On the spinorial representations of $SO(4,4)$

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Dedicated to the memory of Radu Rosca (1908-2005)

Abstract. We study the spinorial representations of the group $SO(4,4)$.

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1 Introduction

It is well known that the orthogonal groups $SO(n)$, $n > 2$ are connected, but are not simply connected, since $\pi_1(SO(n)) \approx \mathbb{Z}_2$. Their simply connected covering groups $Spin(n)$ are obtained by making use of the Clifford algebras C_n.

The pseudo-orthogonal groups $SO(m,m)$ are also connected and not simply connected. The groups $Spin(m,m)$ are also defined by using convenient Clifford algebras. It is worthwhile to note this

Proposition. *When* $m > 2,\ \pi_1(SO(m,m)) \approx \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. The group $SO(m,m)$ acts transitively on the group space $SO(m)$ through homographic transformations: $A \mapsto (aA+b)(cA+d)^{-1}$; this action induces a transitive action on the space W formed by the pairs $(A,B) \in SO(m) \times SO(m)$ with $det(A-B) \neq 0$. The isotropy group H at the point $(I,-I)$ is formed by the matrices $\begin{pmatrix} a & ca \\ ca & a \end{pmatrix} \in SO(m,m)$, where $(I -^t c\ c)a\ ^t a = I$. The space W is diffeomorphic to the tangent space $TSO(m)$. Thus W and H are homotopically equivalent to $SO(m)$ and therefore $\pi_1(W) \approx \pi_1(H) \approx \pi_1(SO(m)) \approx \mathbb{Z}_2$.

The exact sequence of homotopy groups associated with the fibration $H \subset SO(m,m) \to W$ provides the final step of the proof.

2 The spinorial representations of $SO(4,4)$

We denote by G the group defined as the set of real 8×8-matrices S verifying the relations

$$det(S) = 1\ ,\ ^t S\ \Sigma\ S = \Sigma,\ \Sigma = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix}.$$

The group G is isomorphic to $SO(4,4)$.

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Let g be the Lie algebra of the group G.

We denote by $\mathbf{E}$ the real vector space spanned by the eight symbols

$$b_\alpha \ , \ b^\alpha \ , \ (\alpha = 0,1,2,3)$$

and endowed with the quadratic form

$$F(x) = \sum_\alpha x^\alpha x_\alpha \ , \ x = \sum_\alpha (x^\alpha b_\alpha + x_\alpha b^\alpha).$$

The relations $S \in G \ , \ x \in \mathbf{E} \ , \ s \in g$ imply

$$Q(Sx) = Q(x) \ , \ {}^t s \ \Sigma + \Sigma \ s = 0.$$

When b, b' are two of the symbols $b_\alpha \ , b^\alpha$, we denote by b/b' the 8×8-matrix characterized by the properties

$$(b/b')b' = b \ , \ (b/b')b'' = 0 \ , \ (b'' \neq b').$$

Then, whenever $b, b', b'' \in b$, we will have

$$(b/b')(b'/b'') = b/b''.$$

The Lie algebra g is linearly spanned by the 28 matrices

$$B_\alpha^\beta = b_\alpha/b_\beta - b^\beta/b^\alpha$$

$$B_{\alpha\beta} = b_\alpha/b^\beta - b_\beta/b^\alpha \ , \ B^{\alpha\beta} = b^\alpha/b_\beta - b^\beta/b_\alpha \ , \ (\alpha < \beta).$$

As long as we will work with a vector space which is endowed with specific base, we will identify any matrix with the endomorphism associated with that matrix.

Instead of considering the group $SO(4,4)$, we will consider the group G.

Let C_8 be the Clifford algebra spanned by the eight symbols $t_\alpha \ , \ t^\alpha$ subject to the relations

$$t_\alpha t_\beta + t_\beta t_\alpha = t^\alpha t^\beta + t^\beta t^\alpha = 0 \ , \ t_\alpha t^\beta + t^\beta t_\alpha = \delta_\alpha^\beta \ , \ (\alpha, \beta = 1,2,3,4).$$

We denote

$$\varphi = t_4 + t^4.$$

We want to make explicit the two fundamental spinorial representations of the group G. To this end, we denote

$$e_0 = t_1 t_2 t_3 t_4 \ , \ e^0 = -t^1 t^2 t^3 t^4 e_0 \ , \ e_i = t^i t^4 e_0$$

$$e^i = t^j t^k e_0 \ , \ (ijk = 123, 231, 312)$$

$$f_0 = t^4 e_0 = \varphi e_0 \ , \ f^0 = t^1 t^2 t^3 e_0 = \varphi e^0$$

$$f_i = -t^i e_0 = \varphi e_i \ , \ f^i = t^4 t^j t^k e_0 = \varphi e^i$$

$$F_i = t_i t_4 \ , \ G_i = t_j t_k \ , \ F^i = t^i t^4 \ , \ G^i = t^j t^k$$

$$H_a^b = \frac{1}{2}(t_a t^b - t^b t_a) \ , \ H_a = H_a^a, \ (a, b = 1,2,3,4)$$

Then

$$H_a = t_a t^a - \frac{1}{2} = \frac{1}{2} - t^a t_a$$

$$[t_a t_b \ , t^b t^c] = H_a^c \ , \ (a \neq c)$$

$$[t_a t_b \ , \ t^b t^a] = H_a + H_b$$

$$\varphi t_i \varphi = -t_i \ , \ \varphi t^i \varphi = -t^i \ , \ \varphi t_4 \varphi = t^4 \ , \ \varphi t^4 \varphi = t_4$$

$$\varphi F_i \varphi = -F_i^4 \ , \ \varphi F^i \varphi = F_4^i \ , \ \varphi G_i \varphi = G_i \ , \ \varphi G^i \varphi = G^i$$

$$\varphi H_i^j \varphi = H_i^j \ , \ \varphi H_i^4 \varphi = -F_i \ , \ \varphi H_4^i \varphi = -F^i \ , \ \varphi H_4 \varphi = -H_4 \ .$$

