\xi = 2\lambda\kappa\varphi\, \boldsymbol{t}.$$

From this, we obtain

$$\kappa = \text{constant}, \quad \kappa^2 = c - 1 - 2\lambda.$$

Proposition 5.1. *Let γ be a non-geodesic Legendre curve in $\mathcal{M}^3(c)$. Then γ is λ-biminimal if and only if it is a Legendre helix satisfying $\kappa^2 = c - 1 - 2\lambda$.*

As we obatined in [13], γ is biharmonic if and only if its curvature κ satisfies $\kappa^2 = c - 1$. Thus we obtain

Corollary 5.1. *Let γ be a Legendre curve in $\mathcal{M}^3(c)$. Then γ is biharmonic if and only if it is biminimal.*

5.3

Let $\mathcal{M}^3(c)$ be a Sasakian 3-space form and $\pi : \mathcal{M} \to \overline{\mathcal{M}^2}(c+3)$ its fibering. Take a curve $\bar{\gamma}$ and denote by $S = S_{\bar{\gamma}} = \pi^{-1}\{\bar{\gamma}\}$ the Hopf cylinder over γ. The mean curvature vector field of S is $\mathbb{H} = H\boldsymbol{n}$, $H = (\bar{\kappa} \circ \pi)/2$. Here $\bar{\kappa}$ is the signed curvature of $\bar{\gamma}$. Let us denote by ι the inclusion map of S in $\mathcal{M}^3(c)$. The following formulas were obtained in [13]:

$$\bar{\triangle}_\iota \mathbb{H} = \triangle \mathbb{H}, \quad \mathcal{R}_\iota(\mathbb{H}) = (c+1)H\boldsymbol{n}.$$

Hence

$$\tau_2(\iota)^\perp = 2(H'' - 4H^3 + (c-1)H)\boldsymbol{n}.$$

Since $\tau(\iota)^\perp = 2H\boldsymbol{n}$, S is λ-biminimal if and only if

$$H'' - 4H^3 + (c-1)H = \lambda H.$$

This is rewritten as

$$\bar{\kappa}'' - \bar{\kappa}^3 + \{(c-1) - \lambda\}\bar{\kappa} = 0. \tag{5.5.1}$$

The equation (5.5.1) implies the following results.

Theorem 5.1. *A Hopf cylinder S is (-4)-biminimal if and only if the base curve is biminimal.*

Theorem 5.2. *A Hopf cylinder S is c-biminimal if and only if the base curve is $(c+4)$-biminimal.*

Corollary 5.2. *A Hopf cylinder S in S^3 is (-4)-biminimal if and only if the base curve is biminimal in $S^2(4)$.*

In [14], the following result is obtained.

Theorem 5.3. ([14], Theorem 3.1) *Let $\pi : M^3(c) \to \overline{M^2}(\bar{c})$ be a Riemannian submersion with minimal fibers from a space form of constant sectional curvature c to a surface of constant Gaussian curvature $\bar{c}$. Let $\bar{\gamma} : I \subset \mathbb{R} \to \overline{M^2}$ be a curve parametrized by arc-length. Then $S = \pi^{-1}\{\bar{\gamma}\} \subset M^3$ is a biminimal surface if and only if $\bar{\gamma}$ is a $\bar{c}$-biminimal curve.*

In particular, the base curves of biminimal Hopf cyliders in S^3 are 4-biminimal curves in $S^2(4)$. This result for S^3 can be generalized to Sasakian space forms as follows:

Theorem 5.4. *Let $\mathcal{M}^3(c)$ be a Sasakian 3-space form and $\pi : \mathcal{M}^3(c) \to \overline{\mathcal{M}^2}(\bar{c})$ ($\bar{c} = c+3$), its associated fibering. Let $\bar{\gamma} : I \subset \mathbb{R} \to \overline{\mathcal{M}^2}$ be a curve parametrized by arc-length. Then the Hopf cylinder $S = \pi^{-1}\{\bar{\gamma}\}$ is a biminimal surface if and only if $\bar{\gamma}$ is a $\bar{c}$-biminimal curve.*

Proof. A Hopf cylinder $S_{\bar{\gamma}}$ in $\mathcal{M}^3(c)$ is biminimal if and only if $\kappa'' - \kappa^3 = 0$. This is equivalent to

$$\kappa'' - \kappa^3 + (c+3)\kappa = (c+3)\lambda.$$

Namley, $\bar{\gamma}$ is $(c+3)$-biminimal in the base space form. □

Remark 1. The λ-biminimality is different from $\triangle\mathbb{H} = \lambda\mathbb{H}$ or $\triangle^{\perp}\mathbb{H} = \lambda\mathbb{H}$. In fact, the following results are known.

Proposition 5.2. ([13], Theorem 2.1) *A Hopf cylinder satisfies $\triangle\mathbb{H} = \lambda\mathbb{H}$ if and only if the base curve is a geodesic ($\lambda = 0$) or a Riemannian circle ($\lambda \neq 0$). In the latter case, $\lambda = \bar{\kappa}^2 + 2 > 2$.*

Proposition 5.3. ([13], Theorem 2.3, Corollary 2.2) *A Hopf cylinder satisfies $\triangle^{\perp}\mathbb{H} = \lambda\mathbb{H}$ if and only if the base curve is*

(1) $\lambda = 0$: *geodesic, Riemannian circle or a Riemannian clothoid,*

(2) $\lambda > 0$: $\bar{\kappa}(s) = a\cos(\sqrt{\lambda}s) + b\sin(\sqrt{\lambda}s)$,

(3) $\lambda < 0$: $\bar{\kappa}(s) = a\cosh(\sqrt{-\lambda}s) + b\sinh(\sqrt{-\lambda}s)$.

Add in Proof:

(1) A simply connected Sasakian 3-space form $\mathcal{M}^3(c)$ is isomorphic to one of the following model spaces:

- the special unitary group SU(2) if $c > 1$ or $-3 < c < 1$,
- the unit 3-sphere S^3 if $c = 1$,
- the Heisenberg group Nil if $c = -3$,
- the universal covering group $\widetilde{\mathrm{SL}}_2\mathbb{R}$ of the special linear group $\mathrm{SL}_2\mathbb{R}$ if $c < -3$.

Theorem 5.4 for Nil and $\widetilde{\mathrm{SL}}_2\mathbb{R}$ is obtained independently by Loubeau and Montaldo [15].

(2) In Example 4.1, we showed that the only biharmonic Legendre curves in Sol are Legendre geodesics.

Y.-L. Ou and Z.-P. Wang studied biharmonic curves in Sol. In particular, they showed the nonexistence of proper biharmonic helices in Sol [17]. More generally, Caddeo, Montalod, Oniciuc and Piu [6] showed the non-existenece of proper biharmonic curves in Sol parametrised by arclength.

(3) T. Sasahara [20] classified biminimal Legendre surfaces in 5-dimensional Sasakian space forms.

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