Recent research in Affine Differential Geometry

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Abstract

In this paper we emphasize the geometry of affine immersions, widely developed in recent years.

Mathematics Subject Classification: 01-02, 01A65.
Key words: review, affine differential geometry.

The affine differential geometry (ADG) has known a huge development since 1980, by means of study programmes promoted by institutions from different countries all over the world, as for example Max-Planck-Institut für Mathematik Bonn, Technische Universität Berlin, Alexander von Humboldt-Stiftung (Germania), Katholieke Universiteit Leuven (Belgia), Sichuan University (China), Kobe University (Japonia), Brown University (U.S.A.).

In the two ADG monographs published in 1993 and 1994, as well as in the periodical surveys published after 1980, are presented problems and approaches of great diversity and are shown the relationships between ADG and other subdomains of mathematics, physics and computer science. Particularly, it is explained how some problems in ADG are connected to the conformal geometry, the partialy derivatives equations, the spectral theory, the statistical manifolds, the control theory and others.

The directions of development in ADG are:

A. Blaschke hypersurfaces with specific local properties;
B. degenerate hypersurfaces;
C. equiaffine submanifolds of codimension greater than one;
D. centroaffine differential geometry;
E. homogeneous submanifolds;
F. topics in projective differential geometry;
G. partialy derivatives equations methods and ADG;
H. global affine geometry;
I. spectral results and ADG;
J. conformal and projective structures;
K. Weyl geometries and ADG;
L. affine evolutions;
M. statistics and ADG;
N. applications of ADG in computer vision.

Balkan Journal of Geometry and Its Applications, Vol.10, No.1, 2005, pp. 28-31.

In the following we will briefly present some recent results in ADG, using the notations and terminology from [12]: h=the affine metric, ∇= the induced connection, $\hat{\nabla}$=the Levi-Civita connection of h, $\overline{\nabla}$=the conormal connection, S=the shape operator, J=the Pick invariant, C=the cubic shape, R=the curvature tensor, Ric=the Ricci tensor.

A. The main topics refer to conditions on the curvature, conditions on the cubic shape, types of shape operators and Gauss applications, as well as geodesics in ADG.
A complete classification of the surfaces M^2 in R^3 having constant affine mean curvature is given in [5]. Affine spheres M^n in R^{n+1} having null scalar curvature are classified in [8]. The local symmetry under different shapes (∇R=0, $\hat{\nabla}\hat{R}$=0 or $\overline{\nabla}\ \ \overline{R} = 0$) is studied in [21], [18] and [22].
The main result regarding the cubic forms with totally symmetric derivatives shows that ∇C is totally symmetric iff M is an affine sphere ([3]). Nomizu and Pinkall [11] proved that a surface satisfies $\nabla C = 0$ and $C \neq 0$ iff it is the Cayley surface $z = xy + y^3,$ and Magid classified the affine minimal surfaces. It is proved in [12] that every ∇-geodesic of a hypersurface M^n is planar iff M^n is an affine sphere.

B. The theorem of Berwald was generalized to the case of degenerate hypersurfaces in [11], and the affine version of the Beez-Killing theorem is given in [15].

C. In 1993, a new equiaffine transversal plane for surfaces in $\mathbb{R}^4$ was introduced in [14]. This transversal plane is, in general, different from the already known transversal planes introduced by Burstin and Mayer and by Klingenberg.

D. Recent results in centroaffine differential geometry were obtained both for codimension one and two. A classification of positive definite surfaces with planar geodesics with respect to the induced connection was obtained in [17].

E. Affine homogeneous submanifolds have been extensively studied since 1991, in [13]. It was found a new model of equiaffine homogeneous surface in $\mathbf{R^3}$ and thus was completed the classification of equiaffine homogeneous surfaces in $\mathbf{R^3}$ given by Guggenheimer in his book [7].

F. Projective differental geometry has, as is well-known, a long history and has strong relations with different branches of mathematics, say linear differential systems, non-linear holonomic systems, algebraic geometry and complex geometry. The equivalence problem of submanifolds is solved in [19].

G. Many problems from ADG influenced the investigation of higher order partial derivatives equations ([9]). The works [16] and [6] refer to Monge-Ampère equations, affine spheres, affine maximal surfaces and transformation techniques for equations, based on the Codazzi compatibility condition.

H. Among the results of global affine geometry, obtained in the two last decades, we can remark the theorem of the existence and uniqueness of hyperovaloids with prescribed Blaschke conformal connection, given in [16], the theorems for characterization of compact hypersurfaces in terms of affine spectral data [3] and the classification of locally strongly convex, affine-complete spheres.

I. Results on relations between ADG and spectral theory are presented in [3], while [20] includes applications of the spectrum of the Laplacian in ADG.

J. The paper [5] shows the importance of projective flatness in ADG, and [9] contains a conformal classification of locally strongly convex centroaffinely complete Tchebychev hypersurfaces.

K. The group of geometers Bokan-Gilkey-Simon investigated in [3] examples of Weyl geometries in ADG and the asymptotic spectrum of certain second order differential operators, and determined those spectral invariants which are at the same time gauge invariants.

L. An affine evolution $F_t (t \geq 0)$ of a hypersurface $F = F_0$ in the affine space A_{n+1}, with nondegenerate Blaschke metric h is the solution of the equation:

$$\frac{\partial x_t}{\partial t} = y_t = \frac{1}{n} \Delta_h x_t \quad .$$

The paper [4] proved the preservation of strict convexity and the convergence to a point in finite time for the affine evolution of a strictly convex hypersurface.

M. Suppose that M is a given regular family of probability distributions. Then one can determine on M a compatibile pair of a Riemannian metric and a torsion-free affine connection, which give statistical informations of M [2].

N. Computer vision is concerned with the extraction of information about a 3-dimensional scene from one or more images that are obtained by one or more cameras. A plane curve and its pseudo-orthographic projection are related by an affine transformation. Remarcable results in this domain can be found in [10] and [1].

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