A generalization of the holomorphic flag curvature of complex Finsler spaces

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Dedicated to the memory of Radu Rosca (1908-2005)

Abstract. The notion of holomorphic bi-flag curvature for a complex Finsler space (M, F) is defined with respect to the Chern complex linear connection on the pull-back tangent bundle. By means of holomorphic curvature and holomorphic flag curvature of a complex Finsler space, a special approach is devoted to obtain the characterizations of the holomorphic bi-flag curvature. For the class of generalized Einstein complex Finsler spaces some results concerning the holomorphic bi-flag curvature are obtained.

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1 Introduction

In complex Finsler geometry, it is systematically used the notion of holomorphic curvature in η direction, briefly holomorphic curvature, [1]. In the previous papers, [3, 4], we initiated the study of holomorphic curvature of a complex Finsler spaces with respect to the Chern complex linear connection, in brief Chern $(c.l.c)$, as a connection in the holomorphic pull-back tangent bundle $\pi^*(T'M)$. Our goal was to determine the conditions in which a complex Finsler metric has constant holomorphic curvature. With this we marked out a special class of complex Finsler spaces which we called generalized Einstein, $(g.E.)$, for which the question has a favorable answer. In another paper, [5], we gave a generalization of the holomorphic curvature of the complex Finsler spaces named by us holomorphic flag curvature.

Our purpose is to obtain a generalization of the holomorphic flag curvature of a complex Finsler space. The second section of the present paper is devoted to the notion of the holomorphic bi-flag for such a space. We determine the link between the holomorphic bi-flag curvature and the holomorphic flag curvature, (Proposition 2.3). We prove a necessary and sufficient condition that a complex Finsler space has constant holomorphic bi-flag curvature, (Proposition 2.4). In §3 a special approach is dedicated to the holomorphic bi-flag curvature of the $(g.E.)$ complex Finsler spaces. First, for the $(g.E.)$ spaces we find the expression of the holomorphic bi-flag curvature by means of holomorphic curvature (Theorem 3.1). The obtained information is

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used to establish some inequalities between the three kinds of curvature (holomorphic curvature, holomorphic flag curvature and holomorphic bi-flag curvature) of a Kähler complex Finsler space ($g.E.$) with nonzero constant holomorphic curvature, (Propositions 3.2 - 3.7).

In the present section we setting the basic notions which are needed; for more information see [1, 10, 3, 4, 5].

Let M be a complex manifold, $\dim_C M = n$, and $T'M$ the holomorphic tangent bundle in which as a complex manifold the local coordinates will be denoted by (z^k, η^k). The complexified tangent bundle of $T'M$ is decomposed in $T_C(T'M) = T'(T'M) \oplus T''(T'M)$.

Considering the restriction of the projection to $\widetilde{T'M} = T'M \setminus \{0\}$, for pulling the holomorphic tangent bundle $T'M$ back, we obtain a holomorphic tangent bundle $\pi' : \pi^*(T'M) \longrightarrow \widetilde{T'M}$, called *the pull-back tangent bundle* over the slit $\widetilde{T'M}$. We denote by $\left\{\frac{\partial}{\partial z^k}^*, \frac{\partial}{\partial \overline{z}^k}^*\right\}$, and by $\left\{dz^{*k}, d\overline{z}^{*k}\right\}$, the local frame and its dual.

Let $V(T'M) = \ker \pi_* \subset T'(T'M)$ be the vertical bundle, spanned locally by $\{\frac{\partial}{\partial \eta^k}\}$. A complex nonlinear connection, briefly ($c.n.c.$), determines a supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. The adapted frames of the ($c.n.c.$) is $\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}$, where $N_k^j(z,\eta)$ are the coefficients of the ($c.n.c.$). Further on we shall use the abbreviations $\delta_i = \frac{\delta}{\delta z^i}$, $\dot{\partial}_i = \frac{\partial}{\partial \eta^i}$, $\delta_{\bar{i}} = \frac{\delta}{\delta \overline{z}^i}$, $\dot{\partial}_{\bar{i}} = \frac{\partial}{\partial \overline{\eta}^i}$, and theirs conjugates ([1], [2], [10]).

On $T'M$ let $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \overline{\eta}^j}$ be the fundamental metric tensor of a complex Finsler space $(M, F^2 = L)$.

The isomorphism between $\pi^*(T'M)$ and $T'M$ induces an isomorphism of $\pi^*(T_C M)$ and $T_C M$. Thus, $g_{i\bar{j}}$ defines an Hermitian metric structure $\mathcal{G}(z,\eta) := g_{j\overline{k}} dz^{*j} \otimes d\overline{z}^{*k}$on $\pi^*(T_C M)$, with respect to the natural complex structure. Further, the Hermitian metric structure $\mathcal{G}$ on $\pi^*(T'M)$ induces a Hermitian inner product $h(\chi,\gamma) := Re\mathcal{G}(\chi,\overline{\gamma})$ and the angle $\cos(\chi\gamma) = \frac{\mathrm{Re}\mathcal{G}(\chi,\overline{\gamma})}{||\chi|| ||\overline{\gamma}||}$, for any χ, γ the sections on $\pi^*(T'M)$, where $||\chi||^2 = ||\overline{\chi}||^2 = \mathcal{G}(\chi,\overline{\chi})$, see [3].

