Classification of ξ-Ricci-semisymmetric (κ, μ)- manifolds

Mukut Mani Tripathi

Abstract. It is proved that for a non-Sasakian η-Einstein (κ, μ)-manifold M the following three conditions are equivalent: **(a)** M is flat and 3-dimensional, **(b)** M is Ricci-semisymmetric, and **(c)** M is ξ-Ricci-semisymmetric. Then it is proved that an ξ-Ricci-semisymmetric (κ, μ)-manifold M^{2n+1} is either flat and 3-dimensional, or locally isometric to $E^{n+1} \times S^n(4)$, or an Einstein-Sasakian manifold.

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1 Introduction

A Riemannian manifold M is said to be Ricci-semisymmetric (or Ricci-semiparallel) if its Ricci tensor S is semisymmetric, that is, its curvature tensor R satisfies $R(X,Y) \cdot S = 0$, $X, Y \in TM$, where $R(X,Y)$ acts on S as a derivation. Ricci-semisymmetric Riemannian manifolds are natural generalizations of symmetric spaces ($\nabla R = 0$), Einstein spaces, semi-symmetric spaces ($R(X,Y) \cdot R = 0$) and Ricci-symmetric Riemannian manifolds ($\nabla S = 0$). In [6], V.A. Mirzoyan proved that a Riemannian manifold is Ricci-semisymmetric if and only if it is 2-dimensional or an Einstein space or a semi-Einstein space or locally a product of such spaces. Here, a semi-Einstein space is a Riemannian manifold M such that, for each $p \in M$, the tangent space T_pM decomposes as a direct sum $T_p^{(0)} \oplus T_p^{(1)}$, where $T_p^{(0)}$ is the null space of the curvature tensor and on $T_p^{(1)}$ the Ricci tensor is a nonzero multiple of the metric tensor.

In contact geometry, S. Tanno [11] showed that for a K-contact manifold M the following four conditions are equivalent: **(a)** M is an Einstein manifold, **(b)** M is with parallel Ricci tensor (that is, M is Ricci-symmetric), **(c)** M satisfies $R(X,Y) \cdot S = 0$ (that is, M is Ricci-semisymmetric) and **(d)** M satisfies $R(\xi, X) \cdot S = 0$, where ξ is the structure vector field.

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Since a Sasakian manifold is always a K-contact manifold, therefore this result is also true for a Sasakian manifold. Thus, a Ricci-semisymmetric Sasakian manifold is an Einstein manifold. This generalizes a result of M. Okumura [7], which states that a Ricci-symmetric Sasakian manifold is an Einstein manifold.

Both K-contact manifolds and Sasakian manifolds are special classes of contact metric manifolds. In fact, a contact metric manifold is a K-contact manifold if the structure vector filed ξ is Killing; and is a Sasakian manifold if it is normal. Thus, it is a natural motivation to extend this study in contact metric manifolds.

Since we shall need the condition $R(\xi, X) \cdot S = 0$ too many times, we give the following definition.

Definition 1.1. A contact metric manifold is *ξ-Ricci-semisymmetric* if it satisfies $R(\xi, X) \cdot S = 0$.

In [9], D. Perrone proved the following

Theorem 1.2. [9] *Let M^{2n+1} $(2n+1 \geq 5)$ be an ξ-Ricci-semisymmetric contact metric manifold such that*

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) \tag{1.1}$$

for some function κ on M^{2n+1}, then either M^{2n+1} is locally isometric to the Riemannian product $E^{n+1} \times S^n(4)$ or M^{2n+1} is an Einstein-Sasakian manifold.

In [8], B. J. Papantoniou generalized the above result and proved the following

Theorem 1.3. [8] *Let M^{2n+1} be an ξ-Ricci-semisymmetric contact metric manifold such that*

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{1.2}$$

for some $(\kappa, \mu) \in \mathbb{R}^2$. Then M^{2n+1} is either **(a)** *locally isometric to $E^{n+1} \times S^n(4)$, or* **(b)** *an Einstein-Sasakian manifold, or* **(c)** *an η-Einstein manifold if $\kappa^2 + \mu^2(\kappa - 1) \neq 0$.*

However, when we put $\mu = 0$, in the condition (1.2) of Theorem 1.3, we do not get conclusions of Theorem 1.2 directly. Thus, it is necessary to have a closer look into Theorem 1.3. As a result, in this paper we classify ξ-Ricci-semisymmetric (κ, μ)-manifolds completely.

