On subprojective transformations

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Dedicated to the memory of Radu Rosca (1908-2005)

Abstract. The aim of this paper is to study subgeodesically related spaces. Using some results of Levi-Civita and Vrănceanu an example of projectively equivalent Riemann metrics is given. ξ-subcharacteristic vector fields are studied for some deformation algebras and it is also illustrated the relation with the concept of ξ-subgeodesically related connexions.

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1 Introduction

Let M be a connected paracompact, smooth manifold of dimension $n \geq 3$. Let $\mathcal{X}(M)$ be the Lie algebra of vector fields on M, $\mathcal{T}^{(p,q)}(M)$ the $\mathcal{C}^\infty(M)$-module of tensor fields of type (p,q) on M, $\Lambda^p(M)$ the $\mathcal{C}^\infty(M)-$ module of $p-$forms on M and $H^p(M)$ the $p-$th de Rham cohomology group of M.

Let Γ^i_{jk} be the components of an affine symmetric connection ∇ and ξ^i be the components of a vector field ξ. One can associate the differential system of equations

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = a\frac{dx^i}{dt} + b\xi^i, \tag{1.1}$$

a and b being functions of t, which defines the ξ-subgeodesics.

K. Yano introduced the subprojective transformations of connections, which preserve the ξ-subgeodesics

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\omega_k + \delta_k\omega_j + \theta_{jk}\xi^i, \tag{1.2}$$

where ω_i and θ_{jk} are the components of a 1-form and of a symmetric tensor field of type (0,2), respectively.

Two Riemannian spaces (M,g) and $(M,\overline{g})$ are ξ-subgeodesically related, the tensor of correspondence θ_{jk}, being $-g_{jk}$, if the Levi-Civita connections associated to g and $\overline{g}$ satisfy the Yano formulae (1.2). Therefore there exists a diffeomorphism f between these two spaces which maps ξ-subgeodesics onto ξ-subgeodesics. f is called the subgeodesic mapping.

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If $\xi^i = 0$, then the Yano formulae become the Weyl formulae and spaces are geodesically related.

In the present paper subgeodesically and geodesically spaces are considered. The Levi-Civita and Vrănceanu canonical forms are given for certain projectively equivalent metrics on some Weyl manifolds.

It is also illustrated the close ties that exist between the ξ-subcharacteristic vector fields and ξ-subgeodesically related connections.

2 On ξ-subcharacteristic vector fields

Let A be a $(1,2)-$tensor field on M. The $\mathcal{C}^\infty(M)-$ modul $\mathcal{X}(M)$ becomes a $\mathcal{C}^\infty(M)-$algebra if we consider the multiplication rule given by
$X \circ Y = A(X,Y)$, $\forall X, Y \in \mathcal{X}(M)$. This algebra is denoted by $\mathcal{U}(M,A)$ and it is called the algebra associated to A. If ∇ and ∇' are two linear connections on M and $A = \nabla' - \nabla$, then $\mathcal{U}(M,A)$ is called the deformation algebra defined by the pair (∇, ∇').

A vector field $X \in \mathcal{X}(M)$ is called ξ-subcharacteristic in the deformation algebra $\mathcal{U}(M,A)$ if there exists two functions $\lambda, \mu \in \mathcal{C}^\infty(M)$ such that

$$A(X,X) = \lambda X + \mu\xi. \tag{2.1}$$

Remark 2.1 1) If X is a nonvanishing ξ-subcharacteristic vector field i.e. is a vector field of ξ-subcharacteristic direction, then (2.1) is equivalent to

$$A(X,X) \otimes X - X \otimes A(X,X) = \mu(\xi \otimes X - X \otimes \xi).$$

2) The trajectories of vector fields of ξ-subcharacteristic directions, called the ξ-subcharacteristic curves, satisfy the following differential system of equations

$$B^{ijpq}_{ksrh} \frac{dx^k}{dt} \frac{dx^s}{dt} \frac{dx^r}{dt} \frac{dx^h}{dt} = 0, \tag{2.2}$$

where $B^{ijpq}_{ksrh} = (A^i_{ks}\delta^j_r - A^j_{ks}\delta^i_r)(\delta^q_h\xi^p - \delta^q_h\xi^p) - (A^p_{ks}\delta^q_r - A^q_{ks}\delta^p_r)(\delta^i_h\xi^j - \delta^j_h\xi^i)$.