On the other hand, $H(T'M)$ and $\pi^*(T'M)$ are isomorphic. Therefore the structures on $\pi^*(T_C M)$ can be pulled-back to $H(T'M) \oplus \overline{H(T'M)}$. By this isomorphism the natural cobasis dz^{*j} is identified with dz^j. In view of this construction the pull-back tangent bundle $\pi^*(T'M)$ admits a unique complex linear connection ∇, called the Chern ($c.l.c.$), which is metric with respect to $\mathcal{G}$ and of $(1,0)-$ type, [3]:

$$\begin{aligned} \omega_j^i(z,\eta) &= L_{jk}^i(z,\eta) dz^k + C_{jk}^i(z,\eta)\delta\eta^k; \\ L_{jk}^i &= g^{\overline{m}i} \frac{\delta g_{j\overline{m}}}{\delta z^k}\ ;\ C_{jk}^i = g^{\overline{m}i} \frac{\partial g_{j\overline{m}}}{\partial \eta^k}. \end{aligned} \tag{1.1}$$

The Chern ($c.l.c.$) on $\pi^*(T'M)$ determines the Chern-Finsler ($c.n.c.$) on $T'M$, with the coefficients $\overset{CF}{N_k^i} = g^{\overline{m}i} \frac{\partial g_{j\overline{m}}}{\partial z^k} \eta^j$, and its local coefficients of torsion and curvature are

$$T^i_{jk} \; : \; = L^i_{jk} - L^i_{kj}\ ; \tag{1.2}$$
$$R^i_{j\bar{h}k} \; : \; = -\delta_{\bar{h}} L^i_{jk} - \delta_{\bar{h}}(\overset{CF}{N^l_k}) C^i_{jl}\ ;\ \Xi^i_{j\bar{h}k} := -\delta_{\bar{h}} C^i_{jk} = \Xi^i_{k\bar{h}j};$$
$$P^i_{j\bar{h}k} \; : \; = -\dot{\partial}_{\bar{h}} L^i_{jk} - \dot{\partial}_{\bar{h}}(\overset{CF}{N^l_k}) C^i_{jl}\ ;\ S^i_{j\bar{h}k} := -\dot{\partial}_{\bar{h}} C^i_{jk} = S^i_{k\bar{h}j}.$$

The Riemann type tensor

$$\mathbf{R}(W, \overline{Z}, X, \overline{Y}) := \mathcal{G}(R(X, \overline{Y})W, \overline{Z})$$

has the properties:

$$\mathbf{R}(W, \overline{Z}, X, \overline{Y}) = W^i \overline{Z}^j X^k \overline{Y}^h R_{i\bar{j}k\bar{h}};\ \ R_{\bar{j}i\bar{h}k} := R^l_{i\bar{h}k} g_{l\bar{j}}; \tag{1.3}$$
$$R_{i\bar{j}k\bar{h}} = -R_{i\bar{j}\bar{h}k} = \overline{R_{j\bar{i}h\bar{k}}} = R_{\bar{j}i\bar{h}k};$$
$$\text{If } R^i_{j\bar{h}k} = R^i_{k\bar{h}j} \text{ then } R_{i\bar{j}k\bar{h}} = R_{k\bar{j}i\bar{h}} = R_{k\bar{h}i\bar{j}}.$$

We denoting by $R_{\bar{j}k} := R_{i\bar{j}k\bar{h}}\eta^i\overline{\eta}^h = -g_{l\bar{j}}\delta_{\bar{h}}(\overset{CF}{N^l_k})\overline{\eta}^h$ the Ricci tensor, which is 1− homogeneous with respect to η.

According to [1] the complex Finsler space (M, F) is *strongly Kähler* iff $T^i_{jk} = 0$, *Kähler* iff $T^i_{jk}\eta^j = 0$ and *weakly Kähler* iff $g_{i\bar{l}}T^i_{jk}\eta^j\overline{\eta}^l = 0$. Note that a complex Finsler metric which comes from a Hermitian metric on M is called *purely Hermitian metric* in [10], i.e. $g_{i\bar{j}} = g_{i\bar{j}}(z)$, and then the three nuances of Kähler spaces coincide, [13].

The holomorphic flag curvature of F along of the flag (η, χ), with respect to the Chern $(c.l.c.)$, is ([5])

$$\mathcal{K}_F(z, \eta, \chi) := \frac{\mathbf{R}(\eta, \overline{\chi}, \eta, \overline{\chi}) + \mathbf{R}(\chi, \overline{\eta}, \chi, \overline{\eta})}{\mathcal{G}(\eta, \overline{\eta})\mathcal{G}(\chi, \overline{\chi})}, \tag{1.4}$$

where η and χ are local section of $\pi^*(T'M)$. In particular, if η is colinear with χ then we obtain the holomorphic flag curvature from [1]

$$\mathcal{K}_F(z, \eta) := \frac{2\mathbf{R}(\eta, \overline{\eta}, \eta, \overline{\eta})}{\mathcal{G}^2(\eta, \overline{\eta})} = \frac{2\overline{\eta}^j\eta^k R_{\bar{j}k}}{L^2(z, \eta)}. \tag{1.5}$$

From [4], we have

Definition 1.1. *The complex Finsler space (M, F) is called generalized Einstein if $R_{\bar{j}k}$ is proportional to $t_{k\bar{j}}$, i.e. if there exists a real valuated function $K(z, \eta)$, such that*

$$R_{\bar{j}k} = K(z, \eta)t_{k\bar{j}}, \tag{1.6}$$

where $t_{k\bar{j}} := L(z, \eta)g_{k\bar{j}} + \eta_k\overline{\eta}_j$, $\eta_k := \frac{\partial L}{\partial \eta^k}$, $\bar{\eta}_j := \frac{\partial L}{\partial \bar{\eta}^j}$.

