Null 2-type space-like submanifolds of E^5_t with normalized parallel mean curvature vector

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Abstract. The purpose of this article is to classify 3-dimensional null 2-type space-like submanifolds of the pseudo-Euclidean space E^5_t which are constructed from the eigenfunctions of the Laplacian with two eigenvalues 0 and nonzero constant λ, under certain assumptions.

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1 Introduction

A connected submanifold M^n of a pseudo-Euclidean space E^m_t is called of finite type if its position vector field x can be written as a sum of eigenfunctions of its Laplacian; more precisely, M^n is said to be of finite k-type if its position vector field x admits the following spectral decomposition

$$x = x_0 + x_1 + \cdots + x_k, \tag{1.1}$$

where $\Delta x_i = \lambda_i x_i$, $i = 1, 2, \ldots, k$, $\lambda_1 < \cdots < \lambda_k$, x_0 is a constant vector in E^m_t and $x_1, \ldots, x_k$ are non-constant E^m_t-valued maps on M^n. If one of the eigenvalues λ_i vanishes, then M^n is said to be of null k-type (see [1, 2] for detail). We can choose a coordinate system on E^m_t with x_0 as its origin. Then we have the following simple spectral decomposition of x for a null 2-type submanifold M:

$$x = x_1 + x_2, \quad \Delta x_1 = 0, \ \ \Delta x_2 = \lambda x_2. \tag{1.2}$$

In [4, 5], B.Y. Chen gave a classification of null 2-type surfaces in the Euclidean space E^3 and E^4. He proved that circular cylinders and helical cylinders are the only surfaces of null 2-type in E^3 and E^4, respectively. In [5], he also proved that a surface M in the Euclidean space E^4 is of null 2-type with parallel normalized mean curvature vector if and only if M is an open portion of a circular cylinder in a hyperplane of E^4. However, in [12], S.J. LI showed that a surface M in E^m with parallel normalized

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mean curvature vector is of null 2-type if and only if M is an open portion of a circular cylinder.

Later, in [6], B.Y. Chen and H. Song proved that a space-like surface M in $E^4_t,\ (t = 1, 2)$ is of null 2-type with constant mean curvature if and only if M is an open portion of a helical cylinder of the first kind or a helical cylinder of the second kind in $E^4_t,\ (t = 1, 2)$.

Also, in [11], D.S. Kim and Y.H. Kim gave complete classification theorems on null 2-type surfaces in Minkowski space E^4_1. They proved that a Lorentzian surface M in E^4_1 is of null 2-type with constant mean curvature if and only if M is an open portion of a helical cylinder of third kind, a helical cylinder of fourth kind, an extended B-scroll or a cylinder $E^1_1 \times S^1(r),\ S^1_1(r) \times E$.

In the case of the classification of hypersurfaces, the constancy of the mean curvature does not provide enough information to obtain a characterization of null 2-type hypersurfaces of Euclidean spaces and Lorentzian spaces. In [9, 10], A. Ferrandez and P. Lucas studied null 2-type hypersurfaces of Euclidean spaces and null 2-type space-like hypersurfaces of Lorentzian spaces with additional assumption of having at most two distinct principal curvatures. They proved that Euclidean hypersurfaces of null 2-type and having at most two distinct principal curvatures are locally isometric to a generalized spherical cylinder, [9], and a space-like hypersurface of the Lorentzian space E^m_1 with at most two distinct principal curvatures is of null 2-type if and only if it is locally isometric to a generalized hyperbolic cylinder, [10].

The assumptions on hypersurfaces to be of null 2-type are not enough for submanifolds M^n, $n \geq 3$ of the Euclidean spaces E^m and the pseudo-Euclidean spaces E^m_t to be of null 2-type. In [7], the author proved that a 3-dimensional submanifold M of the Euclidean space E^5 having two distinct principal curvatures in the parallel mean curvature direction and having a second fundamental form of a constant square length is of null 2-type if and only if M is locally isometric to one of $E \times S^2 \subset E^4 \subset E^5$, $E^2 \times S^1 \subset E^4 \subset E^5$ or $E \times S^1(a) \times S^1(a)$. However, in [8], the author proved that a 3-dimensional submanifold M of the Euclidean space E^5 with constant mean curvature and non-parallel mean curvature vector is an open portion of a 3-dimensional helical cylinder if and only if M is flat and of null 2-type.