The symbols F, G, H span a Lie algebra, denoted Γ. The algebra Γ is isomorphic to g. Multiplication by φ allows us to transform the action of F, G, H on the vectors f^a, f_a into an action on the vectors e^a, e_a.

The sets $e = (e_0 \ , e_i \ , e^0, e^i)$, respectively $f = (f_0 \ , f_i, \ f^0 \ , f^i)$ span two complex 8-dimensional vector spaces denoted $\mathcal{E}, \ \mathcal{F}$ and multiplications on the left with F, G, H define *two fundamental spinorial representations*

$$\rho_+ : \Gamma \to End(\mathcal{E}) \ , \ \rho_- : \Gamma \to End(\mathcal{F})$$

of the Lie algebra Γ.

One has:

$$H_i \ e_0 = \frac{1}{2} \ e_0 \ , \ H_4 e_0 = \frac{1}{2} e_0 \ , \ H_i \ e^0 = -\frac{1}{2} \ e^0 \ , \ H_4 e^0 = \ \frac{1}{2} e^0$$

$$H_i e_i = -\frac{1}{2} e_i \ , \ H_i e_j = \frac{1}{2} e_j \ , \ H_i e^i = \frac{1}{2} e^i \ , \ H_i e^j = -\frac{1}{2} e^j$$

$$H_i f_0 = \frac{1}{2} f_0 \ , \ H_4 f_0 = -\frac{1}{2} f_0 \ , \ H_i f^0 = -\frac{1}{2} f^0 \ , \ H_4 f^0 = \frac{1}{2} f^0$$

$$H_i f_i = -\frac{1}{2} f_i \ , \ H_i f_j = \frac{1}{2} f_j \ , \ H_i f^i = \frac{1}{2} f^i \ , \ H_i f^j = -\frac{1}{2} f^j$$

$$F_i e_i = -e_0 \ , \ F_i e^0 = e^i \ , \ G_i e^i = -e_0 \ , \ G_i e^0 = e_i$$

$$F^i e_0 = e_i \ , \ F^i e^i = -e^0 \ , \ G^i e_0 = e^i \ , \ G^i e_i = -e^0$$

$$F_i f^j = f_k \ , \ F_i f^k = -f_j \ , \ G_i f^0 = f_i \ , \ G_i f^i = -f_0$$

$$F^i f_j = f^k \ , \ F^i f_k = -f^j \ , \ G^i f_0 = f^i \ , \ G^i f_i = -f^0,$$

where the triple ijk is one of the triples $123, 231, 312$.

Using the same convention, one has:

$$\rho_+(H_i) = \frac{1}{2}(e_0/e_0 - e_i/e_i + e_j/e_j + e_k/e_k - e^0/e^0 + e^i/e^i - e^j/e^j - e^k/e^k)$$

$$\rho_-(H_i) = \frac{1}{2}(f_0/f_0 - f_i/f_i + f_j/f_j + f_k/f_k - f^0/f^0 + f^i/f^i - f^j/f^j - f^k/f^k)$$

$$\rho_+(H_4) = \frac{1}{2}(e_0/e_0 - e_1/e_1 - e_2/e_2 - e_3/e_3 - e^0/e^0 + e^1/e^1 + e^2/e^2 + e^3/e^3)$$

$$\rho_-(H_4) = \frac{1}{2}(f_0/f_0 - f_1/f_1 - f_2/f_2 - f_3/f_3 - e^0/e^0 + f^1/f^1 + f^2/f^2 + f^3/f^3)$$

$$\rho_+(t_it_4) = e^i/e^0 - e_0/e_i \ , \ \rho_-(t_it_4) = f_k/f^j - f_j/f^k$$

$$\rho_+(t_jt_k) = e_i/e^0 - e_0/e^i \ , \ \rho_-(t_jt_k) = f_i/f^0 - f_0/f^i$$

$$\rho_+(t^it^4) = e_i/e_0 - e^0/e^i \ , \ \rho_-(t^it^4) = f^k/f_j - f^j/f_k$$

$$\rho_+(t^jt^k) = e^i/e_0 - e^0/e_i \ , \ \rho_-(t^jt^k) = f^i/f_0 - f^0/f_i$$

3 The vectorial representation

We will now consider the linear representation ρ of the Lie algebra Γ, which is induced by the adjoint representation of the Clifford algebra C_8:

$$(F,t) \mapsto [F,t] \ , \ (G,t) \mapsto [G,t] \ , \ (H,t) \mapsto [H,t].$$

Let $\mathbf{D}$ be the complex vector space spanned by the eight symbols $t_a \ , \ t^a$, $(a = 1,2,3,4)$. We have:

$$[F_i \ ,t^i] = -t_4 \ , \ [F_i \ ,t^4] = t_i \ , \ [G_i \ ,t^j] = -t_k \ , \ [G_i \ ,t^k] = t_j$$

$$[F^i,t_i] = -t^4 \ , \ [F^i,t_4] = t^i \ , \ [G^i,t_j] = -t^k \ , \ [G^i,t_k] = t^j.$$

