Fields associated to Lagrangian dynamical systems

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Abstract. The general theory of uperfields gives us, in particular, fields associated to classical Lagrangean systems. We get a unitary theory.

Mathematics Subject Classification: 53B40, 53B50, 70F20, 70H03, 70H25, 78A25.
Key words: Dynamical Sistems, Mechanics, fields, waves.

I. Uperfields associated to Newtonian dynamical systems

1.1. The Lagrange form. The Lorentz conditions

Let us consider a Newtonian dynamical system described by:

$$\ddot{q}^i = F^i(t, q, \dot{q}),\ (\mathbf{m} = 1),\ (\dot{q}^i = v^i,\ \dot{v}^i = F^i(t, q, v)) \tag{1.1}$$

We associate to it the equivalent system of equations:

$$\begin{aligned} \theta^i &= dq^i - v^i dt = 0, \\ \psi^i &= dv^i - F^i(t, q, v)dt = 0 \end{aligned} \tag{1.1'}$$

and the Lagrange-Gallissot 2-form:

$$\Omega_G = dv^i \wedge dq^i + (F^i dq^i - v^i dv^i) \wedge dt. \tag{1.2}$$

The characteristics of the form Ω_G are the trajectories of the dynamical system (1.1).

In general, a 2-form:

$$\Omega = A_{ij} dv^i \wedge dq^j + (E_i dq^i - P_i dv^i) \wedge dt + \frac{1}{2} B_{ij} dq^i \wedge dq^j + \frac{1}{2} Q_{ij} dv^i \wedge dv^j, \tag{1.2'}$$

with:

$$det \begin{pmatrix} B_{ij} & -A_{ji} \\ A_{ij} & Q_{ij} \end{pmatrix} \neq 0, \tag{1.3}$$

have the characteristics given by:

Balkan Journal of Geometry and Its Applications, Vol.11, No.1, 2006, pp. 107-115.

$$\begin{aligned} B_{ij}dq^j - A_{ji}dv^j + E_i dt = 0,\\ A_{ij}dq^j + Q_{ij}dv^j - P_i dt = 0. \end{aligned} \tag{1.4}$$

Proposition 1. *The characteristics of the* 2*-form* (1.2′) *are the trajectories of the system* (1.1) *if and only if the* **Lorentz conditions***:*

$$\begin{aligned} B_{ij}v^j - A_{ji}F^j + E_i = 0,\\ A_{ij}v^j + Q_{ij}F^j - P_i = 0 \end{aligned} \tag{1.4′}$$

hold.

It follows by (1.1) and (1.4).

The 2-forms (1.2′) which satisfy the condition (1.3) and the Lorentz conditions (1.4′), are called *equivalent* (they admit the same characteristics). There equivalent class is a dynamical notion.

1.2. The behaviour of the coefficients of the form Ω on a change of local chart on the base manifold M

On a change of local chart on M and respectively a change of vectorial chart on TM defined by:

$$\begin{aligned} &\bar{q}^i = \bar{q}^i(q^h), \quad d\bar{q}^i = \frac{\partial \bar{q}^i}{\partial q^h} dq^h,\\ &\bar{v}^i = \frac{\partial \bar{q}^i}{\partial q^h} v^h, \quad d\bar{v}^i = \frac{\partial \bar{q}^i}{\partial q^h} dv^h + \frac{\partial \bar{v}^i}{\partial q^h} dq^h, \end{aligned} \tag{1.5}$$

the coefficients of the form Ω change by the rules:

$$\begin{pmatrix} \overline{B}_{hk} & -\overline{A}_{kh} & \overline{E}_h\\ \overline{A}_{hk} & \overline{Q}_{hk} & -\overline{P}_h\\ -\overline{E}_k & \overline{P}_k & 0 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial q^i}{\partial \bar{q}^h} & \dfrac{\partial v^i}{\partial \bar{q}^h} & 0\\ 0 & \dfrac{\partial v^i}{\partial \bar{v}^h} & 0\\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B_{ij} & -A_{ji} & E_i\\ A_{ij} & Q_{ij} & -P_i\\ -E_j & P_j & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial q^j}{\partial \bar{q}^k} & 0 & 0\\ \dfrac{\partial v^j}{\partial \bar{q}^k} & \dfrac{\partial v^j}{\partial \bar{v}^k} & 0\\ 0 & 0 & 1 \end{pmatrix}$$