A $(g.E.)$ complex Finsler space enjoys of some interesting properties which we collect in:

Theorem 1.1. *Let (M,F) be a $(g.E.)$ complex Finsler space. Then*

i) $K(z,\eta)=\frac{1}{4}\mathcal{K}_F(z,\eta)$; ii) K depends on z alone.

iii) If (M,F) is connected and weakly Kähler, of complex dimension ≥ 2, then it is a space with constant holomorphic curvature.

iv) If the space of nonzero constant holomorphic curvature, then F is weakly Kähler.

v) If the space is Kähler of nonzero constant holomorphic curvature, then F is purely Hermitian. Conversely, a purely Hermitian complex Finsler space, which is Kähler of constant holomorphic curvature, is $(g.E.)$.

Note that for the particular case of the complex Finsler spaces which are Kähler of nonzero constant holomorphic curvature, the notions of $(g.E.)$ and purely Hermitian spaces coincide.

Finally, we recall here that in [5] it is proved

Proposition 1.1. *Let (M,F) be a $(g.E.)$ complex Finsler space. Then*

$$\mathcal{K}_F(z,\eta,\chi)=\frac{\mathcal{K}_F(z)}{L(z,\chi)}\left\{Re\left(C_{\bar{j}\bar{h}}\overline{\chi}^j\overline{\chi}^h\right)+\frac{Re\left[\left(\bar{\eta}_j\overline{\chi}^j\right)^2\right]}{L(z,\eta)}\right\}, \tag{1.7}$$

where $\mathcal{K}_F(z)$ is the holomorphic curvature of (M,F), $L(z,\chi):=g_{i\bar{j}}\chi^i\overline{\chi}^j$ and $C_{\bar{j}\bar{h}}:=C_{i\bar{j}\bar{h}}\eta^i$.

Also, if (M,F) is a $(g.E.)$ complex Finsler space, Kähler with $\mathcal{K}_F(z)=c$, $c\in\mathbf{R}^*$, then

$$Im(\bar{\eta}_j\overline{\chi}^j)=\pm F(z,\eta)F(z,\chi)\sqrt{\cos^2\varphi-\frac{\mathcal{K}_F(z,\eta,\chi)}{c}}, \tag{1.8}$$

where $F(z,\chi)=\sqrt{L(z,\chi)}$ and φ is the angle of η and χ directions.

2 Holomorphic bi-flag curvature

In this section, we introduce a natural generalization of the holomorphic flag curvature, namely the curvature along of two flags, which we call *holomorphic bi-flag curvature.*

We consider $z\in M$ and $\eta\in T'_zM$, $\eta\neq 0$. A flag is given by the tangent vector field η, called flagpole, and another transversal vector field χ, [5]. Let (η,χ) and (η,γ) be two flags of same flagpole (the tangent vector η), but of different transversal vector $\chi(z,\eta)$ and $\gamma\ (z,\eta)$ as sections of $\pi^*(T'M)$.

Definition 2.1. *The holomorphic bi-flag curvature of the complex Finsler metric F, along of the flags (η,χ) and (η,γ), is given by*

$$\aleph_F(z,\eta,\chi,\gamma):=\frac{\mathbf{R}(\eta,\overline{\chi},\eta,\overline{\gamma})+\mathbf{R}(\chi,\overline{\eta},\gamma,\overline{\eta})+\mathbf{R}(\eta,\overline{\gamma},\eta,\overline{\chi})+\mathbf{R}(\gamma,\overline{\eta},\chi,\overline{\eta})}{2\mathcal{G}(\eta,\overline{\eta})\left[\mathcal{G}(\chi,\overline{\chi})\mathcal{G}(\gamma,\overline{\gamma})\right]^{\frac{1}{2}}}, \tag{2.1}$$

where $\mathcal{G}(\chi,\overline{\chi})\neq 0$ and $\mathcal{G}(\gamma,\overline{\gamma})\neq 0$.

The holomorphic bi-flag curvature depends both on the position $z \in M$ and on the flags (η, χ) and (η, γ).

In particular, if $\mathbf{R}$ is symmetric, i.e. $\mathbf{R}(\eta, \overline{\eta}, \chi, \overline{\chi}) = \mathbf{R}(\eta, \overline{\chi}, \chi, \overline{\eta}) = \mathbf{R}(\chi, \bar{\chi}, \eta, \bar{\eta})$ then

$$\aleph_F(z, \eta, \chi, \gamma) = \frac{\mathbf{R}(\eta, \overline{\chi}, \eta, \overline{\gamma}) + \mathbf{R}(\chi, \overline{\eta}, \gamma, \overline{\eta})}{\mathcal{G}(\eta, \overline{\eta}) \left[\mathcal{G}(\chi, \overline{\chi}) \mathcal{G}(\gamma, \overline{\gamma})\right]^{\frac{1}{2}}}. \tag{2.2}$$

Moreover, if $\mathbf{R}$ is symmetric, by Proposition 2.5.2 from [1], p.107, the holomorphic bi-flag curvature completely determines the curvature tensor $R^i_{j\bar{h}k}$.