In this work we study the classification of null 2-type space-like submanifolds of the pseudo-Euclidean spaces. We mainly prove that a 3-dimensional space-like submanifold M of the pseudo-Euclidean space E^5_t with parallel normalized non-null mean curvature vector is of null 2-type having two distinct principal curvatures in the mean curvature direction and having a constant scalar curvature τ if and only if M is locally isometric to one of the following:

1. $S^1(a) \times E^2 \subset E^4 \subset E^5_1$ or $S^2(a) \times E \subset E^4 \subset E^5_1$ when H is space-like,
2. $H^1(a) \times E^2 \subset E^4_1 \subset E^5_1$ or $H^2(a) \times E \subset E^4_1 \subset E^5_1$ when H is time-like, or
3. $H^1(a) \times E^2 \subset E^4_1 \subset E^5_2$, $H^2(a) \times E \subset E^4_1 \subset E^5_2$, or $H^1(a) \times H^1(a) \times E \subset E^5_2$.

The cases (1) and (2) imply that there is no such a submanifold that lies fully in E^5_1.

2 Preliminaries

Let E_t^m be an m-dimensional pseudo-Euclidean space with metric tensor given by

$$g = -\sum_{i=1}^{t} (dx_i)^2 + \sum_{i=t+1}^{m} (dx_i)^2$$

where $(x_1, \ldots, x_m)$ is a rectangular coordinate system of E_t^m. So (E_t^m, g) is a flat pseudo-Riemannian manifold with signature $(t, m-t)$. When $t = 1$, E_1^m is called the Lorentzian space. The hyperbolic space $H^m(a)$ is defined by

$$H^m(a) = \{x \in E_1^{m+1} | \ \langle x, x\rangle = -a^2 \text{ and } x_1 > 0\},$$

where x_1 is the first coordinate in E_1^{m+1}.

Let M be an n-dimensional pseudo-Riemannian submanifold of an m-dimensional pseudo-Euclidean space E_t^m. We denote by h, A, H, ∇ and $\nabla^{\perp}$, the second fundamental form, the Weingarten map, the mean curvature vector, the Riemannian connection and the normal connection of the submanifold M in E_t^m, respectively.

Let $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ be an adapted local orthonormal frame in E_t^m such that $\langle e_A, e_B\rangle = \varepsilon_B \delta_{AB}$, ($\varepsilon_B = \langle e_B, e_B\rangle = \pm 1$), $e_1, \ldots, e_n$ are tangent to M_t^n and $e_{n+1}, \ldots, e_m$ are normal to M_t^n. We use the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq m, \quad 1 \leq i, j, k, \ldots \leq n, \quad n+1 \leq \beta, \nu, \gamma, \ldots \leq m.$$

Let $\{\omega_A\}$ be the dual 1-forms of $\{e_A\}$ defined by $\omega_A(X) = \langle e_A, X\rangle$, $(\omega_A(e_B) = \langle e_B, e_A\rangle = \varepsilon_B \delta_{AB})$. Also, the connection forms ω_A^B are defined by

$$de_A = \sum_{B=1}^{m} \omega_A^B e_B, \quad \varepsilon_B \omega_A^B + \varepsilon_A \omega_B^A = 0.$$

For lifting or lowering indices we use $\omega^A = \varepsilon_A \omega_A$, $\omega_A^B = \varepsilon_B \omega_{AB}$. Then the structure equations of E_t^m are obtained as follows

$$d\omega^A = \sum_{B=1}^{m} \omega^B \wedge \omega_B^A, \quad d\omega_A^B = \sum_{C=1}^{m} \omega_A^C \wedge \omega_C^B. \tag{2.1}$$

Restricting these forms to M we have

$$\omega^\beta = 0, \quad d\omega^\beta = \sum_{i=1}^{m} \omega^i \wedge \omega_i^\beta = 0, \quad \beta = n+1, \ldots, m.$$

By Cartan's Lemma, we can write

$$\omega_i^\beta = \sum_{j=1}^{n} h_{ij}^\beta \omega^j, \quad h_{ij}^\beta = h_{ji}^\beta, \tag{2.2}$$

where h_{ij}^β are coefficients of the second fundamental form in the direction e_β.

The mean curvature vector H is given by

$$H = \frac{1}{n}\sum_{\beta=n+1}^{m} \varepsilon_\beta \mathrm{tr}(h^\beta) e_\beta \tag{2.3}$$

and the scalar curvature τ is given by

$$n(n-1)\tau = n^2|H|^2 - \|h\|^2 \tag{2.4}$$

where $\|h\|^2$ denotes the square of the length of the second fundamental form which is defined by

$$\|h\|^2 = \sum_{\beta} \varepsilon_\beta \mathrm{tr}(h^\beta)^2 = \sum_{i,j,\beta} \varepsilon_\beta \varepsilon_i \varepsilon_j (h^\beta_{ij})^2. \tag{2.5}$$

