Tzitzeica – Bäcklund theorems

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Abstract

We present results about certain congruences of lines and the curvatures of their corresponding focal surfaces. We have generalized to different spaces the Bäcklund's theorem about rectilinear congruences in the Euclidean space E^3. Using Tzitzeica's theorem about rectilinear congruences in $\mathbb{P}^5(\mathbb{R})$ we have obtained a class of surfaces with constant curvature in this projective space. More details can be found in the joint papers [2], [3], [4].

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1. The classical Euclidean case

Let C be a rectilinear congruence (roughly speaking: a family of straight lines which is smoothly dependent of two parameters) in the Euclidean space E^3. It is known that on each straight line d of this congruence there exist two points x and x^* such that x generates a surface S and x^* generates a surface S^*, and the line d is tangent to S at x and to S^* at x^*. The two surfaces S and S^* are called *the focal surfaces* of the respective congruence, and the map $x \to x^*$ defines a transformation from S to S^* called *the Bäcklund transformation.* If the distance $d = d(x, x^*)$ is constant and the angle τ between the normal of S at x and the normal of S^* at x^* is also constant, the congruence is called *pseudospherical (or special) with constants* (d, τ) and the pair (S, S^*) is called a *special Bäcklund pair.*

Bäcklund Theorem. *Let C be a pseudospherical rectilinear congruence with constants (d, τ) and with the focal surfaces S and S^*. Then S and S^* have their Gaussian curvatures constant, negative and equal to $K = -\frac{sin^2\tau}{d^2}$.*

2. The complex spherical case

We have obtained a first generalization of this classical result in 1985 (see [2]) considering the space $\mathbb{C}^4$ endowed with the "complex scalar product" defined by the natural extension to this space of the Euclidean scalar product. We have considered the associated complex sphere Σ^3 and we have defined *spherical surfaces* $f : D \to \Sigma^3$. It has been defined as well *a special Bäcklund pair of spherical surfaces* with constants (φ, c). We have proven the following:

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Theorem 1. *Let* ($f : D \to \Sigma^3$, $f^* : D^* \to \Sigma^3$) *be a special Bäcklund pair of spherical surfaces with constants* (φ, c), $\varphi^2 + 1 \neq 0$ *and* $f(D) = S$, $f^*(D^*) = S^*$. *Then* S *and* S^* *have their Gaussian curvatures constant, negative and equal to* $K = 1 - [(1 - c^2)(1 + \varphi^2)]^{-1}$.

Theorem 2. *Let* (φ, c) *be a pair of complex numbers,* $\varphi^2 + 1 \neq 0$ *and* $f : D \to \Sigma^3$ *a spherical surface with constant Gaussian curvature* $K = 1 - [(1 - c^2)(1 + \varphi^2)]^{-1}$. *Then there exists a spherical surface* $f^* : D^* \to \Sigma^3$ *such that the pair* ($f : D \to \Sigma^3$, $f^* : D^* \to \Sigma^3$) *is a special Bäcklund pair with constants* (φ, c). *When a suitable initial condition is given,* $f^* : D^* \to \Sigma^3$ *is unique.*

3. The general Euclidean case

In 1995 we have published the generalization of Bäcklund's theorem to the Euclidean space E^n. Using Tzitzeica's theorem it follows that this generalization is effective ([3]). It has been considered also the case when one of the surfaces is replaced by a curve. In this case, under suitable similiar conditions for the curve and the surface, it follows that the surface has constant negative Gaussian curvature.

4. The projective case

In 1999 we have published a generalization to the projective real space $\mathbb{P}^5(\mathbb{R})$, which is also related to Tzitzeica's theorem (see [4]). We remind

Tzitzeica Theorem. *Let* $\Sigma \subset \mathbb{P}^3(\mathbb{R})$ *be a surface having the property that through each of its points* x *there pass two distinct assymptotic curves* d_x, d_x^*. *If* f *is the Klein map and* $p_x = f(d_x)$, $p_x^* = f(d_x^*)$ *are generating two surfaces* $S = \{p_x; x \in \Sigma\}$ *and* $S^* = \{p_x^*; x \in \Sigma\}$ *in* $\mathbb{P}^5(\mathbb{R})$, *then these surfaces are the focal surfaces of the rectilinear congruence* $C = \{p_x p_x^*; x \in \Sigma\}$.

We have obtained the following

Theorem. *Let* $\Sigma \subset \mathbb{P}^3(\mathbb{R})$ *be a surface subject to the conditions from Tzitzeica's theorem and having constant Gaussian curvature. Then the coresponding surfaces* S *and* S^* *in* $\mathbb{P}^5(\mathbb{R})$ *have their Gaussian curvatures constant and both equal to 1.*

A degenerate case has been considered too.

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