Proposition 2.1. *i)* $\aleph_F(z, \eta, \chi, \gamma) = \aleph_F(z, \eta, \gamma, \chi)$;
ii) $\aleph_F(z, \eta, \chi, \chi) = \mathcal{K}_F(z, \eta, \chi)$;
iii) $\aleph_F(z, \eta, \chi, \gamma)$ *is real valued;*
iv) $\aleph_F(z, \frac{\eta}{F}, \chi, \gamma) = \aleph_F(z, \eta, \chi, \gamma)$;
v) $\aleph_F(z, \alpha\eta, \beta\chi, \delta\gamma) = \aleph_F(z, \eta, \chi, \gamma)$, *for any* $\alpha, \beta, \delta \in \mathbf{R}_+$.

Further, we propose to determine the relationships between the holomorphic bi-sectional curvature and the holomorphic bi-flag curvature. For this, we consider the unitary flags (l, m_1) and (l, m_2), where $l = \frac{\eta}{F(z,\eta)}$, $m_1 = \frac{\chi}{F(z,\chi)}$, $m_2 = \frac{\gamma}{F(z,\gamma)}$, $F(z, \chi) = \sqrt{L(z, \chi)}$ and $F(z, \gamma) = \sqrt{L(z, \gamma)}$. By means of these, we construct the flags $(l, S_{m_1 m_2})$ and $(l, D_{m_1 m_2})$ of certain flagpole l and of diagonal transversal vectors $S_{m_1 m_2} = m_1 + m_2$ and $D_{m_1 m_2} = m_1 - m_2$. The conjugates are $S_{\bar{m}_1 \bar{m}_2} = \bar{m}_1 + \bar{m}_2$ and $D_{\bar{m}_1 \bar{m}_2} = \bar{m}_1 - \bar{m}_2$.

We denote by φ the angle between the directions of the unitary sections m_1 and m_2. It result that $\cos\varphi = \frac{Re\mathcal{G}(m_1, \bar{m}_2)}{||m_1|| ||\bar{m}_2||} = Re\mathcal{G}(m_1, \bar{m}_2)$ and then

Proposition 2.2. *i)* $\mathcal{G}(S_{m_1 m_2}, S_{\bar{m}_1 \bar{m}_2}) = 4\cos^2 \frac{\varphi}{2}$;
ii) $\mathcal{G}(D_{m_1 m_2}, D_{\bar{m}_1 \bar{m}_2}) = 4\sin^2 \frac{\varphi}{2}$.

Using the above considerations, we shall prove the following

Proposition 2.3. *Let* (M, F) *be a complex Finsler space. Then*

$$\aleph_F(z, \eta, \chi, \gamma) = \mathcal{K}_F(z, \eta, S_{m_1 m_2}) \cos^2 \frac{\varphi}{2} - \mathcal{K}_F(z, \eta, D_{m_1 m_2}) \sin^2 \frac{\varphi}{2}, \tag{2.3}$$

where $\mathcal{K}_F(z, \eta, S_{m_1 m_2})$ *and* $\mathcal{K}_F(z, \eta, D_{m_1 m_2})$ *are the holomorphic flag curvature along of the flags* $(\eta, S_{m_1 m_2})$ *and* $(\eta, D_{m_1 m_2})$, *respectively.*

Proof. Taking into account Proposition 2.1, *iii*) and (2.1) relation, we obtain

$$\begin{aligned} \aleph_F(z, \eta, \chi, \gamma) &= \aleph_F(z, l, m_1, m_2) \\ &= \frac{1}{2}[\mathbf{R}(l, \bar{m}_1, l, \bar{m}_2) + \mathbf{R}(m_1, l, \bar{m}_2, \bar{l}) \\ &\quad + \mathbf{R}(l, \bar{m}_2, l, \bar{m}_1) + \mathbf{R}(m_2, \bar{l}, m_1, \bar{l})]. \end{aligned} \tag{2.4}$$

On other hand, decomposing $\mathbf{R}(l, S_{\bar{m}_1 \bar{m}_2}, l, S_{\bar{m}_1 \bar{m}_2})$, $\mathbf{R}(S_{m_1 m_2}, \bar{l}, S_{m_1 m_2}, \bar{l})$, $\mathbf{R}(l, D_{\bar{m}_1 \bar{m}_2}, l, D_{\bar{m}_1 \bar{m}_2})$ and $\mathbf{R}(D_{m_1 m_2}, \bar{l}, D_{m_1 m_2}, \bar{l})$, a direct computation give:

$\mathbf{R}(l, S_{\bar{m}_1 \bar{m}_2}, l, S_{\bar{m}_1 \bar{m}_2}) + \mathbf{R}(S_{m_1 m_2}, \bar{l}, S_{m_1 m_2}, \bar{l}) - \mathbf{R}(l, D_{\bar{m}_1 \bar{m}_2}, l, D_{\bar{m}_1 \bar{m}_2})$

$-\mathbf{R}(D_{m_1m_2},\bar{l},D_{m_1m_2},\bar{l})=2[\mathbf{R}(l,\bar{m}_1,l,\bar{m}_2)+\mathbf{R}(m_1,l,\bar{m}_2,\bar{l})$
$+\mathbf{R}(l,\bar{m}_2,l,\bar{m}_1)+\mathbf{R}(m_2,\bar{l},m_1,\bar{l})]=4\aleph_F(z,\eta,\chi,\gamma).$