Thus $\mathbf{D}$ is an invariant subspace of C_8 and we get the following endomorphisms of $\mathbf{D}$, defining *the vectorial representation* $\rho : \Gamma \to End(\mathbf{D})$ ::

$$\rho(t_at_b) = t_a/t^b - t_b/t^a \ , \ \rho(t^at^b) = t^a/t_b - t^b/t_a \ , \ \rho(t_at^b) = t_a/t_b - t^b/t^a \ .$$

We resume the results concerning the two spinorial representations $\rho_+ \ ,\rho_-$ and the vectorial representation ρ of the Lie algebra Γ, by composing the following tables:

$\tau \in \Gamma$	$\rho_+(\tau)$	$\rho_-(\tau)$	$\rho(\tau)$
t_it_4	$e^i/e^0 - e_0/e_i$	$e_k/e^j - e_j/e^k$	$e_i/e^0 - e_0/e^i$
t_jt_k	$e_i/e^0 - e_0/e^i$	$e_i/e^0 - e_0/e^i$	$e_j/e^k - e_k/e^j$
t^it^4	$e_i/e_0 - e^0/e^i$	$e^k/e_j - e^j/e_k$	$e^i/e_0 - e^0/e_i$
t^jt^k	$e^i/e_0 - e^0/e_i$	$e^i/e_0 - e^0/e_i$	$e^j/e_k - e^k/e_j$
t_it^j	$e^i/e^j - e_j/e_i$	$e^i/e^j - e_j/e_i$	$e_i/e_j - e^j/e^i$
t_4t^i	$e^k/e_j - e^j/e_k$	$e_i/e_0 - e^0/e^i$	$e_0/e_i - e^i/e^0$
t_it^4	$e_j/e^k - e_k/e^j$	$e_0/e_i - e^i/e^0$	$e_i/e_0 - e^0/e^i$
$t_it^i - t^it_i$	$E_0 - E_i + E_j + E_k$	$E_0 - E_i + E_j + E_k$	$e_i/e_i - e^i/e^i$
$t_4t^4 - t^4t_4$	$E_0 - E_1 - E_2 - E_3$	$-E_0 + E_1 + E_2 + E_3$	$e_0/e_0 - e^0/e^0$

where the following notation has been used: $E_\alpha = e_\alpha/e_\alpha - e^\alpha/e^\alpha$.
Denoting, for $\alpha, \beta = 0,1,2,3, \ \alpha \neq \beta$ and $a = 1,2,3,4$,

$$E^\beta_\alpha = e_\alpha/e_\beta - e^\beta/e^\alpha \ , \ E_{\alpha\beta} = e_\alpha/e^\beta - e_\beta/e^\alpha \ , \ E^{\alpha\beta} = e^\alpha/e_\beta - e^\beta/e_\alpha$$

$$h_a = t_at^a - t^at_a \ ,$$

the following table will define the inverses of the representations $\rho_+,\ \rho_-,\ \rho$:

$E \in g$	$(\rho_+)^{-1}(E)$	$(\rho_-)^{-1}(E)$	$\rho^{-1}(E)$
$-E_0^i$	$t_i t_4$	$-t_i t^4$	$t^i t_4$
E_i^0	$t^i t^4$	$-t^i t_4$	$t_i t^4$
E_{i0}	$t_j t_k$	$t_j t_k$	$t_i t_4$
E^{i0}	$t^j t^k$	$t^j t^k$	$t^i t^4$
$-E_j^i$	$t_i t^j$	$t_i t^j$	$t^i t_j$
E^{jk}	$t^i t_4$	$-t^i t^4$	$t^j t^k$
E_{jk}	$t_i t^4$	$-t_i t_4$	$t_j t_k$
$4E_0$	$h_1 + h_2 + h_3 + h_4$	$h_1 + h_2 + h_3 - h_4$	$4h_4$
$4E_i$	$-h_i + h_j + h_k - h_4$	$-h_i + h_j + h_k + h_4$	$4h_i$

It is interesting to note that $(\rho_+)^{-1}\rho_-$, $\rho^{-1}\rho_-$, $(\rho)^{-1}\rho_-$ are automorphisms of the Lie algebra Γ verifying the following periodicity relations:

$$\left((\lambda')^{-1}\mu'\right)^2 = \left((\nu')^{-1}\mu'\right)^4 = \left((\nu')^{-1}\lambda'\right)^6 = id_\Gamma \ .$$

On the other side, according to the general theory regarding linear representations of orthogonal groups, the G-module $\mathcal{E} \otimes \mathcal{F}$ decomposes into the direct sum of two irreducible submodules of dimensions 8 and 56, with highest weights λ_1 respectively $\lambda_1 + \lambda_2 + \lambda_3$, the first of which being isomorphic to D, while the second is isomorphic to $\Lambda^3\mathbf{R}^8$; as a consequence, there exists a monomorphism of G-modules

$$\psi : D \to \mathcal{E} \otimes \mathcal{F};$$

in our setting, this monomorphism is defined by the formulas

$$\psi(t_l) = e_0 \otimes f^{ijk} - f^{ij} \otimes f^k - f^{jk} \otimes f^i - f^{ki} \otimes f^j$$

$$\psi(t^l) = f^{lijk} \otimes f^l - f^{li} \otimes f^{ljk} - f^{lj} \otimes f^{lki} - f^{lk} \otimes f^{lij},$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$ and

$$f^l = t^l e_0 \ , \ f^{li} = t^l t^i e_0 \ , \ f^{lij} = t^l t^i t^j e_0 \ .$$

For more details concerning the groups $SO(8)$, $Spin(8)$, $SO(4,4)$ and $Spin(4,4)$ see the book *Spin Geometry* [1, p.56].