By this relation follows:

$$\begin{aligned} &\overline{P}_h = \frac{\partial q^i}{\partial \bar{q}^h} P_i, \quad \overline{E}_h = \frac{\partial q^i}{\partial \bar{q}^h} E_i - \frac{\partial v^i}{\partial \bar{q}^h} P_i, \quad \overline{Q}_{hk} = \frac{\partial q^i}{\partial \bar{q}^h} \frac{\partial q^j}{\partial \bar{q}^k} Q_{ij},\\ &\overline{A}_{hk} = \frac{\partial q^i}{\partial \bar{q}^h} \frac{\partial q^j}{\partial \bar{q}^k} A_{ij} + \frac{1}{2}\left(\frac{\partial q^i}{\partial \bar{q}^h}\frac{\partial v^j}{\partial \bar{q}^k} - \frac{\partial q^j}{\partial \bar{q}^h}\frac{\partial v^i}{\partial \bar{q}^k}\right) Q_{ij},\\ &\overline{B}_{hk} = \frac{\partial q^i}{\partial \bar{q}^h}\frac{\partial q^j}{\partial \bar{q}^k} B_{ij} + \left(\frac{\partial v^i}{\partial \bar{q}^h}\frac{\partial q^j}{\partial \bar{q}^k} - \frac{\partial v^i}{\partial \bar{q}^k}\frac{\partial q^j}{\partial \bar{q}^h}\right) A_{ij} + \frac{1}{2}\left(\frac{\partial v^i}{\partial \bar{q}^h}\frac{\partial v^j}{\partial \bar{q}^k} - \frac{\partial v^j}{\partial \bar{q}^h}\frac{\partial v^i}{\partial \bar{q}^k}\right) Q_{ij}. \end{aligned} \tag{1.6}$$

1.3. The Maxwell's principle.

We say that a 2-form (1.2′) for which the condition (1.3) holds, satisfies the Maxwell's principle ([4]) if it is closed. By $d\Omega = 0$, the Maxwell's equations:

$$
\begin{aligned}
&\frac{\partial A_{hi}}{\partial q^j} - \frac{\partial A_{hj}}{\partial q^i} + \frac{\partial B_{ij}}{\partial v^h} = 0, \quad \frac{\partial A_{jh}}{\partial v^i} - \frac{\partial A_{ih}}{\partial v^j} + \frac{\partial Q_{ij}}{\partial q^h} = 0,\\
&\frac{\partial A_{ij}}{\partial t} + \frac{\partial E_j}{\partial v^i} + \frac{\partial P_i}{\partial q^j} = 0, \qquad \frac{\partial B_{ij}}{\partial t} + \frac{\partial E_j}{\partial q^i} - \frac{\partial E_i}{\partial q^j} = 0,\\
&\frac{\partial Q_{ij}}{\partial t} + \frac{\partial P_i}{\partial v^j} - \frac{\partial P_j}{\partial v^i} = 0, \quad \sum_{(i,j,h)} \frac{\partial B_{ij}}{\partial q^h} = 0, \quad \sum_{(i,j,h)} \frac{\partial Q_{ij}}{\partial v^h} = 0
\end{aligned}
\tag{1.7}
$$

follow.

The coefficients A_{ij}, B_{ij}, Q_{ij}, E_i, P_i, which satisfy the condition (1.3), the Lorentz conditions (1.4) and the Maxwell's equations (1.7), are called *coefficients of the uperfield*, the form Ω is called *uperfield form*.

Given a solution of the equations (1.7), any other system of functions is a solution if and only if the difference between the two solutions is a first integral ([2]). For given initial conditions, the solution of the equations (1.7) exists and it is unique ([2]).

II. Field theory

2.1. The canonical isomorphism

Given the dynamical system (1.1), we associate to it the Lagrange 2-form (1.2′).