The first equation of (2.1) gives

$$d\omega^i = \sum_{j=1}^{m} \omega^j \wedge \omega^i_j, \quad \varepsilon_i \omega^i_j + \varepsilon_j \omega^j_i = 0, \tag{2.6}$$

where $\{\omega^i_j\}$ is the connection forms on M and uniquely determined by these equations. However, from the second equation of (2.1) we can have the Gauss and Codazzi equations, respectively, as

$$d\omega^j_i = \sum_{k=1}^{n} \omega^k_i \wedge \omega^j_k + \sum_{\beta=n+1}^{m} \omega^\beta_i \wedge \omega^j_\beta \tag{2.7}$$

and

$$d\omega^\beta_i = \sum_{k=1}^{n} \omega^k_i \wedge \omega^\beta_k + \sum_{\nu=n+1}^{m} \omega^\nu_i \wedge \omega^\beta_\nu. \tag{2.8}$$

Using (2.2) and the connection equations $\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega^k_j(e_i) e_k$ we can restate the equations of Gauss (2.7) and Codazzi (2.8) relative to the basis $e_1, \ldots, e_n$, respectively, as follows

$$\begin{aligned} e_\ell(\omega^j_i(e_k)) - e_k(\omega^j_i(e_\ell)) &= \sum_{r=1}^{n} \{\omega^r_i(e_\ell)\omega^j_r(e_k) - \omega^r_i(e_k)\omega^j_r(e_\ell) \\ &+ \omega^j_i(e_r)[\omega^r_k(e_\ell) - \omega^r_\ell(e_k)]\} + \sum_{\nu=n+1}^{m} \varepsilon_j \varepsilon_\nu (\varepsilon_k h^\nu_{ik} h^\nu_{j\ell} - \varepsilon_\ell h^\nu_{jk} h^\nu_{i\ell}), \\ &1 \le i < j \le n, \ \ 1 \le \ell < k \le n \end{aligned} \tag{2.9}$$

and

$$e_j(h^\nu_{ik}) - e_k(h^\nu_{ij}) = \sum_{r=1}^{n}\{\, h^\nu_{ir}[\omega^r_k(e_j) - \omega^r_j(e_k)] + h^\nu_{rk}\omega^r_i(e_j) - h^\nu_{rj}\omega^r_i(e_k)\} \tag{2.10}$$
$$+ \sum_{\beta=n+1}^{m} (h^\beta_{ij}\omega^\nu_\beta(e_k) - h^\beta_{ik}\omega^\nu_\beta(e_j)),$$
$$\nu = n+1,\dots,m,\;\; i = 1,\dots,n,\;\; 1 \le j < k \le n.$$

If the normal space of M in E^m_t is flat, then we can choose a parallel orthonormal normal basis on M. Therefore we have $\omega^\nu_\beta = 0$. Hence the equations of Codazzi become

$$e_j(h^\nu_{ii}) = \varepsilon_j\varepsilon_i(h^\nu_{ii} - h^\nu_{jj})\omega^j_i(e_i),\;\; i \ne j. \tag{2.11}$$

and

$$\varepsilon_j(h^\nu_{ii} - h^\nu_{kk})\omega^i_k(e_j) + \varepsilon_k(h^\nu_{jj} - h^\nu_{ii})\omega^i_j(e_k) = 0,\;\; i \ne j \ne k \ne i. \tag{2.12}$$

3 Some Basic Lemmas

We need the following some well known formulas and lemmas (for details see [1, 2, 3, 5]).

Lemma 3.1. *Let M be an n-dimensional pseudo-Riemannian submanifold of a pseudo-Euclidiean space E^m_t. Then we have*

$$\Delta H = \Delta^{\nabla^\perp} H + \sum_{i=1}^{n} \varepsilon_i\{(\nabla_{e_i}A_H)e_i + A_{\nabla^\perp_{e_i}H}e_i + h(A_He_i, e_i)\}, \tag{3.1}$$

where $\Delta^{\nabla^\perp} = -\sum_{i=1}^n \varepsilon_i\{\nabla^\perp{}_{e_i}\nabla^\perp{}_{e_i} - \nabla^\perp{}_{\nabla_{e_i}e_i}\}$ is the Laplacian operator associated with the induced normal connection $\nabla^\perp$.

Lemma 3.2. *Let M be an n-dimensional pseudo-Riemannian submanifold of a pseudo-Euclidiean space E^m_t. Then we have*

$$\mathrm{tr}(\nabla A_H) = \sum_{i=1}^{n} \varepsilon_i(\nabla_{e_i}A_H)e_i = \frac{n}{2}\nabla\langle H, H\rangle + \mathrm{tr}(A_{\nabla^\perp H}), \tag{3.2}$$

where $\nabla\langle H, H\rangle$ is the gradient of $\langle H, H\rangle$ and $\mathrm{tr}(A_{\nabla^\perp H}) = \sum_{i=1}^n \varepsilon_i A_{\nabla^\perp_{e_i}H}e_i$.