The geometric interpretation of vector fields of ξ-subcharacteristic direction is given by the following result

Proposition A[8] *Let ∇ and ∇' be two symmetric linear connections on M and $\xi \in \mathcal{X}(M)$. Let $X \in \mathcal{X}(M), X_p \neq 0, \forall p \in M$ such that X and ξ are either independent $\forall p \in M$ or dependent $\forall p \in M$. The following assertions are equivalent:*

1) X is a vector field of ξ -subcharacteristic direction in the deformation algebra $\mathcal{U}(M, \nabla' - \nabla)$.

2) Let any $p \in M$. If c is a (ξ, ∇)-subgeodesic verifying

$$c(t_0) = p, \frac{dc}{dt} \mid_{t_0} = aX_p, a \in \mathbf{R}^*,$$

then the point p is (ξ, ∇')-subgeodesic i.e. ξ_p belongs to the osculating plane of the curve c at p.

The following result illustrates the relation between the ξ-subcharacteristic vector fields and the ξ-subgeodesically related connections:

Proposition B [8] *Let ∇ and ∇' be two symmetric linear connections on M and $\xi \in \mathcal{X}(M)$. The following assertions are equivalent:*

1) All the elements of the algebra $\mathcal{U}(M, \nabla' - \nabla)$ are ξ-subcharacteristic vector fields.

2) In every point $p \in M$ there exists a curve ξ-subcharacteristic tangent to a given direction.

3) There exists a symmetric (0,2)-tensor field θ and a 1-form ω on M such that

$$\nabla'_X Y - \nabla_X Y = \omega(X)Y + \omega(Y)X + \theta(X,Y)\xi, \forall X, Y \in \mathcal{X}(M).$$

4) ∇' and ∇ have the same ξ-subgeodesics.

3 On geodesically and subgeodesically related Riemann spaces

Let g be a Riemannian metric on M. A Weyl manifold is a triple $(M, \widehat{g}, W)$, where $\widehat{g} = \{e^u g \mid u \in \mathcal{C}^\infty(M)\}$ is the conformal class defined by g and $W : \widehat{g} \longrightarrow \Lambda^1(M)$ is a Weyl structure on the conformal manifold $(M, \widehat{g})$, hence

$$W(e^u g) = W(g) - du, \forall u \in \mathcal{C}^\infty(M). \tag{3.1}$$

A linear connection ∇ on M is compatible with the Weyl structure W if

$$\overset{W}{\nabla} g + W(g) \otimes g = 0. \tag{3.2}$$

There exists a unique torsion free linear connection $\overset{W}{\nabla}$, verifying (3.2), given by the formula:

$$\begin{aligned} 2g(\overset{W}{\nabla}_X Y, Z) &= X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + \\ &+ W(g)(X)g(Y,Z) + W(g)(Y)g(X,Z) - W(g)(Z)g(X,Y) + \\ &+ g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X), \ \forall X, Y, Z \in \mathcal{X}(M). \end{aligned} \tag{3.3}$$

$\overset{W}{\nabla}$ is called the Weyl conformal connection. This connection is invariant under a "gauge transformation" $g \longrightarrow e^u g$. So, the $1-$form $W(g)$ is required to change by (1.1).

Weyl introduced a $2-$form $\psi(W)$ on M by setting $\psi(W) = dW(g)$, $g \in \widehat{g}$, and called it the distance curvature. This is a gauge invariant. If $\psi(W) = 0$, then by (1.1), the cohomology class $[W(g)] \in H^1(M)$ of the closed form $W(g)$ does not depend on the choice of a metric in $\widehat{g}$. For simplicity, we write $ch(W) = [W(g)]$. The $2-$form $\psi(W)$ and the class $ch(W)$ are the obstructions for a Weyl structure to be a Riemannian structure. Indeed:

Proposition C [2] *Let $(M, \widehat{g}, W)$ be a Weyl manifold and $\overset{W}{\nabla}$ be the Weyl conformal connection. Then the following two conditions are equivalent:*

1) $\psi(W) = 0$ and $ch(W) = 0$;

2) There is a Riemann metric in $\widehat{g}$ *such that* $\overset{W}{\nabla} g = 0$.