In view of Definition 2.1 and Proposition 2.1, the last relation becomes $\mathcal{K}_F(z,l,S_{m_1m_2})=\mathcal{K}_F(z,\eta,S_{m_1m_2})$ and $\mathcal{K}_F(z,l,D_{m_1m_2})=\mathcal{K}_F(z,\eta,D_{m_1m_2})$, that is (2.3). $\Box$

Colorallary 2.1. *Let (M,F) be a complex Finsler space. Then*

$$\begin{aligned}\aleph_F(z,\eta,\chi,\gamma) &= 2\mathcal{K}_F(z,\eta,S_{m_1m_2})\cos^2\frac{\varphi}{2}\\ &\quad -\frac{1}{2}\left[\mathcal{K}_F(z,\eta,\chi)+\mathcal{K}_F(z,\eta,\gamma)\right];\\ \aleph_F(z,\eta,\chi,\gamma) &= -2\mathcal{K}_F(z,\eta,D_{m_1m_2})\sin^2\frac{\varphi}{2}\\ &\quad +\frac{1}{2}\left[\mathcal{K}_F(z,\eta,\chi)+\mathcal{K}_F(z,\eta,\gamma)\right].\end{aligned} \tag{2.5}$$

It is natural to determine the conditions in which the holomorphic bi-flag curvature of a complex Finsler space along of any two flags (η,χ) and (η,γ) is a constant.

Proposition 2.4. *Let (M,F) be a complex Finsler space of constant holomorphic flag curvature along of any flag (η,χ), i.e. $\mathcal{K}_F(z,\eta,\chi)=c$, $c\in\mathbb{R}$. Then*

i)

$$\aleph_F(z,\eta,\chi,\gamma)=c\cdot\cos\varphi. \tag{2.6}$$

ii) (M,F) has the constant holomorphic bi-flag curvature along of any two flags if and only if φ is a constant.

Proof. *i)* Because for any flag (η,χ) we have $\mathcal{K}_F(z,\eta,\chi)=c$, $c\in\mathbb{R}$, the (2.3) relation became

$\aleph_F(z,\eta,\chi,\gamma)=c\left(\cos^2\frac{\varphi}{2}-\sin^2\frac{\varphi}{2}\right)=c\cos\varphi.$

ii) results immediately in view of (2.6). $\Box$

Colorallary 2.2. *Let (M,F) be a complex Finsler space. If*

$$|\mathcal{K}_F(z,\eta,\chi)|\le c,\ \ c\in\mathbb{R},\ \ c>0,$$

along of any flag (η,χ), then

$$|\aleph_F(z,\eta,\chi,\gamma)|\le c.$$

Proof. Indeed,

$|\aleph_F(z,\eta,\chi,\gamma)|=\left|\mathcal{K}_F(z,\eta,S_{m_1m_2})\cos^2\frac{\varphi}{2}-\mathcal{K}_F(z,\eta,D_{m_1m_2})\sin^2\frac{\varphi}{2}\right|$
$\le|\mathcal{K}_F(z,\eta,S_{m_1m_2})|\cos^2\frac{\varphi}{2}+|\mathcal{K}_F(z,\eta,D_{m_1m_2})|\sin^2\frac{\varphi}{2}$
$\le c\left(\cos^2\frac{\varphi}{2}+\sin^2\frac{\varphi}{2}\right)=c.$ $\Box$

Colorallary 2.3. *Let (M,F) be a complex Finsler space of zero holomorphic bi-flag curvature. Then*

$$\begin{aligned}\mathcal{K}_F(z,\eta,S_{m_1m_2}) &= \frac{1}{4}(1+tg^2\frac{\varphi}{2})\left[\mathcal{K}_F(z,\eta,\chi)+\mathcal{K}_F(z,\eta,\gamma)\right];\\ \mathcal{K}_F(z,\eta,D_{m_1m_2}) &= \frac{1}{4}(1+ctg^2\frac{\varphi}{2})\left[\mathcal{K}_F(z,\eta,\chi)+\mathcal{K}_F(z,\eta,\gamma)\right],\end{aligned} \tag{2.7}$$

The proof follows from (2.5).

3 The holomorphic bi-flag curvature of $(g.E.)$ complex Finsler spaces

For the beginning, let us express the holomorphic bi-flag curvature of a $(g.E.)$ complex Finsler space by means of the holomorphic curvature of the same space.

In locally coordinates, the holomorphic bi-flag curvature of the complex Finsler metric F along of the flags (η, χ) and (η, γ) is given by

$$\begin{aligned} \aleph_F(z,\eta,\chi,\gamma) &= \frac{1}{2L(z,\eta)F(z,\chi)F(z,\gamma)}(\eta^i\overline{\chi}^j\eta^k\overline{\gamma}^h + \chi^i\overline{\eta}^j\gamma^k\overline{\eta}^h \\ &\quad +\eta^i\overline{\gamma}^j\eta^k\overline{\chi}^h + \gamma^i\overline{\eta}^j\chi^k\overline{\eta}^h)R_{i\bar{j}k\bar{h}}, \end{aligned} \tag{3.1}$$

with $F(z,\chi) = \sqrt{g_{i\bar{j}}\chi^i\overline{\chi}^j} \neq 0$, $F(z,\gamma) = \sqrt{g_{i\bar{j}}\gamma^i\overline{\gamma}^j} \neq 0$, and the angle φ between the directions of η and χ is $\cos\varphi = \dfrac{\eta_i\chi^i + \bar{\eta}_j\bar{\chi}^j}{2\sqrt{L(z,\eta)L(z,\chi)}}$.