To achieve our goal, we organize the paper as follows. Section 2 contains a brief introduction to contact metric manifolds, (κ, μ)-manifolds and $N(\kappa)$-contact metric manifolds. Section 3 contains some basic results. In section 4, we give an improved version of Theorem 1.2. Next, in Section 5 we give a brief account of η-Einstein (κ, μ)-manifolds. Then we prove a structure theorem for an ξ-Ricci-semisymmetric non-Sasakian η-Einstein (κ, μ)-manifold. In the last section, using the results of sections 3, 4 and 5 we prove the main result, which is as follows:

Theorem 1.4. *Let M^{2n+1} be an ξ-Ricci-semisymmetric (κ, μ)-manifold. Then one of the following statements is true.*
(a) *M^{2n+1} is flat and 3-dimensional.*
(b) *M^{2n+1} is locally isometric to $E^{n+1} \times S^n(4)$.*
(c) *M^{2n+1} is an Einstein-Sasakian manifold.*

2 Contact metric manifolds

A differentiable 1-form η on a $(2n+1)$-dimensional differentiable manifold M is called a *contact form* if $\eta \wedge (d\eta)^n \neq 0$ everywhere on M, and M equipped with a contact form is a *contact manifold*. Since rank of $d\eta$ is $2n$ on the Grassmann algebra $\bigwedge T_p^* M$ at each point $p \in M$, therefore there exists a unique global vector field ξ, called the *characteristic vector field*, such that

$$\eta(\xi) = 1, \qquad \text{and} \qquad d\eta(\xi, \cdot) = 0. \tag{2.1}$$

Moreover, it is well-known that there exist a $(1,1)$-tensor field φ and a Riemannian metric g such that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(X) = g(X, \xi), \tag{2.2}$$

$$\varphi^2 = -I + \eta \otimes \xi, \quad d\eta(X, Y) = g(X, \varphi Y), \tag{2.3}$$

$$g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y) \tag{2.4}$$

for $X, Y \in TM$. The structure (η, ξ, φ, g) is called a *contact metric structure* and the manifold M endowed with such a structure is said to be a *contact metric manifold*.

The contact metric structure (η, ξ, φ, g) on M gives rise to a natural almost Hermitian structure on the product manifold $M \times \mathbb{R}$. If this structure is integrable, then M is said to be a *Sasakian manifold*. A Sasakian manifold is characterized by the condition

$$\nabla_X \varphi = R_0(\xi, X), \qquad X \in TM, \tag{2.5}$$

where ∇ is Levi-Civita connection and

$$R_0(X, Y) Z = g(Y, Z) X - g(X, Z) Y, \qquad X, Y, Z \in TM.$$

Also, a contact metric manifold M is Sasakian if and only if the curvature tensor R satisfies

$$R(X, Y)\xi = R_0(X, Y)\xi, \quad X, Y \in TM. \tag{2.6}$$

In a contact metric manifold M, the $(1,1)$-tensor field h is defined by half of the Lie derivative of φ in the direction ξ. The tensor field h is symmetric and satisfies

$$h\xi = 0, \;\; h\varphi + \varphi h = 0, \;\; \nabla\xi = -\varphi - \varphi h, \;\; \mathrm{trace}(h) = \mathrm{trace}(\varphi h) = 0. \tag{2.7}$$

The (κ, μ)-*nullity distribution* $N(\kappa, \mu)$ ([2],[8]) of a contact metric manifold M is defined by

$$N(\kappa, \mu) : p \to N_p(\kappa, \mu) = \{Z \in T_p M \mid R(X, Y)Z = (\kappa I + \mu h) R_0(X, Y) Z\}$$

for all $X, Y \in TM$, where $(\kappa, \mu) \in \mathbb{R}^2$. A contact metric manifold M with $\xi \in N(\kappa, \mu)$ is called a (κ, μ)-*manifold*. In this case, we have $h^2 = (\kappa - 1)\varphi^2$. In fact,