Let π be a 1−form on M. We denote by $\overset{L}{\nabla}$ the connection compatible with the Weyl structure W, which is π-semi-symmetric i.e. the torsion tensor is required to be $\overset{L}{T}(X,Y) = \pi(Y)X - \pi(X)Y,\ \forall X,Y \in \mathcal{X}(M)$ and

$$\begin{gathered} 2g(\overset{L}{\nabla}_X Y, Z) = X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + \\ +W(g)(X)g(Y,Z) + W(g)(Y)g(X,Z) - \\ -W(g)(Z)g(X,Y) + 2\pi(Y)g(X,Z) - \\ -2\pi(Z)g(X,Y) + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X) \end{gathered} \tag{3.4}$$

holds. The relation between these two connections is given by

$$\overset{L}{\nabla}_X Y = \overset{W}{\nabla}_X Y + \pi(Y)X - g(X,Y)\pi^\sharp, \tag{3.5}$$

where $g(Z,\pi^\sharp) = \pi(Z),\ \forall Z \in \mathcal{X}(M)$.

We denote by $\overset{L}{\overline{\nabla}}$ the transposed connection of $\overset{L}{\nabla}$ i.e.

$$\overset{L}{\overline{\nabla}}_X Y = \overset{L}{\nabla}_Y X + [X,Y]. \tag{3.6}$$

The relations (3.5) and (3.6) lead to

$$\overset{L}{\overline{\nabla}}_X Y = \overset{W}{\nabla}_X Y + \pi(X)Y - g(X,Y)\pi^\sharp. \tag{3.7}$$

Let us denote by $\overset{s}{\nabla}$ the symmetric connection associated to $\overset{L}{\nabla}$ i.e. $\overset{s}{\nabla} = \frac{1}{2}(\overset{L}{\nabla} + \overset{L}{\overline{\nabla}})$. Hence

$$\overset{s}{\nabla}_X Y = \overset{W}{\nabla}_X Y + \frac{1}{2}\pi(X)Y + \frac{1}{2}\pi(Y)X - g(X,Y)\pi^\sharp. \tag{3.8}$$

Let (M,g) be a Riemanian manifold. Let $(M,\widehat{g},W)$ be a Weyl manifold and $\pi \in \wedge^1(M)$. Let $\overset{\circ}{\nabla}$ be the Levi-Civita connection associated to g. From (3.3) one gets

$$\overset{W}{\nabla}_X Y = \overset{\circ}{\nabla}_X Y + \varphi(X)Y + \varphi(Y)X - g(X,Y)\varphi^\sharp, \tag{3.9}$$

where $2\varphi = W(g)$ and $g(\varphi^\sharp, X) = \varphi(X), \forall X \in \mathcal{X}(M)$. The relation (3.8) leads to

$$\overset{s}{\nabla}_X Y = \overset{\circ}{\nabla}_X Y + (\varphi + \frac{1}{2}\pi)(X)Y + (\varphi + \frac{1}{2}\pi)(Y)X - g(X,Y)(\pi + \varphi)^\sharp. \tag{3.10}$$

Let us suppose that $\overset{s}{\nabla}$ is the Levi-Civita connection associated to another Riemannian metric $\tilde{g}$ on M. Let $g_{ij}, \tilde{g}_{ij}, \varphi_i, \pi_i$ be the local components of $g, \tilde{g}, \varphi$ and π respectively, in a local system of coordinates $(x^1, \dots, x^n)$. We denote with $\begin{vmatrix} i \\ jk \end{vmatrix}, \widetilde{\begin{vmatrix} i \\ jk \end{vmatrix}}$ the Christoffel symbols of the metrics

$$ds^2 = g_{ij}dx^i dx^j, d\tilde{s}^2 = \tilde{g}_{ij}dx^i dx^j. \tag{3.11}$$

The relation (3.10) becomes

$$\widetilde{\left| \begin{matrix} i \\ jk \end{matrix} \right|} = \left| \begin{matrix} i \\ jk \end{matrix} \right| + \delta^i_j(\varphi_k + \frac{1}{2}\pi_k) + \delta^i_k(\varphi_j + \frac{1}{2}\pi_j) - g_{jk}(\pi^i + \varphi^i), \tag{3.10$'$}$$

where $\pi^i = g^{ij}\pi_j, \varphi^i = g^{ij}\varphi_j$. Considering $i = j$ in (3.10)$'$ and summing, one gets

$$n\varphi_k + \frac{n-1}{2}\pi_k = \widetilde{\left| \begin{matrix} i \\ ik \end{matrix} \right|} - \left| \begin{matrix} i \\ ik \end{matrix} \right| = \frac{\partial}{\partial x^k}\left(ln\sqrt{\frac{det(\tilde{g}_{ij})}{det(g_{ij})}}\right). \tag{3.10$''$}$$