1-type pseudo-Riemannian submanifold of a pseudo-Euclidiean space E^m_t were completely classified in [3]. They are minimal submanifolds of E^m_t, minimal submanifolds of a pseudo-Riemannian sphere in E^m_t or minimal submanifolds of a pseudo-hyperbolic space in E^m_t.

For a null 2-type submanifold M of E^m_t, using $\Delta x = -nH$ the definition (1.2) implies

$$\Delta H = \lambda H. \tag{3.3}$$

Lemma 3.3. *Let M be an n-dimensional pseudo-Riemannian submanifold of a pseudo-Euclidiean space E^m_t. Then, there is a constant $\lambda \neq 0$ such that $\Delta H = \lambda H$ holds if and only if M is either of 1-type or of null 2-type.*

If the mean curvature vector H is non-null, that is, $\langle H, H\rangle \neq 0$, then there is an orthonormal normal frame $e_{n+1}, \dots, e_m$ such that $H = \alpha e_{n+1}$, where $\alpha^2 = \varepsilon_{n+1}\langle H, H\rangle$.

Lemma 3.4. *Let M be an n-dimensional pseudo-Riemannian submanifold of a pseudo-Euclidiean space E^m_t. If M is not of 1-type, then M is of null 2-type if and only if*

$$\operatorname{tr}(\bar\nabla A_H) = \operatorname{tr}(\nabla A_H) + \operatorname{tr}(A_{\nabla^\perp H}) = 0 \tag{3.4}$$

and

$$\Delta^{\nabla^\perp} H + \sum_{i=1}^{n} \varepsilon_i h(A_H e_i, e_i) = \lambda H, \tag{3.5}$$

for some nonzero constant λ.

From the definition of $\Delta^{\nabla^\perp} H$ we have

$$\begin{aligned} \Delta^{\nabla^\perp} H = {} & (\Delta\alpha + \sum_{\nu=n+2}^{m}\sum_{i=1}^{n} \varepsilon_i\varepsilon_\nu\varepsilon_{n+1}\alpha(\omega^\nu_{n+1}(e_i))^2)e_{n+1} \\ & - \sum_{\nu=n+2}^{m}\{2\omega^\nu_{n+1}(\nabla\alpha) + \alpha\operatorname{tr}(\nabla\omega^\nu_{n+1}) + \sum_{i=1}^{n}\sum_{\beta=n+2}^{m}\alpha\varepsilon_i\omega^\beta_{n+1}(e_i)\omega^\nu_\beta(e_i)\}e_\nu, \end{aligned} \tag{3.6}$$

where $\nabla\alpha = \sum_{i=1}^{n}\varepsilon_i(e_i\alpha)e_i$ and $\operatorname{tr}(\nabla\omega^\beta_{n+1}) = \sum_{i=1}^{n}\varepsilon_i(\nabla_{e_i}\omega^\beta_{n+1})(e_i)$.

Lemma 3.5. *Let M be an n-dimensional pseudo-Riemannian submanifold of E^m_t. If M is not of 1-type and $H = \alpha e_{n+1}$ is non-null, then M is of null 2-type if and only if we have*

$$\operatorname{tr}(\bar\nabla A_H) = \frac{n}{2}\nabla\langle H, H\rangle + 2\operatorname{tr}(A_{\nabla^\perp H}) = 0 \tag{3.7}$$

$$\Delta\alpha = \lambda\alpha - \alpha\varepsilon_{n+1}\|A_{n+1}\|^2 - \alpha\sum_{i=1}^{n}\sum_{\nu=n+2}^{m}\varepsilon_i\varepsilon_\nu\varepsilon_{n+1}(\omega^\nu_{n+1}(e_i))^2, \tag{3.8}$$

$$\varepsilon_\beta\operatorname{tr}(A_H A_\beta) = 2\omega^\beta_{n+1}(\nabla\alpha) + \alpha\operatorname{tr}(\nabla\omega^\beta_{n+1}) + \alpha\sum_{i=1}^{n}\sum_{\nu=n+2}^{m}\varepsilon_i\omega^\nu_{n+1}(e_i)\omega^\beta_\nu(e_i), \tag{3.9}$$

where λ is a nonzero constant, $\beta = n+2, \dots, m$ and $\|A_{n+1}\|^2 = \sum_{i=1}^{n}\varepsilon_i\langle A_{n+1}e_i, A_{n+1}e_i\rangle$.