Let us consider the non-singular matrix Δ with its inverse Δ^{-1}:

$$
\Delta = \begin{pmatrix} B_{ij} & -A_{ji} \\ A_{ij} & Q_{ij} \end{pmatrix}, \quad \Delta^{-1} = \begin{pmatrix} Q^{ij} & A^{ij} \\ -A^{ji} & B^{ij} \end{pmatrix}. \tag{2.1}
$$

By $\Delta \cdot \Delta^{-1} = I$ we obtain the relations:

$$
\begin{aligned}
&B_{ij}Q^{jh} + A^{hj}A_{ji} = \delta_i^h, \quad B_{ij}A^{jh} + B^{hj}A_{ji} = 0,\\
&A_{ij}A^{jh} + Q_{ij}B^{jh} = \delta_i^h, \quad A_{ij}Q^{jh} + A^{hj}Q_{ji} = 0,
\end{aligned}
\tag{2.1′}
$$

which define the components of the inverse matrix.

We can now define a natural isomorphism, denoted by $\hat{\Delta}$, $\hat{\Delta} : T(\mathbf{R} \times TM) \to T^*(\mathbf{R} \times TM)$, locally expressed by:

$$
\hat{\Delta}=(\Delta_{ab})=\begin{pmatrix} B_{ij} & -A_{ji} & E_i \\ A_{ij} & Q_{ij} & -P_i \\ -E_j & P_j & 1 \end{pmatrix} : \begin{pmatrix} X^i \\ Y^i \\ Z \end{pmatrix} \to \begin{pmatrix} B_{ij} & -A_{ji} & E_i \\ A_{ij} & Q_{ij} & -P_i \\ -E_j & P_j & 1 \end{pmatrix} \begin{pmatrix} X^j \\ Y^j \\ Z \end{pmatrix} = \begin{pmatrix} a_i \\ b_i \\ c \end{pmatrix}
$$

$(a, b = \overline{1, 2m+1})$, where:

$$
a_i = B_{ij}X^j - A_{ji}Y^j + E_iZ, \quad b_i = A_{ij}X^j + Q_{ij}Y^j - P_iZ, \quad c = -E_iX^i + P_iY^i + Z.
$$

Proposition 2. *The necessary and sufficient condition so that the non-autonomous semispray* $S = \dfrac{\partial}{\partial t} + v^i \dfrac{\partial}{\partial q^i} + F^i \dfrac{\partial}{\partial v^i}$ *to become, by the canonical isomorphism, dt, ($\Delta(S) = dt$), is that the Lorentz conditions to be satisfied.*

Indeed, this follows by:

$$
\hat{\Delta} : \begin{pmatrix} v^i \\ F^i \\ 1 \end{pmatrix} \to \begin{pmatrix} B_{ij} & -A_{ji} & E_i \\ A_{ij} & Q_{ij} & -P_i \\ -E_j & P_j & 1 \end{pmatrix} \begin{pmatrix} v^j \\ F^j \\ 1 \end{pmatrix} = \begin{pmatrix} B_{ij}v^j - A_{ji}F^j + E_i \\ A_{ij}v^j + Q_{ij}F^j - P_i \\ -E_jv^j + P_jF^j + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

2.2 The uperfield equations. The waves equation

1. The volume form

On $\mathbf{R} \times TM$ we build the following volume form:

$$\eta = \sqrt{| \, det\Delta \, |} \; dq^1 \wedge \cdots \wedge dq^m \wedge dv^1 \wedge \cdots \wedge dv^m \wedge dt,$$

where the matrix Δ is defined by (2.1).

2. Differential operators

a) The Hodge-de Rham adjunction operator is defined, for any p-form $\alpha \in \Lambda^p(\mathbf{R} \times TM)$, by associating to it a $(n-p)$-form $*\alpha \in \Lambda^{n-p}(\mathbf{R} \times TM)$, where $n = 2m+1$, given by:

$$(*\alpha)(X_1, \ldots, X_{n-p})\eta = \alpha \wedge \Delta X_1 \wedge \cdots \wedge \Delta X_{n-p}.$$

Locally, this operator is expressed by:

$$(*\alpha)_{a_{p+1},\ldots,a_n} = \frac{1}{p!}\eta_{a_1,\ldots,a_p,a_{p+1},\ldots,a_n}\alpha^{a_1,\ldots,a_p},$$

where $\alpha^{a_1,\ldots,a_p} = \Delta^{a_1 b_1} \ldots \Delta^{a_p b_p}\alpha_{b_1,\ldots,b_p}$, $(a_i, b_i = \overline{1,n})$, are the contravariant components of the p-form α.