Let us denote with $\xi = (\pi + \varphi)^\sharp$. The formula (3.10)$'$ implies

$$\widetilde{\left| \begin{matrix} i \\ jk \end{matrix} \right|} = \left| \begin{matrix} i \\ jk \end{matrix} \right| + \delta^i_j\omega_k + \delta^i_k\omega_j - g_{jk}\xi^i, \tag{3.12}$$

where $\omega_i = \varphi_i + \frac{1}{2}\pi_i, \xi^i = \varphi^i + \pi^i$. Therefore the metrics (3.11) are ξ^i-subgeodesically related. There exist differentiable mappings u and h, with variables $(x^1, \ldots, x^n)$, such that $\xi_i = \frac{\partial u}{\partial x^i}$ and $\omega_i = \frac{\partial h}{\partial x^i}$ [9].

Therefore $\varphi_i = \frac{1}{2}\frac{\partial(u+h)}{\partial x^i}, \pi_i = \frac{1}{2}\frac{\partial(u-h)}{\partial x^i}, \forall i = \overline{1,n}$.

We consider $\tilde{\tilde{g}} = e^{2u}g$. One has

$$\widetilde{\widetilde{\left| \begin{matrix} i \\ jk \end{matrix} \right|}} = \left| \begin{matrix} i \\ jk \end{matrix} \right| + \delta^i_j\xi_k + \delta^i_k\xi_j - g_{jk}\xi^i, \tag{3.13}$$

where $\widetilde{\widetilde{\left| \begin{matrix} i \\ jk \end{matrix} \right|}}$ are the Christoffel symbols associated to $\tilde{\tilde{g}}$. Therefore one gets

$$\widetilde{\left| \begin{matrix} i \\ jk \end{matrix} \right|} = \widetilde{\widetilde{\left| \begin{matrix} i \\ jk \end{matrix} \right|}} + \delta^i_j\sigma_k + \delta^i_k\sigma_j, \tag{3.14}$$

where $\sigma_i = \omega_i - \xi_i$. Hence the metrics

$$d\tilde{\tilde{s}}^2 = \tilde{\tilde{g}}_{ij}dx^i dx^j, d\tilde{s}^2 = \tilde{g}_{ij}dx^i dx^j \tag{3.15}$$

are geodesically related. So, the metrics (3.15) can be reduced to the canonical forms of Levi-Civita and Vrănceanu (according to the fact that the Riemann space $(M, \tilde{\tilde{g}})$ is of cathegory n or cathegory $m < n$).

$$\begin{gathered} dV^2 = a_1(x^1)f'(x^1)(dx^1)^2 + \ldots + a_n(x^n)f'(x^n)(dx^n)^2, \\ dL^2 = \tfrac{1}{x^1 \ldots x^n}\{\tfrac{a_1(x^1)f'(x^1)}{x^1}(dx^1)^2 + \ldots + \tfrac{a_n(x^n)f'(x^n)}{x^n}(dx^n)^2\}, \end{gathered} \tag{3.16}$$

where $f(x) = (x - x^1) \cdot \ldots \cdot (x - x^n)$ or

$$\begin{gathered} dV^2 = a_i(x^i)F'(x^i)(dx^i)^2 + F(c^2)c_{\lambda\mu}(x^{m+1}, \ldots, x^p)dx^\lambda dx^\mu + \\ +F(k^2)c_{\alpha'\beta'}(x^{p+1}, \ldots, x^n)dx^{\alpha'}dx^{\beta'}, \\ dL^2 = \tfrac{1}{x^1 \ldots x^m}\{\tfrac{a_i(x^i)F'(x^i)}{x^i}(dx^i)^2 + \tfrac{F(c^2)}{c^2}c_{\lambda\mu}(x^{m+1}, \ldots, x^p)dx^\lambda dx^\mu + \\ +\frac{F(k^2)}{k^2}c_{\alpha'\beta'}(x^{p+1}, \ldots x^n)dx^{\alpha'}dx^{\beta'}\}, \end{gathered} \tag{3.17}$$

where $F(x) = (x - x^1) \cdot \ldots \cdot (x - x^m), 1 \le i \le m, m+1 \le \lambda, \mu \le p, p+1 \le \alpha', \beta' \le n$ and c^2 and k^2 are nonvanishing constants. Therefore the metrics (3.11) can be reduced to

$$ds^2 = e^{-2u(x^1,\ldots,x^n)} dV^2, d\tilde{s}^2 = dL^2.$$

Hence we obtain:

Theorem 3.1 *Let (M, g) be a Riemannian space and W a Weyl structure on the conformal manifold $(M, \widehat{g})$. Let π be a 1-form on M, $\overset{L}{\nabla}$ be the π-semi-symmetric conformal connection, $\overset{s}{\nabla}$ be the symmetric connection associated to $\overset{L}{\nabla}$. We suppose that $\overset{s}{\nabla}$ is the Levi-Civita connection associated to another Riemannian metric $\tilde{g}$ on M. Then*

i) The 1-forms $W(g)$ and π are exact.

ii) The metrics (3.11) can be reduced to $ds^2 = e^{-2u(x^1,\ldots,x^n)} dV^2, d\tilde{s} = dL^2$, where dV^2 and dL^2 are the canonical forms of Levi-Civita and Vrănceanu, given by (3.16) or (3.17), according to the case when the equation $det(\tilde{g}_{ij} - r^2 g_{ij}) = 0$ has distinct roots or has $m < n$ equal roots.

Remark 3.1. Let us consider the first formula (3.17) for $c = k$. Multiplying all the variables $x^1, \ldots, x^n$ with the same constant, we can suppose that c is the unit. Therefore the metric dV^2 can be written

$$dV^2 = a_i(x^i) F'(x^i)(dx^i)^2 + F(1) c_{\alpha\beta} dx^\alpha dx^\beta. \tag{3.18}$$

One gets the next result, under the same hypothesis of the previous theorem:

Theorem 3.2. *The metric $ds^2 = g_{ij} dx^i dx^j$ can be written $ds^2 = e^{-2u(x^1,\ldots,x^n)} dV^2$, where dV^2 is given by the first formula of (3.16) or by the expression (3.18), if the equation $det(\tilde{g}_{ij} - r^2 g_{ij}) = 0$ has distinct roots or has $m < n$ equal roots, respectively.*

The last result underlines the connection between the concept of ξ - subcharacteristic vector fields and of those of deformation algebra on Weyl manifolds:

Theorem 3.3. *Let (M, g) be a Riemannian space and W a Weyl structure on the conformal manifold $(M, \widehat{g})$. Let π be a 1-form on M, $\overset{L}{\nabla}$ be the π-semi-symmetric conformal connection, $\overset{s}{\nabla}$ be the symmetric connection associated to $\overset{L}{\nabla}$. We suppose that $\overset{s}{\nabla}$ is the Levi-Civita connection associated to another Riemannian metric $\tilde{g}$ on M. Let $\tilde{\tilde{\nabla}}$ be a connection conformally related to the Levi-Civita connection $\overset{\circ}{\nabla}$.*

Then the deformation algebras $\mathcal{U}(M, \overset{s}{\nabla} - \overset{\circ}{\nabla})$ and $\mathcal{U}(M, \tilde{\tilde{\nabla}} - \overset{\circ}{\nabla})$ have the same ξ-subcharacteristic vector fields, where $\xi = (\pi + \frac{1}{2} W(g))^\sharp$.

Proof. One considers $\tilde{A} = \overset{s}{\nabla} - \overset{\circ}{\nabla}$ and $\tilde{\tilde{A}} = \tilde{\tilde{\nabla}} - \overset{\circ}{\nabla}$.

$\tilde{\tilde{\nabla}}$ and $\overset{s}{\nabla}$ being geodesically related, one has

$\tilde{\tilde{A}}(X, Y) - \tilde{A}(X, Y) = \frac{1}{2}\pi(X)Y + \frac{1}{2}\pi(Y)X.$

Let $X \in \mathcal{U}(M, \tilde{A})$ be a ξ-subcharacteristic vector field. So, there exist $\lambda, \mu \in \mathcal{C}^\infty(M)$ such that $\tilde{A}(X, X) = \lambda X + \mu\xi$, where $\lambda = (W(g) + \pi)(X)$ and $\mu = -g(X, X)$.

Therefore $\tilde{\tilde{A}}(X,X) = \nu X + \mu\xi$, where $\nu = (W(g) + \frac{3}{2}\pi)(X)$ and X is a ξ-subcharacteristic vector field of the algebra $\mathcal{U}(M, \tilde{\tilde{A}})$.

The converse inclusion is analogous.

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