4 Octets

Let $\mathbf{H}$ be the skew-field of quaternions and denote $\mathbf{Q} = \mathbf{H} \times \mathbf{H}$.

We shall introduce in $\mathbf{Q}$ a new multiplication law, by performing a slight modification of the multiplication rules governing the Cayley algebra.

We will get a link between the so modified Cayley algebra and the fundamental spinorial representations of the group $SO(4,4)$.

We denote by $1, i, j, k$ the standard quaternions satisfying the relations

$$i^2 = j^2 = k^2 = -1 \ , \ ij = -ji = k.$$

The pairs $(x,y) \in \mathbf{Q} = \mathbf{H} \times \mathbf{H}$ will be named *octets*. The product of two octets, the scalar product of two octets, the norm of an octet, the conjugate of an octet and the inverse of a non vanishing octet are defined by the formulas:

$$(x,y)(u,v) = (xu + \bar{v}y, vx + y\bar{u})$$

$$< (x,y),(u,v) >=< x,u > - < y,v > \ , \ || \ (x,y) \ ||^2 =| \ x \ |^2 - | \ y \ |^2$$

$$(x,y)^* = (\bar{x}, -y) \ , \ (x,y)^{-1} = \frac{(\bar{x}, -y)}{| \ x \ |^2 - | \ y \ |^2} \ .$$

Then we will have

$$| \ (x,y)(u,v) \ |^2 =| \ (x,y) \ |^2 \ | \ (u,v) \ |^2 \ .$$

The last formula shows that multiplication either on the left or on the right with objects $(a,b) \in \mathbf{Q}$ having $| \ a \ |^2 - | \ b \ |^2 = 1$ defines linear transformations that keep invariant the quadratic form

$$P(x,y) =| \ x \ |^2 - | \ y \ |^2 \ .$$

We introduce the following octets, forming two bases of the real vector space $\mathbf{Q}$:

$$1' = (1,0) \ , \ i' = (i,0) \ , \ j' = (j,0) \ , \ k' = (k,0)$$

$$1'' = (0,1) \ , \ i'' = (0,i) \ , \ j'' = (0,j) \ , \ k'' = (0,k)$$

$$e_0 = \frac{1' - 1''}{2} \ , \ e_1 = \frac{i' + i''}{2} \ , \ e_2 = \frac{j' + j''}{2} \ , \ e_3 = \frac{k' + k''}{2}$$

$$e^0 = \frac{1' + 1''}{2} \ , \ e^1 = \frac{i' - i''}{2} \ , \ e^2 = \frac{j' - j''}{2} \ , \ e^3 = \frac{k' - k''}{2} \ .$$

Then we get:

$$1'^2 = 1' \ , \ i'^2 = j'^2 = k'^2 = -1' \ , \ i'j' = -j'i' = k'$$

$$1''^2 = i''^2 = j''^2 = k''^2 = 1' \ , \ i''j'' = k'$$

$$i'j'' = -j''i' = i''j' = -j'i'' = -k''$$

$$(e_0)^2 = e_0 \ , \ (e^0)^2 = e^0 \ , \ e_0 e^0 = e^0 e_0 = 0$$

$$e_0 e_a = e_a e^0 = e_a \ , \ e^0 e^a = e^a e_0 = e^a \ , \ (a = 1,2,3)$$

$$e^0 e_a = e_a e_0 = e_0 e^a = e^a e^0 = 0$$

$$(e_a)^2 = (e^a)^2 = 0 \ , \ e_a e^a = -e_0 \ , \ e^a e_a = -e^0$$

$$e^a e^b = -e^b e^a = e_c \ , \ e_a e_b = -e_b e_a = e^c \ , \ (abc = 123, 231, 312)$$

$$e_a e^b = e^b e_a = 0$$

$$\bar{1}' = 1' \ , \ \bar{1}'' = 1'' \ , \ \bar{i}' = -i' \ , \ \bar{i}'' = i''$$

$$\bar{e}_0 = e_0 \ , \ \bar{e}^0 = e^0 \ , \ \bar{e}_a = -e^a \ , \ \bar{e}^a = -e_a.$$

The formula

$$E(x)=\sum_{\alpha=0}^{3}(x^\alpha e_\alpha + x_\alpha e^\alpha)$$

defines a map $E:\mathbf{R}^8\to\mathbf{Q}$. One has:

$$E(x)E(y)=(x^0y^0-\sum_a x^a y_a)e_0+(x_0y_0-\sum_a x_a y^a)e^0$$

$$+\sum_a(x^0y^a+x^ay_0+x_by_c-x_cy_b)e_a+\sum_a(x_0y_a+x_ay^0+x^by^c-x^cy^b)e^a.$$