By direct calculation the equation (3.7) becomes

$$\mathrm{tr}(\bar{\nabla}A_H) = \frac{n}{2}\varepsilon_{n+1}\nabla(\alpha^2) + 2A_{n+1}(\nabla\alpha) + 2\alpha\sum_{i=1}^{n}\sum_{\nu=n+2}^{m}\varepsilon_i\omega_{n+1}^{\nu}(e_i)A_{e_\nu}(e_i) = 0.$$

Using this equation we can have the following corollary from Lemma 3.5.

Corollary 3.1. *Let M be an n-dimensional pseudo-Riemannian submanifold of E_t^{n+2}. If M is not of 1-type, $H = \alpha e_{n+1}$ is non-null and the normalized mean curvature vector, e_{n+1}, is parallel, then M is of null 2-type if and only if we have*

$$A_{n+1}(\nabla\alpha^2) + \frac{n\alpha\varepsilon_{n+1}}{2}\nabla(\alpha^2) = 0, \tag{3.10}$$

$$\Delta\alpha = \lambda\alpha - \alpha\varepsilon_{n+1}\|A_{n+1}\|^2, \tag{3.11}$$

$$\mathrm{tr}(A_{n+1}A_{n+2}) = 0, \tag{3.12}$$

where λ is a nonzero constant.

In [9, 10], the following theorems are given on null 2-type hypersurfaces of Euclidean spaces and null 2-type space-like hypersurfaces of Lorentzian space.

Theorem 3.1. ([9]) *Let M be a Euclidean hypersurface with at most two distinct principal curvature. Then, M is of null 2-type if and only if it is locally isometric to $E^p \times S^{n-p}(a)$.*

Theorem 3.2. ([10]) *Let M^n be a space-like hypersurface of the Lorentzian spaces E_1^n with at most two distinct principal curvature. Then, M^n is of null 2-type if and only if it is locally isometric to $E^p \times H^{n-p}(a)$.*

4 Null 2-type space-like submanifolds of E_t^5

We prove the followings.

Proposition 4.1. *Let M be a 3-dimensional space-like submanifold of the pseudo-Euclidean space E_t^5 with parallel normalized mean curvature vector such that M is not of 1-type. If M is of null 2-type with the Weingarten map in the direction of the mean curvature vector H has two distinct eigenvalues, then the mean curvature α is constant on M.*

Proof. As the codimension is 2 and the normalized mean curvature vector, $e_4 = H/\alpha$, $\alpha^2 = \epsilon_4\langle H, H\rangle$, is parallel, then the unit normal vector e_5 is also parallel. Therefore the normal space is flat, i.e., $\omega_4^5 \equiv 0$ on M. Hence we can have the diagonalized Weingarten maps in the direction e_4 and e_5. Since A_4 has two distinct eigenvalues, say, $h_{11}^4 \neq h_{22}^4 = h_{33}^4$. We can write

$$A_4 = \mathrm{diag}(h_{11}^4, h_{22}^4, h_{22}^4) \quad \text{and} \quad A_5 = \mathrm{diag}(h_{11}^5, h_{22}^5, h_{33}^5)$$

with $h^5_{11} + h^5_{22} + h^5_{33} = 0$. However, from (3.12) we get

$$\mathrm{tr}(A_4A_5) = (h^4_{11} - h^4_{22})h^5_{11} = 0 \tag{4.1}$$

because of $h^5_{11} + h^5_{22} + h^5_{33} = 0$. As $h^4_{11} - h^4_{22} \neq 0$ we have $h^5_{11} = 0$ and $h^5_{22} = -h^5_{33}$.

Assume that α is not constant. Let $V = \{p \in M : \nabla\alpha^2(p) \neq 0\}$ which is open in M. From (3.10) it is seen that the vector $\nabla\alpha^2$ is an eigenvector of A_4 corresponding to the eigenvalue $-\dfrac{3\alpha\varepsilon_4}{2}$. Then we may say that $\nabla\alpha^2$ is parallel to e_1 or e_3 (the same as e_2). For the last case it could also be proved that the mean curvature α is constant by using the same way as in the first case. Thus $h^4_{11} = -\frac{3\alpha\varepsilon_4}{2}$ and $h^4_{22} = h^4_{33} = \frac{9\alpha\varepsilon_4}{4}$ because of $3\alpha\varepsilon_4 = h^4_{11} + 2h^4_{22}$. Then we have

$$\omega^4_1 = -\frac{3\alpha\varepsilon_4}{2}\,\omega^1, \quad \omega^4_2 = \frac{9\alpha\varepsilon_4}{4}\omega^2, \quad \omega^4_3 = \frac{9\alpha\varepsilon_4}{4}\omega^3. \tag{4.2}$$