(κ,μ)-manifolds exist for all values of $\kappa \leq 1$ and all μ. The class of (κ,μ)-manifolds contains Sasakian manifolds for $\kappa = 1$ and $h = 0$. For $\kappa < 1$, the curvature is completely determined for (κ,μ)-manifolds; in particular, they have constant scalar curvature. Characteristic examples of non-Sasakian (κ,μ)-manifolds are the tangent sphere bundles of Riemannian manifolds of constant sectional curvature not equal to one and certain Lie groups [4]. If the dimension of a contact metric manifold M is greater than three and in the definition of (κ,μ)-manifold we assume that κ and μ are some smooth functions on M independent of the choice of vector fields X and Y, then the functions κ and μ must be constant [5]. If $\mu = 0$, the (κ,μ)-nullity distribution $N(\kappa,\mu)$ is reduced to the *κ-nullity distribution* $N(\kappa)$ [12]. If $\xi \in N(\kappa)$, then we call a contact metric manifold M an *$N(\kappa)$-contact metric manifold* [12]. For more details we refer to [1].

3 Some basic results

For a (κ,μ)-manifold M^{2n+1}, we have

$$S(X,\xi) = 2n\kappa\eta(X), \qquad X \in TM, \tag{3.1}$$

$$R(\xi,X) = R_0(\xi,(\kappa I + \mu h)X) \qquad X \in TM. \tag{3.2}$$

From (3.2) it follows that

$$\eta(R(\xi,X)Y) = \kappa(g(X,Y) - \eta(X)\eta(Y)) + \mu g(hX,Y), \tag{3.3}$$

$$R(\xi,X)\xi = \kappa(\eta(X)\xi - X) - \mu hX. \tag{3.4}$$

From (3.1) and (3.3) we get

$$S(R(\xi,X)Y,\xi) = 2n\kappa(\kappa(g(X,Y) - \eta(X)\eta(Y)) + \mu g(hX,Y)), \tag{3.5}$$

and from (3.1) and (3.4), it follows that

$$S(R(\xi,X)\xi,Y) = 2n\kappa^2\eta(X)\eta(Y) - \kappa S(X,Y) - \mu S(hX,Y). \tag{3.6}$$

Lemma 3.1. *Let M^{2n+1} be an ξ-Ricci-semisymmetric (κ,μ)-manifold. Then*

$$S((\kappa I + \mu h)X,Y) - 2n\kappa g((\kappa I + \mu h)X,Y) = 0. \tag{3.7}$$

Proof. The condition $R(\xi,X)\cdot S = 0$ implies that

$$S(R(\xi,X)Y,\xi) + S(Y,R(\xi,X)\xi) = 0, \tag{3.8}$$

which in view of (3.5) and (3.6) gives (3.7). □

In a non-Sasakian (κ,μ)-manifold M^{2n+1} the Ricci operator Q is given by [2]

$$\begin{aligned} Q &= (2(n-1) - n\mu)I + (2(n-1) + \mu)h \\ &+ (2(1-n) + n(2\kappa + \mu))\eta \otimes \xi. \end{aligned} \tag{3.9}$$

Consequently, the Ricci tensor S and the scalar curvature r are given by

$$\begin{aligned} S(X,Y) &= (2(n-1)-n\mu)\,g(X,Y) \\ &+ (2(n-1)+\mu)\,g(hX,Y) \\ &+ (2(1-n)+n(2\kappa+\mu))\,\eta(X)\,\eta(Y), \end{aligned} \tag{3.10}$$

$$r = 2n(2n-2+\kappa-n\mu). \tag{3.11}$$

From (3.10), we have

$$\begin{aligned} S(hX,Y) &= (2(n-1)-n\mu)\,g(hX,Y) \\ &- (\kappa-1)(2(n-1)+\mu)\,g(X,Y) \\ &+ (\kappa-1)(2(n-1)+\mu)\,\eta(X)\,\eta(Y), \end{aligned} \tag{3.12}$$

where $\eta \circ h = 0$, $h^2 = (\kappa-1)\,\varphi^2$ and (2.4) are used.

We also recall the following three theorems for later use.

Theorem 3.2. ([1], p. 101) *Let M^{2n+1} be a contact metric manifold satisfying $R(X,Y)\xi = 0$. Then, M^{2n+1} is locally isometric to $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

Theorem 3.3. [13] *A Ricci flat (κ,μ)-manifold is flat and 3-dimensional.*

Theorem 3.4. [13] *A non-Sasakian Einstein (κ,μ)-manifold is 3-dimensional and flat.*

The above theorem is a generalization of a result of S. Tanno [12], which states that if an $N(\kappa)$-contact metric manifold of dimension ≥ 5 is Einstein, then it is necessarily Sasakian.