When we denote

$$\bar{E}(x)=x_0e_0+x^0e^0-\sum_{a=1}^{3}(x^ae_a+x_ae^a)\ ,\ Q(x,y)=\sum_{\alpha=0}^{3}x_\alpha y^\alpha,$$

we get

$$E(x)\bar{E}(y)=Q(x,y)\ e^0+Q(y,x)\ e_0$$

$$+\sum_{a=1}^{3}\Big((x^ay^0-x^0y^a-x^by^c+x^cy^b)e_a+(x_ay_0-x_0y_a-x_by_c+x_cy_b)e^a\Big)$$

$$E(x)\bar{E}(y)+E(y)\bar{E}(x)=\Big(Q(x,y)+Q(y,x)\Big)\ 1'\ ,\ E(x)\bar{E}(x)=Q(x,x)\ 1'.$$

When $x_0+x^0=y_0+y^0=0$, we also have

$$E(x)E(y)+E(y)E(x)=-\Big(Q(x,y)+Q(y,x)\Big)\ 1'.$$

Multiplication on the left $w\mapsto E(x)w$ defines a linear map $\mathbf{Q}\to\mathbf{Q}$. Using the basis $(e_0\ ,...,e_3\ ,e^0,...,e^3)$, this linear map is represented by the matrix

$$E_l(x)=\begin{pmatrix} x^0I_4 & X_l \\ X_l^! & x_0I_4\end{pmatrix},$$

where

$$X_l=\begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x^3 & -x^2 \\ -x_2 & -x^3 & 0 & x^1 \\ -x_3 & x^2 & -x^1 & 0\end{pmatrix}\ ,\ X_l^!=\begin{pmatrix} 0 & x^1 & x^2 & x^3 \\ -x^1 & 0 & x_3 & -x_2 \\ -x^2 & -x_3 & 0 & x_1 \\ -x^3 & x_2 & -x_1 & 0\end{pmatrix}.$$

We shall have, for each vector $w\in\mathbf{R}^8$,

$$E\Big(E_l(x)w\Big)=E(x)E(w).$$

Similarly, the multiplication on the right $w\mapsto wE(x)$ is described by the matrix

$$E_r(x)=\begin{pmatrix} X_r' & X_r'' \\ X_r^{''!} & X_r^{'!}\end{pmatrix},$$

where

$$X'_r = \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \\ -x_1 & x_0 & 0 & 0 \\ -x_2 & 0 & x_0 & 0 \\ -x_3 & 0 & 0 & x_0 \end{pmatrix} , \ X''_r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -x^3 & x^2 \\ 0 & x^3 & 0 & -x^1 \\ 0 & -x^2 & x^1 & 0 \end{pmatrix}$$

$$X'^!_r = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ -x^1 & x^0 & 0 & 0 \\ -x^2 & 0 & x^0 & 0 \\ -x^3 & 0 & 0 & x^0 \end{pmatrix} , \ X''^!_r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -x_3 & x_2 \\ 0 & x_3 & 0 & -x_1 \\ 0 & -x_2 & x_1 & 0 \end{pmatrix} .$$

One has

$$X_l X_l^! = -(\sum_{i=1}^{3} x_i x^i) I_4 \ .$$

Let us denote

$$F(x) = \sum_{\alpha=0}^{3} x_\alpha x^\alpha = Q(x,x).$$

Then G is the linear group formed by the real 8×8-matrices A which verify the relations $det(A) = 1$, $F(Aw) = F(w)$.

When $F(x) = 1$, any of the relations $E(w') = E_l(x)E(w)$, $E(w') = E(w)E_r(x)$ implies $F(w') = F(w)$. This means that

When $F(x) = 1$, the matrices $E_l(x)$, $E_r(x)$ belong to the group G.

More generaly, denoting

$$E'_l(x) = -\begin{pmatrix} -x_0 I_4 & X_l \\ X_l^! & -x^0 I_4 \end{pmatrix},$$

we get

$$E_l(x) E'_l(x) = \Big(\sum_\alpha x_\alpha x^\alpha\Big) I_8$$

In particular, when $x_0 + x^0 = 0$, one has $E'_l(x) = -E_l(x)$ and

$$\Big(E_l(x)\Big)^2 = -\Big(\sum_{\alpha=0}^{3} x_\alpha x^\alpha\Big) I_8 \ .$$

Similar relations hold for the matrix $E_r(x)$.

Let us now consider the matrices r_α , r^α, s_α , s^α verifying the relations

$$E_l(x) = \sum_{\alpha=0}^{3} (-x^\alpha r_\alpha + x_\alpha r^\alpha) \ , \ E_r(y) = \sum_{\alpha=0}^{3} (y^\alpha s_\alpha - y_\alpha s^\alpha)$$

and denote

$$r = r^0 - r_0 \ , \ s = s^0 - s_0$$

$$k_a = \frac{1}{2}(r_a r^a - r^a r_a) \ , \ h_a = \frac{1}{2}(s_a s^a - s^a s_a)$$

We shall have, for $w \in \mathbf{R}^8$ and $\alpha = 0,1,2,3$

$$E(r_\alpha w) = e_\alpha E(w)\ ,\ E(r^\alpha w) = e^\alpha E(w).$$

Using the relation $E(x)\bar{E}(y) + E(y)\bar{E}(x) = \Big(Q(x,y) + Q(y,x)\Big)1'$, we get, for $a, b = 1, 2, 3$,

$$r^2 = s^2 = I_8\ ,\ r_a r^b + r^b r_a = \delta_a^b I_8\ ,\ s_a s^b + s^b s_a = \delta_a^b\ I_8$$

$$r r_a + r_a r = r r^a + r^a r = r_a r_b + r_b r_a = r^a r^b + r^b r^a = 0$$

$$s s_a + s_a s = \mathfrak{s} s^a + s^a s = s_a s_b + s_b s_a = s^a s^b + s^b s^a = 0.$$