Since $\nabla\alpha^2$ is parallel to e_1 we can have $e_2(\alpha) = e_3(\alpha) = 0$, that is, $e_2(h^4_{11}) = e_3(h^4_{11}) = 0$, and

$$d\alpha = e_1(\alpha)\omega^1. \tag{4.3}$$

However, by using the equation of Codazzi (2.11) for $\nu = 4$ if $i = 1$ we have

$$\omega^2_1(e_1) = \omega^3_1(e_1) = 0, \tag{4.4}$$

because of $h^4_{11} - h^4_{22} \neq 0$, and if $j = 1$, considering $h^4_{22} = h^4_{33} = \dfrac{9\alpha\varepsilon_4}{4}$, then we obtain

$$\omega^1_2(e_2) = \omega^1_3(e_3) = \frac{3}{5}\frac{e_1(\alpha)}{\alpha}. \tag{4.5}$$

Also, the equation of Codazzi (2.12) for $\nu = 4$ and $j = 1$ implies that

$$\omega^1_2(e_3) = \omega^1_3(e_2) = 0. \tag{4.6}$$

Applying the structure equations and using (4.6), it can be shown that $d\omega^1 = 0$. Hence we have locally

$$\omega^1 = du, \tag{4.7}$$

where u is a local coordinate on U. From (4.3) and (4.7) we have $d\alpha \wedge du = 0$. This shows that α is a function of u, i.e., $\alpha = \alpha(u)$ and $d\alpha = \alpha'(u)du$. Thus, by (4.5) we have

$$\omega^1_2(e_2) = \omega^1_3(e_3) = \frac{3\alpha'}{5\alpha}. \tag{4.8}$$

Considering (4.4) and (4.6), from the equation of Gauss (2.9) for $i = \ell = 1, j = k = 2$ we get

$$e_1(\omega^1_2(e_2)) = (\omega^1_2(e_2))^2 + \varepsilon_4 h^4_{11}h^4_{22}. \tag{4.9}$$

Using (4.8), the equation (4.9) turns into

$$40\alpha\alpha'' - 64(\alpha')^2 + 225\varepsilon_4\alpha^4 = 0. \tag{4.10}$$

Let $y = (\alpha')^2$. Then the above the equation can be reduced to the following first order differential equation:

$$2\alpha y' - 64y + 225\varepsilon_4\alpha^4 = 0, \tag{4.11}$$

where y' denotes the first derivative of y with respect to α. For this equation we obtain the solution

$$(\alpha')^2 = C\alpha^{16/5} - \varepsilon_4\left(\frac{225}{16}\right)^2\alpha^4, \tag{4.12}$$

where C is a constant.

When we use the definition of $\Delta\alpha$, the fact that $\nabla\alpha^2$ is parallel to e_1 and the equation (4.8) we obtain

$$\Delta\alpha = \frac{6(\alpha')^2}{5\alpha} - \alpha''. \tag{4.13}$$

Also, since $\|A_4\|^2 = \dfrac{99\alpha^2}{8}$, considering (4.13) and the second equation (3.11) of Corollary 3.1 we get

$$40\alpha\alpha'' - 48(\alpha')^2 + 40\lambda\alpha^2 - 495\varepsilon_4\alpha^4 = 0. \tag{4.14}$$

Combining (4.10) and (4.14) we obtain

$$(\alpha')^2 = 45\varepsilon_4\alpha^4 - \frac{5}{2}\lambda\alpha^2. \tag{4.15}$$

As a result, using (4.12) and (4.15) we deduce that α is locally constant on V which is a contradiction with the definition of M. Therefore α is constant on M. $\square$

Let $H^1(a) \times H^1(a) \times E = \{(x_1, x_2, \dots, x_5) : -x_1^2 + x_3^2 = -a^2,\ -x_2^2 + x_4^2 = -a^2\}$. For later use we need the connection forms ω_A^B of $H^1(a) \times H^1(a) \times E \subset E_2^5$. By a suitable choice of the Euclidean coordinates, its equation takes the following form

$$x(u_1, u_2, u_3) = (a\cosh u_2, a\cosh u_3, a\sinh u_2, a\sinh u_3, u_1),$$

where a is a nonzero constant. If we put

$$e_1 = \frac{\partial}{\partial u_1} = (0,0,0,0,1), \quad e_2 = \frac{1}{a}\frac{\partial}{\partial u_2} = (\sinh u_2, 0, \cosh u_2, 0, 0),$$

$$e_3 = \frac{1}{a}\frac{\partial}{\partial u_2} = (0, \sinh u_3, 0, \cosh u_3, 0),$$