4 $N(\kappa)$-contact metric manifolds

Let M^{2n+1} be a contact metric manifold. If $\mu = 0$, the (κ,μ)-nullity distribution $N(\kappa,\mu)$ is reduced to the *κ-nullity distribution* $N(\kappa)$ [12]. If $\xi \in N(\kappa)$, then we call a contact metric manifold M an *$N(\kappa)$-contact metric manifold*. The condition (1.1) of Theorem 1.2 is the condition for a contact metric manifold to be an $N(\kappa)$-contact metric manifold. If the dimension of a contact metric manifold is greater than three, then in the condition (1.1) of Theorem 1.2 the function κ must be constant [5]. Now, we give an improved version of Theorem 1.2 as follows.

Theorem 4.1. *Let M^{2n+1} be an ξ-Ricci-semisymmetric $N(\kappa)$-contact metric manifold. Then either*
(a) *M^{2n+1} is flat and 3-dimensional, or*
(b) *M^{2n+1} is locally isometric to $E^{n+1} \times S^n(4)$, or*
(c) *M^{2n+1} is an Einstein-Sasakian manifold.*

Proof. Let M^{2n+1} be an ξ-Ricci-semisymmetric $N(\kappa)$-contact metric manifold. Then, in view of Lemma 3.1, we have

$$\kappa \left(S(X,Y) - 2n\kappa g(X,Y)\right) = 0. \tag{4.1}$$

Therefore, either $S = 2n\kappa g$ or $\kappa = 0$. In the first case M^{2n+1} reduces to an Einstein manifold. Therefore in view of Theorem 3.4, we have either the statement **(a)** or the statement **(c)**. If $\kappa = 0$, in view of Theorem 3.2, we get either the statement **(a)** or the statement **(b)**. The converse is straightforward. □

As an immediate consequence of Theorem 4.1, we have Theorem 3 of Sharma and Koufogiorgos [10] as the following

Corollary 4.2. *Let M^{2n+1} $(n > 1)$ be an $N(\kappa)$-contact metric manifold. If M^{2n+1} is Ricci-symmetric then either*
(a) *M^{2n+1} is locally isometric to $E^{n+1} \times S^n(4)$, or*
(b) *M^{2n+1} is an Einstein-Sasakian manifold.*

5 Non-Sasakian η-Einstein (κ, μ)-manifolds

A contact metric manifold M is said to be *η-Einstein* ([7] or see [1] p. 105) if the Ricci operator Q satisfies

$$Q = aI + b\eta \otimes \xi, \tag{5.1}$$

where a and b are some smooth functions on the manifold. In particular if $b = 0$, then M becomes an *Einstein manifold*. In dimensions ≥ 5 it is known that for any η-Einstein K-contact manifold, a and b are constants [11].

In [3], it is proved that a 3-dimensional contact metric manifold is η-Einstein if and only if it is an $N(\kappa)$-contact metric manifold. More precisely, in a 3-dimensional $N(\kappa)$-contact metric manifold, we have

$$Q = \left(\frac{r}{2} - \kappa\right) I + \left(3\kappa - \frac{r}{2}\right) \eta \otimes \xi. \tag{5.2}$$

From (3.9) and (5.1), we see that a non-Sasakian (κ, μ)-manifold M^{2n+1} is η-Einstein if and only if $\mu = -2(n-1)$. In a non-Sasakian η-Einstein (κ, μ)-manifold M^{2n+1}, we have

$$Q = 2\left(n^2 - 1\right) I + 2\left(1 + n\kappa - n^2\right) \eta \otimes \xi, \tag{5.3}$$

$$S = 2\left(n^2 - 1\right) g + 2\left(1 + n\kappa - n^2\right) \eta \otimes \eta, \tag{5.4}$$

$$r = 2n\left(\kappa + 2(n-1)(n+1)\right), \tag{5.5}$$

$$S(hX, Y) = 2\left(n^2 - 1\right) g(hX, Y). \tag{5.6}$$

Example 5.1. A contact metric manifold, obtained by a $\mathcal{D}$-homothetic deformation of the contact metric structure on the tangent sphere bundle of a Riemannian manifold M^{n+1} of constant curvature $\frac{n^2 \pm 2n+1}{n^2-1}$, is a non-Sasakian η-Einstein (κ,μ)-manifold.