To get more precise formulas, let us denote by e/e' the matrix having a single non vanishing entry, equal to 1 and situated on the line e and the column e'; then we have

$$(e_1/e)(e'/e_2) = \delta_{ee'}\ e_1/e_2$$

and, under the restrictions $a \neq b,\ abc = 123, 231, 312$,

$$r_0 = \sum_{\alpha=0}^{3} e_\alpha/e_\alpha\ ,\ r^0 = \sum_{\alpha=0}^{3} e^\alpha/e^\alpha\ ,\ r = \sum_{\alpha=0}^{3} \Big(e^\alpha/e^\alpha - e_\alpha/e_\alpha\Big)$$

$$r_a = -(e_b/e^c - e_c/e^b + e^0/e_a - e^a/e_0)$$
$$r^a = e_0/e^a - e_a/e^0 + e^b/e_c - e^c/e_b$$
$$r_0 r_a = r_a r^0 = -(e_b/e^c - e_c/e^b) = -t_a t^4$$
$$r^0 r_a = r_a r_0 = -(e^0/e_a - e^a/e_0) = t^b t^c$$
$$r_0 r^a = r^a r^0 = e_0/e^a - e_a/e^0 = -t_b t_c$$
$$r^0 r^a = r^a r_0 = e^b/e_c - e^c/e_b = t^a t_4$$
$$(r_0)^2 = r_0\ ,\ (r^0)^2 = r^0\ ,\ r_0 r^0 = r^0 r_0 = (r_a)^2 = (r^a)^2 = 0$$
$$r_a r^a = e^0/e^0 + e^a/e^a + e_b/e_b + e_c/e_c$$
$$r^a r_a = e_0/e_0 + e_a/e_a + e^b/e^b + e^c/e^c$$
$$k_a = -\frac{1}{2}(e_0/e_0 + e_a/e_a - e_b/e_b - e_c/e_c - e^0/e^0 - e^a/e^a + e^b/e^b + e^c/e^c).$$

When $abc = 123, 231, 312$, we also get

$$r_a r^b = -r^b r_a = -(e_b/e_a - e^a/e^b)$$

$$r_a r_b = -r_b r_a = e_c/e_0 - e^0/e^c = t^c t^4$$
$$r^a r^b = -r^b r^a = e^c/e^0 - e_0/e_c = t_c t_4\ .$$

When we denote

$$R = -r_1 r_2 r_3\ ,\ R' = r^1 r^2 r^3$$

we get

$$R = e^0/e_0\ ,\ R' = e_0/e^0\ ,\ R'R = e_0/e_0\ \ ,\ RR' = e^0/e^0$$
$$RR'R = R\ ,\ R'RR' = R'$$

$$r^a\ R = -e_a/e_0\ ,\ R\ r^a = e^0/e^a\ ,\ r^a\ R\ r^{c'} = -e_a/e^{c'}$$
$$r^a r^b\ R = e^c/e_0\ ,\ R\ r^a r^b = -e^0/e_c\ ,\ r^a r^b\ R\ r^{a'} r^{b'} = -e^c/e_{c'}$$
$$r^a\ R\ r^{a'} r^{b'} = e_a/e_{c'}\ ,\ r^a r^b\ R\ r^{a'} = e^c/e^{a'}\ ,\ r^a\ RR' = -e_a/e^0$$
$$r^a r^b\ RR' = e^c/e^0\ ,\ R'R\ r^{a'} = e_0/e^{a'}\ ,\ R'R\ r^{a'} r^{b'} = -e_0/e_{c'}\ .$$

Let us redenote the basis of $\mathbf{R}^8$ as follows

$$e_0^- = e_0\ ,\ e_i^- = e_i\ ,\ e_0^+ = e^0\ ,\ e_i^+ = e^i\ ,\ (i = 1, 2, 3) \tag{4.1}$$

and denote by e/e' the matrix verifying the formula

$$(e/e')e'' = \delta_{e'e''}\ e.$$

The matrix $\mathcal{E}$, with matricial entries

$$(e_0^-\ , e_1^-\ , e_2^-\ , e_3^-\ , e_0^+\ , e_1^+\ , e_2^+\ , e_3^+)/(e_0^-\ , e_1^-\ , e_2^-\ , e_3^-\ , e_0^+\ ,_1^+\ , e_2^+\ , e_3^+\ )$$

writes:

$$\mathcal{E} =$$
$$\begin{pmatrix}
R'R & -R'Rr^2r^3 & -R'Rr^3r^1 & -R'Rr^1r^2 & R' & R'Rr^1 & R'Rr^2 & R'Rr^3 \\
-r^1R & r^1Rr^2r^3 & r^1Rr^3r^1 & r^1Rr^1r^2 & -r^1RR' & -r^1Rr^1 & -r^1Rr^2 & -r^1Rr^3 \\
-r^2R & r^2Rr^2r^3 & r^2Rr^3r^1 & r^2Rr^1r^2 & -r^2RR' & -r^2Rr^1 & -r^2Rr^2 & -r^2Rr^3 \\
-r^3R & r^3Rr^2r^3 & r^3Rr^3r^1 & r^3Rr^1r^2 & -r^3RR' & -r^3Rr^1 & -r^3Rr^2 & -r^3Rr^3 \\
R & -Rr^2r^3 & -Rr^3r^1 & -Rr^1r^2 & RR' & Rr^1 & Rr^2 & Rr^3 \\
r^2r^3R & -r^2r^3Rr^2r^3 & -r^2r^3Rr^3r^1 & -r^2r^3Rr^1r^2 & r^2r^3RR' & r^2r^3Rr^1 & r^2r^3Rr^2 & r^2r^3Rr^3 \\
r^3r^1R & -r^3r^1Rr^2r^3 & -r^3r^1Rr^3r^1 & -r^3r^1Rr^1r^2 & r^3r^1RR' & r^3r^1Rr^1 & r^3r^1Tr^2 & r^3r^1Rr^3 \\
r^1r^2T & -r^1r^2Rr^2r^3 & -r^1r^2Rr^3r^1 & -r^1r^2Rr^1r^2 & r^1r^2RR' & r^1r^2Rr^1 & r^1r^2Rr^2 & r^1r^2Rr^3
\end{pmatrix}.$$

Using the relation

$$e/e' = (e/e_0)(e_0/e')$$

we can write

$$\mathcal{E} = \begin{pmatrix} r^1r^2r^3 \\ -r^1 \\ -r^2 \\ -r^3 \\ 1 \\ r^2r^3 \\ r^3r^1 \\ r^1r^2 \end{pmatrix} (R) \begin{pmatrix} 1 & -r^2r^3 & -r^3r^1 & -r^1r^2 & r^1r^2r^3 & r^1 & r^2 & r^3 \end{pmatrix}.$$

Denoting by $\mathcal{R}$ the column matrix on the left and by $\mathcal{R}^!$ the matrix obtained by transposing $\mathcal{R}$ and by applying the reversing operator to each entry, we can write

$$\mathcal{E} = \mathcal{R}\ (R)\ \mathcal{R}^!\ J\ ,\ J = \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix}.$$

As a consequence, we get the following relation:

$$\mathcal{E}^! = -J\ \mathcal{E}\ J.$$

As far as concerns the matrices s, we get the following formulas:

$$s_0 = e_0/e_0 + \sum_{a=1}^{3} e^a/e^a \ , \ s^0 = e^0/e^0 + \sum_{a=1}^{3} e_a/e_a$$

$$s = e^0/e^0 - e_0/e_0 + \sum_{a=1}^{3} \Big(e_a/e_a - e^a/e^a \Big)$$

$$s_a = e_0/e_a - e^a/e^0 - e_b/e^c + e_c/e^b$$

$$s^a = e_a/e_0 - e^0/e^a + e^b/e_c - e^c/e_b$$

$$s_0 s_a = s_a s^0 = e_0/e_a - e^a/e^0 = -r^b r^c$$

$$s^0 s_a = s_a s_0 = e_c/e^b - e_b/e^c = r_0 r_a$$

$$s_0 s^a = s^a s^0 = -e^c/e_b + e^b/e_c = -r^0 r_a$$

$$s^0 s^a = s^a s_0 = -e^0/e^a + e_a/e_0 = r_b r_c$$

$$(s_0)^2 = s_0 \ , \ (s^0)^2 = s^0 \ , \ s_0 s^0 = s^0 s_0 = (s_a)^2 = (s^a)^2 = 0 \ ,$$

$$s_a s_b = -s_b s_a = e_0/e^c - e_c/e^0 = r_0 r^c$$

$$s^a s^b = -s^b s^a = e^0/e_c - e^c/e_0 = r^0 r_c$$

$$s_a s^a = (e_0/e_0 + e^a/e^a + e_b/e_b + e_c/e_c)$$

$$s^a s_a = (e^0/e^0 + e_a/e_a + e^b/e^b + e^c/e^c)$$

$$s_a s^b = -s^b s_a = e_b/e_a - e^a/e^b \ , \ (a \neq b)$$

$$h_a = \frac{1}{2}(e_0/e_0 - e_a/e_a + e_b/e_b + e_c/e_c - e^0/e^0 + e^a/e^a - e^b/e^b - e^c/e^c)$$

$$S' = -s^1 s^2 s^3 = -e^0/e_0 = -R \ , \ S = s_1 s_2 s_3 = -e_0/e^0 = -R'$$

$$SS' = e_0/e_0 \ , \ S'S = e^0/e^0 \ , \ SS'S = S \ , \ S'SS' = S'$$

$$s_a S' = -e^a/e_0 \ , \ s_a s_b S' = e_c/e_0 \ , \ S' s_a = e^0/e_a \ , \ S' s_a s_b = -e^0/e^c$$

$$SS' s_a = -e_0/e_a \ , \ SS' s_a s_b = e_0/e^c \ , \ s_a S' S = e^a/e^0 \ , \ s_a s_b S' S = -e_c/e^0 \ ..$$