$$e_4 = \frac{1}{\sqrt{2}}(\cosh u_2, \cosh u_3, \sinh u_2, \sinh u_3, 0),$$

$$e_5 = \frac{1}{\sqrt{2}}(\cosh u_2, -\cosh u_3, \sinh u_2, -\sinh u_3, 0),$$

then, by a straight forward calculation we obtain

$$\omega^1 = du_1, \quad \omega^2 = adu_2, \quad \omega^3 = adu_3, \quad \omega_2^1 = \omega_3^1 = \omega_3^2 = \omega_1^4 = \omega_1^5 = \omega_5^4 = 0,$$
$$\omega_2^4 = -\frac{1}{a\sqrt{2}}\,\omega^2, \quad \omega_3^4 = -\frac{1}{a\sqrt{2}}\,\omega^3, \quad \omega_2^5 = -\frac{1}{a\sqrt{2}}\,\omega^2, \quad \omega_3^5 = \frac{1}{a\sqrt{2}}\,\omega^3. \tag{4.16}$$

Theorem 4.1. *Let M be a 3-dimensional space-like submanifold of the pseudo-Euclidean space E_t^5 with parallel normalized non-null mean curvature vector such that M is not of 1-type. Then M is of null 2-type having two distinct principal curvatures in the mean curvature direction and having a constant scalar curvature τ if and only if M is locally isometric to one of the following:*

1. $S^1(a) \times E^2 \subset E^4 \subset E_1^5$ *or* $S^2(a) \times E \subset E^4 \subset E_1^5$ *when H is space-like,*
2. $H^1(a) \times E^2 \subset E_1^4 \subset E_1^5$ *or* $H^2(a) \times E \subset E_1^4 \subset E_1^5$ *when H is time-like, or*
3. $H^1(a) \times E^2 \subset E_1^4 \subset E_2^5$, $H^2(a) \times E \subset E_1^4 \subset E_2^5$, *or* $H^1(a) \times H^1(a) \times E \subset E_2^5$.

Proof. As the codimension is 2 and the normalized mean curvature vector, $e_4 = H/\alpha$, is parallel, then the normal space is flat. Let M be of null 2-type and let the Weingarten map in the direction H has two distinct principal curvatures. Then the mean curvature α on M is constant by Proposition 4.1. However, as in the proof of Proposition 4.1 we can have

$$A_4 = \mathrm{diag}(h_{11}^4, h_{22}^4, h_{22}^4) \quad \text{and} \quad A_5 = \mathrm{diag}(0, h_{22}^5, -h_{22}^5).$$

By using (3.11) we have $\|A_4\|^2 = (h_{11}^4)^2 + 2(h_{11}^4)^2 = \lambda$ which is constant. Hence, as α is constant, it is easily seen that the eigenvalues h_{11}^4 and h_{22}^4 of A_4 are constant. Since the scalar curvature and the eigenvalues of A_4 are constant, by using (2.4) and (2.5) we obtain h_{22}^5 =const.

Using the fact that $h_{11}^4 \neq h_{22}^4 = h_{33}^4$, $h_{11}^5 = 0$, $h_{22}^5 = -h_{33}^5$ and all h_{ij}^ν's are constant, from the equations of Codazzi (2.11) and (2.12) for $\nu = 4$ we obtain

$$\omega_j^1(e_i) = 0, \quad i = 1, 2, 3, \;\; j = 2, 3 \tag{4.17}$$

and for $\nu = 5$ from (2.12) we get

$$h_{22}^5 \omega_3^2(e_i) = 0, \quad i = 1, 2, 3. \tag{4.18}$$

However, by using the equations of Gauss (2.9), for $i = \ell = 1, j = k = 2$ and for $i = \ell = 2, j = k = 3$, we obtain, respectively,

$$h_{11}^4 h_{22}^4 = 0, \tag{4.19}$$

and

$$e_2(\omega_3^2(e_3)) - e_3(\omega_3^2(e_2)) = (\omega_3^2(e_2))^2 + (\omega_3^2(e_3))^2 + \varepsilon_4(h_{22}^4)^2 - \varepsilon_5(h_{55}^5)^2. \tag{4.20}$$

Since A_4 has two distinct eigenvalues, one of h_{11}^4 and h_{22}^4 is a non-zero constant. Considering the equations (4.18), (4.19) and (4.20) we have the following classifications.

Let $t = 1$, that is, $\varepsilon_4\varepsilon_5 = -1$.

Case 1. $h^4_{11} \neq 0$ and $h^4_{22} = 0$. Then, by (4.18) we get $h^5_{22} = 0$ or $\omega^2_3(e_i) = 0$, $i = 1,2,3$. Using the second part, the equation (4.20) implies that $h^5_{22} = 0$. Thus, A_5 vanishes. Since the normal space is flat and $A_5 \equiv 0$, then M is contained in a hyperplane P of E^5_1.

If H is space-like, then P is a space-like hyperplane of E^5_1. Therefore, by Theorem 3.1 M is locally isometric to the circular cylinder $S^1(a) \times E^2 \subset E^4 \subset E^5_1$.