b) The codifferentiation operator is given by:

$$\delta = (-1)^{p+1} *^{-1} d \, * .$$

c) The Laplace d'Alembert operator is defined by:

$$\Box = \delta d + d\delta \; : \; \alpha \to \Box\alpha.$$

3. The propagation equation of the uperfield (the waves equation)

If Ω is the (closed) Lagrange 2-form, the propagation equation of the uperfield is:

$$\Box\Omega = d\delta\Omega.$$

Remarks. By the rules of transformations (1.6) of the components of the 2-form Ω, follows that the functions Q_{ij} and P_i give us the distinguished objects: $P = P_i dq^i$ (a covector - the impulse form) and $Q = \frac{1}{2}Q_{ij}dq^i \wedge dq^j$ (a 2-form). The functions P_i and E_i are the components of a covector $\varepsilon = E_i dq^i - P_i dv^i$ on TM (parameterized by t); $\pi = Q - P \wedge dt$ is a 2-d-form on $\mathbf{R} \times M$.

We have the following special cases:

a. If $m = 2p$ and $det(Q_{ij}) \neq 0$, the 2-form $Q = \frac{1}{2}Q_{ij}dq^i \wedge dq^j$ has the property that $\eta = Q^p = \sqrt{| \, det(Q_{ij}) \, |} \; dq^1 \wedge \cdots \wedge dq^m$ is a distinguished volume form on M, parameterized by t.

b. If $m = 2p+1$ and $det \begin{pmatrix} Q_{ij} & -P_i \\ P_j & 0 \end{pmatrix} \neq 0$, the 2-form $\pi = Q - P \wedge dt$ has the property that $\eta_t = \pi^{p+1} = \sqrt{\left| det \begin{pmatrix} Q_{ij} & -P_i \\ P_j & 0 \end{pmatrix} \right|} \, dq^1 \wedge \cdots \wedge dq^m \wedge dt$ is a distinguished volume form on $\mathbf{R} \times M$.

2.3 The transformations of the uperfield coefficients on a family of differentiable changes of chart

Let us consider a change of chart on the manifold $\mathbf{R} \times TM$, given by:

$$\begin{aligned} &\bar{q}^i = \bar{q}^i(t, q^h), && d\bar{q}^i = \frac{\partial \bar{q}^i}{\partial q^h} dq^h + \frac{\partial \bar{q}^i}{\partial t} dt, \\ &\bar{v}^i = \frac{\partial \bar{q}^i}{\partial q^h} v^h + \frac{\partial \bar{q}^i}{\partial t}, && d\bar{v}^i = \frac{\partial \bar{q}^i}{\partial q^h} dv^h + \frac{\partial \bar{v}^i}{\partial q^h} dq^h + \frac{\partial \bar{v}^i}{\partial t} dt. \end{aligned} \tag{2.2}$$

The coefficients of the uperfield change by the rules:

$$\begin{pmatrix} \overline{B}_{hk} & -\overline{A}_{kh} & \overline{E}_h \\ \overline{A}_{hk} & \overline{Q}_{hk} & -\overline{P}_h \\ -\overline{E}_k & \overline{P}_k & 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial q^i}{\partial \bar{q}^h} & \frac{\partial v^i}{\partial \bar{q}^h} & 0 \\ 0 & \frac{\partial v^i}{\partial \bar{v}^h} & 0 \\ \frac{\partial q^i}{\partial t} & \frac{\partial v^i}{\partial t} & 1 \end{pmatrix} \begin{pmatrix} B_{ij} & -A_{ji} & E_i \\ A_{ij} & Q_{ij} & -P_i \\ -E_j & P_j & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial q^j}{\partial \bar{q}^k} & 0 & \frac{\partial q^j}{\partial t} \\ \frac{\partial v^j}{\partial \bar{q}^k} & \frac{\partial v^j}{\partial \bar{v}^k} & \frac{\partial v^j}{\partial t} \\ 0 & 0 & 1 \end{pmatrix}$$