Now, we prove the following

Theorem 5.2. *Let M^{2n+1} be a non-Sasakian η-Einstein (κ,μ)-manifold. Then the following conditions are equivalent:*
(a) *M^{2n+1} is flat and 3-dimensional.*
(b) *M^{2n+1} is Ricci-semisymmetric.*
(c) *M^{2n+1} is ξ-Ricci-semisymmetric.*

Proof. Let M^{2n+1} be a non-Sasakian η-Einstein (κ,μ)-manifold. Then **(a)** $\rightarrow$ **(b)** $\rightarrow$ **(c)** is obvious. Now, we assume the condition **(c)**. From (3.5), we get

(5.7) $$S(R(\xi,X)Y,\xi) = 2n\kappa^2(g(X,Y)-\eta(X)\eta(Y)) - 4n(n-1)\kappa g(hX,Y).$$

In view of (5.4) and (3.6), we get

(5.8) $$\begin{aligned} S\left(R\left(\xi,X\right)\xi,Y\right) &= -2\left(n^2-1\right)\kappa\left(g\left(X,Y\right)-\eta(X)\eta(Y)\right) \\ &+ 4\left(n-1\right)\left(n^2-1\right)g\left(hX,Y\right). \end{aligned}$$

If M satisfies $R\left(\xi,X\right)\cdot S=0$, it follows that

$$S\left(R\left(\xi,X\right)Y,\xi\right)+S\left(Y,R\left(\xi,X\right)\xi\right)=0,$$

which in view of (5.7) and (5.8) gives

(5.9) $$\begin{aligned} 0 &= 2\left(1+n\kappa-n^2\right)\kappa\left(g\left(X,Y\right)-\eta(X)\eta(Y)\right) \\ &- 4\left(n-1\right)\left(1+n\kappa-n^2\right)g\left(hX,Y\right). \end{aligned}$$

Contracting the above equation and using $\operatorname{trace}(h)=0$, we get

(5.10) $$4n\left(1+n\kappa-n^2\right)\kappa=0.$$

Then, in view of (5.10), we get either $1+n\kappa-n^2=0$ or $\kappa=0$. If $1+n\kappa-n^2=0$, in view of (5.4) M^{2n+1} reduces to an Einstein manifold. Therefore in view of Theorem 3.4, we get the condition **(a)**. If $\kappa=0$, then from (5.9), we get

$$4\left(n-1\right)^2\left(n+1\right)g\left(hX,Y\right)=0.$$

Then either $n=1$ or $h=0$. If $n=1$, we again get the condition **(a)**. Since for a (κ,μ)-manifold, the conditions of being a Sasakian manifold, a K-contact manifold, $\kappa=1$ and $h=0$ are all equivalent; therefore $h=0$ is not possible. This completes the proof. □

6 ξ-Ricci-semisymmetric (κ,μ)-manifolds

In this section we prove our main theorem as follows:

Proof of Theorem 1.4. Let M be an ξ-Ricci-semisymmetric (κ, μ)-manifold of dimension $(2n+1)$. We have following cases.

Case 1. Let $\mu = 0$. In view of Theorem 4.1, we have one of the statements **(a)**, **(b)** and **(c)**.

Case 2. Let $\mu \neq 0$ and $\kappa = 1$. Since a Ricci-semisymmetric Sasakian manifold is an Einstein manifold, in this case we have the statement **(c)**.

Case 3. Let $\mu \neq 0$, $\kappa = 0$. Then from (3.7) and $h^2 = -\varphi^2$, we get $S = 0$. Now, in view of Theorem 3.3 we get $\mu = 0$, which is a contradiction. Thus, the **Case 3** is not possible.

Case 4. Let $\mu \neq 0$, $0 \neq \kappa < 1$. After eliminating $g(hX, Y)$ and $S(hX, Y)$ from (3.10), (3.7) and (3.12); we get

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$$

for some suitable a and b. Thus, M^{2n+1} is a non-Sasakian η-Einstein (κ, μ)-manifold. Then in view of Theorem 5.2, we have $n = 1$ and $\mu = -2(n-1) = 0$, which is a contradiction. Thus the **Case 4** is not possible. □

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Author's address:

Mukut Mani Tripathi
Department of Mathematics and Astronomy,
Lucknow University, Lucknow 226 007, India.
email: mmtripathi66@yahoo.com