Theorem 3.1. *Let (M,F) be a $(g.E.)$ complex Finsler space. Then*

$$\aleph_F(z,\eta,\chi,\gamma) = \frac{\mathcal{K}_F(z)}{F(z,\chi)F(z,\gamma)}\left[Re\left(C_{\bar{j}\bar{h}}\overline{\chi}^j\overline{\gamma}^h\right) + \frac{Re\left(\bar{\eta}_j\overline{\chi}^j\bar{\eta}_h\overline{\gamma}^h\right)}{L(z,\eta)}\right]. \tag{3.2}$$

Proof. Because (M,F) is a $(g.E.)$ complex Finsler space, by relation (3.8) from Proposition 2.3, *iii*) and Proposition 2.4 from [4], we obtain:
$R_{\bar{j}l\bar{h}k}\eta^l\eta^k = 2K(z)\left(\bar{\eta}_h\bar{\eta}_j + L(z,\eta)C_{\bar{j}\bar{h}}\right)$;
$R_{\bar{j}l\bar{h}k}\bar{\eta}^j\bar{\eta}^h = 2K(z)\left(\eta_l\eta_k + L(z,\eta)C_{lk}\right)$. Plugging into (3.1) it results:
$\aleph_F(z,\eta,\chi,\gamma) = \frac{2K(z)}{L(z,\eta)F(z,\chi)F(z,\gamma)}\left(\bar{\eta}_j\overline{\chi}^j\bar{\eta}_h\overline{\gamma}^h + \eta_l\chi^l\eta_k\gamma^k\right) +$
$+\frac{2K(z)}{F(z,\chi)F(z,\gamma)}\left(C_{\bar{j}\bar{h}}\overline{\chi}^j\overline{\gamma}^h + C_{kl}\chi^l\chi^k\right)$
$= \frac{4K(z)}{L(z,\eta)F(z,\chi)F(z,\gamma)}Re\left(\bar{\eta}_j\overline{\chi}^j\bar{\eta}_h\overline{\gamma}^h\right) + \frac{4K(z)}{F(z,\chi)F(z,\gamma)}Re\left(C_{\bar{j}\bar{h}}\overline{\chi}^j\overline{\gamma}^h\right)$.
But, $K(z) = \frac{1}{4}\mathcal{K}_F(z)$, so the last relation is (3.2). □

In the following we establish some inequalities between the holomorphic bi-flag curvature and holomorphic curvature of a $(g.E.)$ complex Finsler space. The approach are related to the Kähler case. In order to reduce the clutter, let us make the abbreviations $\mathcal{K}_F(X) := \mathcal{K}_F(z,\eta,X)$, $\aleph_F(\chi,\gamma) := \aleph_F(z,\eta,\chi,\gamma)$,where $X \in \{\chi,\gamma\}$, $\Omega_1 := \cos^2\theta_1 - \frac{\mathcal{K}_F(\chi)}{c}$ and $\Omega_2 := \cos^2\theta_2 - \frac{\mathcal{K}_F(\gamma)}{c}$.

Proposition 3.1. *Let (M,F) be a $(g.E.)$ complex Finsler space, Kähler with $\mathcal{K}_F(z) = c$, $c \in \mathbb{R}^*$. If $Im\left(\bar{\eta}_j\overline{\chi}^j\right)Im\left(\bar{\eta}_j\overline{\gamma}^j\right) \geq 0$ then*

$$\frac{\aleph_F(\chi,\gamma)}{c} = -\sqrt{\Omega_1\Omega_2} + \cos\theta_1\cos\theta_2, \tag{3.3}$$

where θ_1 (θ_2) is the angle of η and χ (η and γ) directions.

If $Im\left(\bar{\eta}_j\overline{\chi}^j\right)Im\left(\bar{\eta}_j\overline{\gamma}^j\right) \leq 0$ then the sign in front of the brackets is positive.

Proof. Because (M,F) is a $(g.E.)$ complex Finsler space, Kähler with $\mathcal{K}_F(z) = c$, $c \in \mathbb{R}^*$, by (3.2) we obtain $\frac{\aleph_F(\chi,\gamma)}{c} = \frac{Re\left(\bar{\eta}_j\overline{\chi}^j\bar{\eta}_h\overline{\gamma}^h\right)}{L(z,\eta)F(z,\chi)F(z,\gamma)}$.

But,
$Re\left(\bar{\eta}_j\overline{\chi}^j\bar{\eta}_h\overline{\gamma}^h\right) = Re\left(\bar{\eta}_j\overline{\chi}^j\right) Re\left(\bar{\eta}_h\overline{\gamma}^h\right) - Im\left(\bar{\eta}_j\overline{\chi}^j\right) Im\left(\bar{\eta}_h\overline{\gamma}^h\right).$
If $Im\left(\bar{\eta}_j\overline{\chi}^j\right) Im\left(\bar{\eta}_j\overline{\gamma}^j\right) \geq 0$ then taking into account (2.2) and (1.8) we have $\frac{Re\left(\bar{\eta}_j\overline{\chi}^j\bar{\eta}_h\overline{\gamma}^h\right)}{L(z,\eta)F(z,\chi)F(z,\gamma)} = \cos\theta_1\cos\theta_2 - \sqrt{\Omega_1\Omega_2}$. $\square$