By these relations follows that the functions A_{ij}, B_{ij} and Q_{ij} change by the same rules as given in (1.6) and the functions E_i and P_i change by the rules:

$$\begin{aligned} \overline{P}_h &= \frac{\partial q^i}{\partial \bar{q}^h} \left(P_i - \frac{\partial q^j}{\partial t} A_{ij} - \frac{\partial v^j}{\partial t} Q_{ij} \right), \\ \overline{E}_h &= \frac{\partial q^i}{\partial \bar{q}^h} E_i - \frac{\partial v^i}{\partial \bar{q}^h} P_i + \frac{\partial q^i}{\partial \bar{q}^h} \frac{\partial q^j}{\partial t} B_{ij} + \\ &+ \left(\frac{\partial v^i}{\partial \bar{q}^h} \frac{\partial q^j}{\partial t} - \frac{\partial q^j}{\partial \bar{q}^h} \frac{\partial v^i}{\partial t} \right) A_{ij} + \frac{\partial v^i}{\partial \bar{q}^h} \frac{\partial v^j}{\partial t} Q_{ij}. \end{aligned} \tag{2.3}$$

III. Fields associated to classical Lagrangean dynamical systems

Proposition 3. *A necessary condition so that a second order dynamical system to be Lagrangean is that the system is written in main form:*

$$F_k = A_{ki}(t, q, v)\dot{v}^i + B_k(t, q, v) = 0, \ (v = \dot{q}). \tag{3.1}$$

The system is non-degenerated if the property:

$$det \left(\frac{\partial F_i}{\partial \dot{v}^j} \right) = det(A_{ij}) \neq 0$$

holds.

Such a system of equations, which is a model of the dynamics of a real system, is not unique. We say that two systems $F_i = 0$ and $G_i = 0$ are equivalent if they admit the same solutions. We consider the class of systems $G_i = F_j D_i^j = 0$, where $D_i^j = D_i^j(t, q, v)$, $det(D_i^j) \neq 0$, $\forall (t, q, v) \in \mathbf{R} \times TM$.

Without loosing the generality, we can assume that the system we consider from the above equivalence class, satisfies $A_{ij} = A_{ji}$.

The functions A_{ij} are the components of a d-metric on M, they change, on a change of local chart, by the rules:

$$\overline{A}_{hk} = \frac{\partial q^i}{\partial \bar{q}^h} \frac{\partial q^j}{\partial \bar{q}^k} A_{ij}.$$

If (A^{ij}) is the inverse of the matrix (A_{ij}) and if we multiply (3.1) by A^{hi}, we obtain the equations (1.1), where:

$$F^i = -A^{ij} B_j.$$

The coefficients $-B_i = A_{ij} F^j$ are the covariant components of the field of force F^i, with respect to the metric given by A_{ij}. We have:

Proposition 4. *A dynamical system* (3.1) *is Lagrangean if and only if it is self-adjoint ([3]).*

The self-adjointness conditions are:

$$\begin{aligned}
&det(A_{ij}) \neq 0,\ A_{ij} = A_{ji},\ \frac{\partial A_{ik}}{\partial v^j} = \frac{A_{jk}}{\partial v^i}, \\
&\frac{\partial B_i}{\partial v^j} + \frac{\partial B_j}{\partial v^i} = 2\left(\frac{\partial}{\partial t} + v^k \frac{\partial}{\partial q^k}\right) A_{ij}, \\
&\frac{\partial B_i}{\partial q^j} - \frac{\partial B_j}{\partial q^i} = \frac{1}{2}\left(\frac{\partial}{\partial t} + v^k \frac{\partial}{\partial q^k}\right)\left(\frac{\partial B_i}{\partial v^j} - \frac{\partial B_j}{\partial v^i}\right).
\end{aligned} \tag{3.2}$$

Proposition 5. *A dynamical system* (1.1) *is equivalent with a selfadjoint (Lagrangean) system if and only if the Lagrange* 2*-form* (1.2′) *has the components* $Q_{ij} \equiv 0$ *([2]).*

Indeed, in this case, the Maxwell's equations (1.7) become the Helmholtz equations (3.2).

Let us associate to the system (3.1) the Pfaff forms:

$$\begin{aligned}
\theta^i &= dq^i - v^i dt, \\
\psi_i &= A_{ij} dv^j + B_i dt.
\end{aligned} \tag{3.1′}$$

The corresponding Lagrange 2-form is:

$$\Omega = \psi_i \wedge \theta^i + \frac{1}{2} B_{ij} \theta^i \wedge \theta^j, \tag{3.3}$$

where the coefficients B_{ij} are, for the moment, arbitrary.