Resuming, we can give the matrix $\mathcal{E}$ the following expressions:

$$\mathcal{E} = \begin{pmatrix} s_1 s_2 s_3 \\ s_2 s_3 \\ s_3 s_1 \\ s_1 s_2 \\ -1 \\ -s_1 \\ -s_2 \\ -s_3 \end{pmatrix} (-R) \begin{pmatrix} 1 & -s_1 & -s_2 & -s_3 & -s_1 s_2 s_3 & s_2 s_3 & s_3 s_1 & s_1 s_2 \end{pmatrix} =$$

$$= \begin{pmatrix} r^1r^2r^3 \\ -r^1 \\ -r^2 \\ -r^3 \\ 1 \\ r^2r^3 \\ r^3r^1 \\ r^1r^2 \end{pmatrix} (R) \begin{pmatrix} 1 & -r^2r^3 & -r^3r^1 & -r^1r^2 & r^1r^2r^3 & r^1 & r^2 & r^3 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \\ -s^1 \\ -s^2 \\ -s^3 \\ -s^1s^2s^3 \\ -s^2s^3 \\ -s^3s^1 \\ -s^1s^2 \end{pmatrix} (R') \begin{pmatrix} -s^1s^2s^3 & s^2s^3 & s^3s^1 & s^1s^2 & 1 & s^1 & s^2 & s^3 \end{pmatrix}.$$

We add the relations

$$k_a\ (e/e') = \frac{1}{2}\ e/e'\ ,\ (e'/e)\ k_a = \frac{1}{2}\ e'/e\ , \text{ valid for } e = e_0^+\ , e_a^+\ , e_b^-\ , e_c^-$$

$$k_a\ (e/e') = -\frac{1}{2}\ e/e'\ ,\ (e'/e)k_a = -\frac{1}{2}\ e'/e\ , \text{ valid for } \ e = e_b^+\ , e_c^+\ , e_0^-\ , e_a^-$$

$$h_a\ (e/e') = \frac{1}{2}\ e/e'\ ,\ (e'/e)\ h_a = \frac{1}{2}\ e'/e\ , \text{ valid for } e = e_a^-\ , e_0^+\ , e_b^+\ , e_c^+$$

$$h_a\ (e/e') = -\frac{1}{2}\ e/e'\ ,\ (e'/e)\ h_a = -\frac{1}{2}\ e'/e\ , \text{ valid for } e = e_0^-\ , e_b^-\ , e_c^-\ , e_a^+.$$

We also have:

$$e_0^- = e_0 = (e_0/e_0)e_0 = R'Re_0 = SS'e_0$$
$$e_i^- = e_i = (e_i/e_0)e_0 = -r^iRe_0 = s_js_kS'e_0$$
$$e_0^+ = e^0 = (e^0/e_0)e_0 = Re_0 = -S'e_0$$
$$e_i^+ = e^i = (e^i/e_0)e_0 = r^jr^kRe_0 = -s_iS'e_0$$
$$(ijk = 123, 231, 312).$$

5 Summary: the spinorial G-modules

The group G is formed by the real 8×8-matrices A verifying the relations $det(A) = 1\ ,\ F(Ax) = F(x)$, where F is quadratic form

$$F(x) = \sum_{\alpha=0}^{3} x_\alpha\ x^\alpha,$$

is spanned by the following 28 matrices:

$$e_0/e_0 - e^0/e^0 = r_0 - r^0$$

$$e_a/e_a - e^a/e^a = k_b + k_c + r_0 - r^0 = h_b + h_c - r_0 + r^0$$

$$e_0/e_a - e^a/e^0 = -r^b r^c = s_0 s_a$$

$$e_0/e^a - e_a/e^0 = -r_0 r^a = s_b s_c$$

$$e_a/e_b - e^b/e^a = -r_b r^a = s_b s^a$$

$$e_a/e_0 - e^0/e^a = r_b r_c = -s^0 s^a$$

$$e_a/e^b - e_b/e^a = -r_0 r_c = -s^0 s_c$$

$$e^a/e_b - e^b/e_a = r^0 r^c = -s_0 s^c.$$

We produced four spinorial representations of the Lie algebra Γ, namely ρ_+ , ρ_- , ρ', ρ''. It is not difficult to prove that ρ' is equivalent to ρ_+, while ρ'' is equivalent to ρ_-.

The six matrices r_a , r^a generate algebraically a Clifford algebra C_6 associated with the quadratic form

$$F_0(x) = \sum_{a=1}^{3} x_a x^a.$$

The matrices r, r_a , r^a generate the Clifford algebra C_7 associated with the quadratic form $F_0(x) + z^2$.

For each $e \in \{e_0 , e_1 , e_2 , e_3 , e^0, e^1, e^2, e^3\}$, the vector space V_e, which is linearly spanned by the eight matrices e_α/e , e^α/e with $\alpha = 0, 1, 2, 3$, provides the fundamental spinorial representation of the pseudo-orthogonal group $G' \approx SO(3, 4)$ associated with the quadratic form $F_0(x) + (x^0)^2$ and also the two fundamental spinorial representations of the pseudo-orthogonal group $G'' \approx SO(3, 3)$ associated with $F_0(x)$.

The 15 matrices $r_a r_b$, $(a \neq b)$, $r_a r^b - r^b r_a$, $r^a r^b$, $(a \neq b)$ span the Lie algebra of the group G''.

The 21 matrices rr_a , $rr^a, r_a r_b$, $r_a r^b - r^b r_a$, $r^a r^b$ span the Lie algebra of the group G'.

The 28 matrices $t_0 t_a$, $t_0 t^a, t^0 t_a, t^0 t^a, t_a t_b$, $t_a t^b, t^a t^b$ span the Lie algebra g of the pseudo-orthogonal group $G \approx SO(4, 4)$ associated with the quadratic form $F(x) = \sum_{\alpha=0}^{3} x_\alpha x^\alpha$.

References

[1] H.B. Lawson, M.-L. Michelson, *Spin Geometry*, Princeton, 1997.

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