If H is time-like, then P is a Lorentzian hyperplane of E^5_1. Therefore, by Theorem 3.2 M is locally isometric to the hyperbolic cylinder $H^1(a) \times E^2 \subset E^4_1 \subset E^5_1$.

Case 2. $h^4_{11} = 0$ and $h^4_{22} \neq 0$. Then, by (4.18) we have $h^5_{22} = 0$ or $\omega^2_3(e_i) = 0$, $i = 1,2,3$. Suppose that $\omega^2_3(e_i) = 0$ for $i = 1,2,3$. Thus, from (4.20) we get $\varepsilon_4(h^4_{22})^2 - \varepsilon_5(h^5_{55})^2 = 0$ which implies that $h^4_{22} = h^5_{55} = 0$ as $\varepsilon_4\varepsilon_5 = -1$. This is a contradiction because $h^4_{22} \neq 0$. Therefore $\omega^2_3(e_i) \neq 0$ at least for one $i \in \{1,2,3\}$ and $h^5_{55} = 0$, and hence A_5 vanishes on M. Considering that the normal space is flat, M lies in a hyperplane P of E^5_1.

If H is space-like, then P is a space-like hyperplane of E^5_1. Therefore, by Theorem 3.1 M is locally isometric to $S^2(a) \times E^1 \subset E^4 \subset E^5_1$.

If H is time-like, then P is a Lorentzian hyperplane of E^5_1. Therefore, by Theorem 3.2 M is locally isometric to $H^2(a) \times E^1 \subset E^4_1 \subset E^5_1$.

Let $t = 2$, that is, $\varepsilon_4\varepsilon_5 = 1$. Then the normal space is time-like.

Case 3. $h^4_{11} \neq 0$ and $h^4_{22} = 0$. Then, by (4.18) we get $h^5_{22} = 0$ or $\omega^2_3(e_i) = 0$, $i = 1,2,3$. Using the second part, the equation (4.20) implies that $h^5_{22} = 0$. Therefore A_5 vanishes. Since the normal space is flat and $A_5 \equiv 0$, then M is contained in a Lorentzian hyperplane P of E^5_2. Therefore, by Theorem 3.2 M is locally isometric to $H^1(a) \times E^2 \subset E^4_1 \subset E^5_2$.

Case 4. $h^4_{11} = 0$ and $h^4_{22} \neq 0$. Then, by (4.18) we get $h^5_{22} = 0$ or $\omega^2_3(e_i) = 0$, $i = 1,2,3$.

Subcase 4-a. $\omega^2_3(e_i) \neq 0$ for at least one $i \in \{1,2,3\}$ and $h^5_{22} = 0$. Hence, we have $A_4 = \mathrm{diag}(0, h^4_{22}, h^4_{22})$ and $A_5 \equiv 0$. Considering that the normal space is flat, M lies in a Lorentzian hyperplane P of E^5_2. Therefore, M is locally isometric to $H^2(a) \times E \subset E^4_1 \subset E^5_2$ by Theorem 3.2 .

Subcase 4-b. $h^5_{22} \neq 0$ and $\omega^2_3(e_i) = 0$, $i = 1,2,3$. From (4.20) we get $\varepsilon_4(h^4_{22})^2 - \varepsilon_5(h^5_{55})^2 = 0$ which implies that $h^4_{22} = \mp h^5_{55} \neq 0$ as $\varepsilon_4\varepsilon_5 = 1$. Putting $\mu_0 = h^4_{22} = -\frac{3\alpha}{2}$ we have $A_4 = \mathrm{diag}(0, \mu_0, \mu_0)$ and $A_5 = \mathrm{diag}(0, \mp\mu_0, \pm\mu_0)$. Considering $\omega^2_3(e_i) = 0$, $i = 1,2,3$ and (4.17) it is seen that M is flat. Also, we can write

$$\omega^4_1 = 0,\ \ \omega^4_2 = \mu_0\,\omega^2, \quad \omega^4_3 = \mu_0\,\omega^3,\ \ \omega^5_1 = 0,\ \ \omega^5_2 = \pm\mu_0\,\omega^2, \quad \omega^5_3 = \mp\mu_0\,\omega^3$$

Since M has a flat normal connection it is seen that the connection forms ω^A_B coincide with the connection forms of $H^1(a) \times H^1(a) \times E$ given in (4.16). Therefore, from the fundamental theorem of submanifolds, M is locally isometric to $H^1(a) \times H^1(a) \times E \subset E^5_2$.

The converses of all these cases are trivial. □

Remark : The cases (1) and (2) show that in the case $t = 1$ there is no such a submanifold that lies fully in E^5_1.

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