We have:

$$\Omega = A_{ij} dv^i \wedge dq^j + (E_i dq^i - P_i dv^i) \wedge dt + \frac{1}{2} B_{ij} dq^i \wedge dq^j. \tag{3.3′}$$

The Lorentz conditions (1.4′) become:

$$\begin{aligned}
E_i + B_{ij} v^j &= -B_i, \\
P_i - A_{ij} v^j &= 0.
\end{aligned} \tag{3.4}$$

The characteristics of the above 2-form are the trajectories of the system (3.1). By the Maxwell's equations we obtain the coefficients B_{ij}.

The first set of the relations (3.4) tells us that the field of force of components $-B_i$ is a Lorentz field with respect to the coefficients of the field (E_i, B_{ij}), the field of force being $F^i = A^{ih}(E_h + B_{hk}v^k)$.

The second set of the relations (3.4) tells us that the functions P_i are the covariant components of the field of velocities with respect to the d-metric A_{ij}. We call them *impulse functions*.

The field of force F can be considered of mechanical (newtonian) nature as being contravariant (spray), or as a Lorentz field of force, of electromagnetic nature as being covariant.

We have the properties: $det(A_{ij}) \neq 0$ and $A_{ij} = A_{ji}$. The Maxwell's equations are:

$$\begin{aligned}
&\frac{\partial A_{ih}}{\partial v^j} - \frac{\partial A_{jh}}{\partial v^i} = 0, \quad \frac{\partial B_{ij}}{\partial v^h} + \frac{\partial A_{hi}}{\partial q^j} - \frac{\partial A_{hj}}{\partial q^i} = 0,\\
&\frac{\partial A_{ij}}{\partial t} + \frac{\partial E_j}{\partial v^i} + \frac{\partial P_i}{\partial q^j} = 0, \quad \sum_{(i,j,h)} \frac{\partial B_{ij}}{\partial q^h} = 0,\\
&\frac{\partial B_{ij}}{\partial t} + \frac{\partial E_j}{\partial q^i} - \frac{\partial E_i}{\partial q^j} = 0, \quad \frac{\partial P_j}{\partial v^i} - \frac{\partial P_i}{\partial v^j} = 0.
\end{aligned} \tag{3.5}$$

By the second Lorentz relation (3.4), the last Maxwell equation and the property $A_{ij} = A_{ji}$ follows the first relation of (3.5). The forth and fifth relations of (3.5) can be written as: $rot\ E + \dfrac{\partial B}{\partial t} = 0$, $div\ B = 0$; they are the well-known Maxwell equations for the electric field E_i and the magnetic induction B_{ij}.

The magnetic induction is connected to the metric by the second relation (3.5) and the electric field is connected to the metric by the third relation (3.5) and the second Lorentz condition.

We call the subset (E_i, B_{ij}) *field*.

The transformation rules (1.6) of the coefficients of field become:

$$\begin{aligned}
&\overline{P}_h = \frac{\partial q^i}{\partial \bar{q}^h} P_i, \quad \overline{E}_h = \frac{\partial q^i}{\partial \bar{q}^h} E_i - \frac{\partial v^i}{\partial \bar{q}^h} P_i,\\
&\overline{A}_{hk} = \frac{\partial q^i}{\partial \bar{q}^h} \frac{\partial q^j}{\partial \bar{q}^k} A_{ij},\\
&\overline{B}_{hk} = \frac{\partial q^i}{\partial \bar{q}^h} \frac{\partial q^j}{\partial \bar{q}^k} B_{ij} + \left(\frac{\partial v^i}{\partial \bar{q}^h} \frac{\partial q^j}{\partial \bar{q}^k} - \frac{\partial v^i}{\partial \bar{q}^k} \frac{\partial q^j}{\partial \bar{q}^h} \right) A_{ij}.
\end{aligned} \tag{3.6}$$

Remarks. In this special case, the uperfield has two components:

1. A geometric component which contains a d-covector: the impulse P_i and a non-degenerated and symmetric two times covariant second order d-tensor: A_{ij}. They endow the configuration space with a Lagrangean structure.

2. A component of "field" (E_i, B_{ij}) which satisfies the classical electromagnetic field equations.

3. The field (E_i, B_{ij}) and the metric A_{ij} allow us to build a (classical) field theory, where the Maxwell's equations hold.

4. Together, the coefficients E_i and P_i can be considered as a field of covectors on TM: $E_i dq^i - P_i dv^i$. On the evolution, this Pfaff form lead us to a conservation law.