Lemma 3.1. *Let (M, F) be a (g.E.) complex Finsler space, Kähler with $\mathcal{K}_F(z) = c$, $c \in \mathbb{R}^*$.*
i) If $\frac{\mathcal{K}_F(X)}{c} \geq 0$, then $\sqrt{\Omega_1\Omega_2} \leq 1$;
ii) If $\frac{\mathcal{K}_F(X)}{c} \leq 0$, then $\sqrt{\Omega_1\Omega_2} \geq \frac{\sqrt{\mathcal{K}_F(\chi)\mathcal{K}_F(\gamma)}}{|c|}$.
iii) If $\frac{\mathcal{K}_F(\chi)}{c} \leq 0$ and $\frac{\mathcal{K}_F(\gamma)}{c} \geq 0$ (or $\frac{\mathcal{K}_F(\chi)}{c} \geq 0$ and $\frac{\mathcal{K}_F(\gamma)}{c} \leq 0$) then

$$\sqrt{\Omega_1\Omega_2} \leq \sqrt{1 - \frac{\mathcal{K}_F(\chi)}{c}} \ \text{ or } \sqrt{\Omega_1\Omega_2} \leq \sqrt{1 - \frac{\mathcal{K}_F(\gamma)}{c}}. \tag{3.4}$$

Proof. *i)* and *iii)* immediately result from the inequality
$\sqrt{\Omega_1\Omega_2} \leq \left(1 - \frac{\mathcal{K}_F(\chi)}{c}\right)^{\frac{1}{2}} \left(1 - \frac{\mathcal{K}_F(\gamma)}{c}\right)^{\frac{1}{2}}$.
The inequality $-\frac{\mathcal{K}_F(X)}{c} \leq \cos^2\theta_1 - \frac{\mathcal{K}_F(X)}{c}$ leads to *ii)*. $\square$

Taking into account Proposition 3.1 and Lemma 3.1 we can prove:

Proposition 3.2. *Let (M, F) be a (g.E.) complex Finsler space, Kähler with $\mathcal{K}_F(z) = c$, $c \in \mathbb{R}^*$, and $Im\left(\bar{\eta}_j\overline{\chi}^j\right) Im\left(\bar{\eta}_j\overline{\gamma}^j\right) \geq 0$.*
i) If $c > 0$ then $\aleph_F(\chi, \gamma) \leq c$;
ii) If $c < 0$ then $\aleph_F(\chi, \gamma) \geq c$.
iii) If $c > 0$ and $\mathcal{K}_F(X) \leq 0$, then $\aleph_F(\chi, \gamma) \leq c - \sqrt{\mathcal{K}_F(\chi)\mathcal{K}_F(\gamma)}$;
iv) If $c < 0$ and $\mathcal{K}_F(X) \geq 0$, then $\aleph_F(\chi, \gamma) \geq c + \sqrt{\mathcal{K}_F(\chi)\mathcal{K}_F(\gamma)}$.

Proposition 3.3. *Let (M, F) be a (g.E.) complex Finsler space, Kähler with $\mathcal{K}_F(z) = c$, $c \in \mathbb{R}^*$, $Im\left(\bar{\eta}_j\overline{\chi}^j\right) Im\left(\bar{\eta}_j\overline{\gamma}^j\right) \geq 0$ and $Re\left(\bar{\eta}_j\overline{\chi}^j\right) Re\left(\bar{\eta}_j\overline{\gamma}^j\right) \geq 0$.*
i) If $c > 0$ and $\mathcal{K}_F(X) \geq 0$ (or $c < 0$ and $\mathcal{K}_F(X) \leq 0$), where $X \in \{\chi, \gamma\}$, then $|\aleph_F(\chi, \gamma)| \leq |c|$;
ii) If $c > 0$, $\mathcal{K}_F(\chi) \leq 0$ and $\mathcal{K}_F(\gamma) \geq 0$ (or $\mathcal{K}_F(\chi) \geq 0$ and $\mathcal{K}_F(\gamma) \leq 0$) then $c \geq \aleph_F(\chi, \gamma) \geq -c + \frac{\mathcal{K}_F(\chi)}{2}$ (or $-c + \frac{\mathcal{K}_F(\gamma)}{2}$);
iii) If $c < 0$, $\mathcal{K}_F(\chi) \geq 0$ and $\mathcal{K}_F(\gamma) \leq 0$ (or $\mathcal{K}_F(\chi) \leq 0$ and $\mathcal{K}_F(\gamma) \geq 0$) then $c \leq \aleph_F(\chi, \gamma) \leq -c + \frac{\mathcal{K}_F(\chi)}{2}$ (or $-c + \frac{\mathcal{K}_F(\gamma)}{2}$).

Proposition 3.4. *Let (M, F) be a (g.E.) complex Finsler space, Kähler with $\mathcal{K}_F(z) = c$, $c \in \mathbb{R}^*$, $Im\left(\bar{\eta}_j\overline{\chi}^j\right) Im\left(\bar{\eta}_j\overline{\gamma}^j\right) \geq 0$ and $Re\left(\bar{\eta}_j\overline{\chi}^j\right) Re\left(\bar{\eta}_j\overline{\gamma}^j\right) \leq 0$.*
i) If $c > 0$ then $\aleph_F(\chi, \gamma) \leq 0$;
ii) If $c < 0$ then $\aleph_F(\chi, \gamma) \geq 0$;
iii) If $c > 0$ and $\mathcal{K}_F(X) \leq 0$, then $\aleph_F(\chi, \gamma) \leq -\sqrt{\mathcal{K}_F(\chi)\mathcal{K}_F(\gamma)}$;
iv) If $c < 0$ and $\mathcal{K}_F(X) \geq 0$, then $\aleph_F(\chi, \gamma) \geq \sqrt{\mathcal{K}_F(\chi)\mathcal{K}_F(\gamma)}$.