5. The field of force F_i, considered as a semispray, let us to associate a nonlinear connection to the structure of the space.

6. By the rules of transformations (3.6) of the coefficients of the form Ω, on a change of local chart on M (respectively on TM), follows that by the uperfield's interpretation we obtain a unitary electro-gravitational theory.

7. This point of view holds if we consider a change of coordinates on $\mathbf{R} \times TM$ (2.2). In this case we have the relations:

$$
\begin{aligned}
\overline{P}_h &= \frac{\partial q^i}{\partial \bar{q}^h}\left(P_i - \frac{\partial q^j}{\partial t}A_{ij}\right),\\
\overline{E}_h &= \frac{\partial q^i}{\partial \bar{q}^h}E_i - \frac{\partial v^i}{\partial \bar{q}^h}P_i + \frac{\partial q^i}{\partial \bar{q}^h}\frac{\partial q^j}{\partial t}B_{ij} + \left(\frac{\partial v^i}{\partial \bar{q}^h}\frac{\partial q^j}{\partial t} - \frac{\partial q^j}{\partial \bar{q}^h}\frac{\partial v^i}{\partial t}\right)A_{ij}.
\end{aligned}
\tag{3.6'}
$$

8. By the relations (2.1′), if $Q_{ij} = 0$, follows for the components of the inverse matrix Δ^{-1} of the field that: (A^{ij}) is the inverse of the matrix (A_{ij}) and the functions B^{ij} are the contravariant components of B_{ij}, lifted with the components of the inverse matrix (A^{ij}).

IV. The case of a dynamical system given by the Lagrange function

Given the Lagrange function $L = L(t, q, v)$, the Euler-Lagrange equations are:

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad (dq^i = v^i dt).
$$

We have:

$$
\frac{\partial^2 L}{\partial v^i \partial v^j}\dot{v}^j + \frac{\partial^2 L}{\partial v^i \partial q^j}v^j + \frac{\partial^2 L}{\partial v^i \partial t} - \frac{\partial L}{\partial q^i} = 0.
$$

This system is written in the main form (3.1), where:

$$
\begin{aligned}
A_{ij} &= \frac{\partial^2 L}{\partial v^i \partial v^j},\\
B_i &= \frac{\partial^2 L}{\partial v^i \partial q^j}v^j + \frac{\partial^2 L}{\partial v^i \partial t} - \frac{\partial L}{\partial q^i}.
\end{aligned}
$$

The Lagrange form is:

$$
\begin{aligned}
\Omega_L = {}& \frac{\partial^2 L}{\partial v^i \partial v^j}dv^i \wedge dq^j + \frac{1}{2}\left(\frac{\partial^2 L}{\partial q^i \partial v^j} - \frac{\partial^2 L}{\partial v^i \partial q^j}\right)dq^i \wedge dq^j\\
&+ \left[\left(\frac{\partial L}{\partial q^i} - v^h\frac{\partial^2 L}{\partial v^h \partial q^i} - \frac{\partial^2 L}{\partial v^i \partial t}\right)dq^i - v^h\frac{\partial^2 L}{\partial v^h \partial v^i}dv^i\right] \wedge dt.
\end{aligned}
$$

In this case, it follows for the components of the field the expressions: $A_{ij} = \dfrac{\partial^2 L}{\partial v^i \partial v^j}$ (*the metric*), $B_{ij} = \dfrac{\partial^2 L}{\partial q^i \partial v^j} - \dfrac{\partial^2 L}{\partial v^i \partial q^j}$ (*the magnetic induction*), $P_i = \dfrac{\partial^2 L}{\partial v^i \partial v^j}v^j$ (*the* *impulse*), $E_i = \dfrac{\partial L}{\partial q^i} - \dfrac{\partial^2 L}{\partial v^i \partial t} - v^h \dfrac{\partial^2 L}{\partial v^h \partial q^i}$ (*the electric field*). Obviously, this values satisfy the Maxwell's equations.

By the Lagrange equations, the field of force with the Lorentz expression: $F^i = A^{ih}(E_h + B_{hk} v^k)$, lead us to: $F_i = \dfrac{\partial L}{\partial q^i} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial v^i}\right) + F^h A_{hi} = A_{ih} F^h$, which show that the Lorentz force F_i is the covariant expression of the Newtonian field F^i, with respect to the canonical d-metric A_{ij}.

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