Proposition 3.5. *Let (M, F) be a (g.E.) complex Finsler space, Kähler with $\mathcal{K}_F(z) = c$, $c \in \mathbb{R}^*$ and $Im\left(\bar{\eta}_j\overline{\chi}^j\right) Im\left(\bar{\eta}_j\overline{\gamma}^j\right) \leq 0$.*
i) If $c > 0$ and $\mathcal{K}_F(X) \geq 0$, where $X \in \{\chi, \gamma\}$, then $\aleph_F(\chi, \gamma) \leq 2c$;
ii) If $c < 0$ and $\mathcal{K}_F(X) \leq 0$, where $X \in \{\chi, \gamma\}$, then $\aleph_F(\chi, \gamma) \geq 2c$.

iii) If $c > 0$, $\mathcal{K}_F(\chi) \leq 0$ *and* $\mathcal{K}_F(\gamma) \geq 0$ *(or* $\mathcal{K}_F(\chi) \geq 0$ *and* $\mathcal{K}_F(\gamma) \leq 0$*) then* $2c \geq \aleph_F(\chi,\gamma) \geq 2c - \frac{\mathcal{K}_F(\chi)}{2}$ *(or* $2c - \frac{\mathcal{K}_F(\gamma)}{2}$*);*

iv) If $c < 0$, $\mathcal{K}_F(\chi) \geq 0$ *and* $\mathcal{K}_F(\gamma) \leq 0$ *(or* $\mathcal{K}_F(\chi) \leq 0$ *and* $\mathcal{K}_F(\gamma) \geq 0$*) then* $2c \leq \aleph_F(\chi,\gamma) \leq 2c - \frac{\mathcal{K}_F(\chi)}{2}$ *(or* $2c - \frac{\mathcal{K}_F(\gamma)}{2}$*).*

Proposition 3.6. *Let* (M,F) *be a (g.E.) complex Finsler space, Kähler with* $\mathcal{K}_F(z) = c$, $c \in \mathbb{R}^*$, $Im\left(\bar{\eta}_j\overline{\chi}^j\right) Im\left(\bar{\eta}_j\overline{\gamma}^j\right) \leq 0$ *and* $Re\left(\bar{\eta}_j\overline{\chi}^j\right) Re\left(\bar{\eta}_j\overline{\gamma}^j\right) \geq 0$.

i) If $c > 0$ *then* $\aleph_F(\chi,\gamma) \geq 0$;

ii) If $c < 0$ *then* $\aleph_F(\chi,\gamma) \leq 0$;

iii) If $c > 0$ *and* $\mathcal{K}_F(X) \leq 0$, *then* $\aleph_F(\chi,\gamma) \geq \sqrt{\mathcal{K}_F(\chi)\mathcal{K}_F(\gamma)}$;

iv) If $c < 0$ *and* $\mathcal{K}_F(X) \geq 0$, *then* $\aleph_F(\chi,\gamma) \leq -\sqrt{\mathcal{K}_F(\chi)\mathcal{K}_F(\gamma)}$.

Proposition 3.7. *Let* (M,F) *be a (g.E.) complex Finsler space, Kähler with* $\mathcal{K}_F(z) = c$, $c \in \mathbb{R}^*$, $Im\left(\bar{\eta}_j\overline{\chi}^j\right) Im\left(\bar{\eta}_j\overline{\gamma}^j\right) \leq 0$ *and* $Re\left(\bar{\eta}_j\overline{\chi}^j\right) Re\left(\bar{\eta}_j\overline{\gamma}^j\right) \leq 0$.

i) If $c > 0$ *and* $\mathcal{K}_F(X) \geq 0$, *where* $X \in \{\chi,\gamma\}$,*then* $\aleph_F(\chi,\gamma) \leq c$;

ii) If $c < 0$ *and* $\mathcal{K}_F(X) \leq 0$, *where* $X \in \{\chi,\gamma\}$, *then* $\aleph_F(\chi,\gamma) \geq c$;

iii) If $c > 0$, $\mathcal{K}_F(\chi) \leq 0$ *and* $\mathcal{K}_F(\gamma) \geq 0$ *(or* $\mathcal{K}_F(\chi) \geq 0$ *and* $\mathcal{K}_F(\gamma) \leq 0$*) then* $\aleph_F(\chi,\gamma) \leq c - \frac{\mathcal{K}_F(\chi)}{2}$ *(or* $c - \frac{\mathcal{K}_F(\gamma)}{2}$*);*

iv) If $c < 0$, $\mathcal{K}_F(\chi) \geq 0$ *and* $\mathcal{K}_F(\gamma) \leq 0$ *(or* $\mathcal{K}_F(\chi) \leq 0$ *and* $\mathcal{K}_F(\gamma) \geq 0$*) then* $\aleph_F(\chi,\gamma) \geq c - \frac{\mathcal{K}_F(\chi)}{2}$ *(or* $c - \frac{\mathcal{K}_F(\gamma)}{2}$*).*

It is clear that the holomorphic bi-flag curvature is an important generalization of the holomorphic flag curvature, however we will prove in a coming paper that it is not the corespondent of the holomorphic bisectional curvature from Hermitian geometry.

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