Research works of Romanian mathematicians on Centro-Affine Geometry

Vasile Cruceanu

Abstract

The aim of this paper is to review some Romanian researches in centro-affine geometry, field pioneered by Gh. Tzitzeica.

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At the end of the last century, some of the important problems of Euclidean differential geometry were completely settled. The main results and methods were excellently synthesized by G. Darboux and L. Bianchi in their celebrated books. Later, starting from problems of Euclidean geometry, researchers were guided to properties and configurations with larger character as for example: affine, projective or conformal. Then it appears the ideas of a systematic study of differential geometries with respect to other fundamental groups different from the Euclidean one. Thus, F. Klein in the further editions of his famous Program from Erlangen, with respect to the study of the geometry corresponding to projective group and its subgroups said: "these differential geometries provide a rich source of truths which has been not reach yet".

The ideas of F. Klein were continued in Romania by Gh. Tzitzeica. Thus in 1912, at the Conference in Cambridge, he lectured about the so-called general method as follows:"Let be given a class of figures in plane or space (curves, surfaces, congruences). In order to study their infinitesimal properties which are invariant with respect to the transformations of a group, we have to find certain system of coordinates and determine an arbitrary element of the figure. Then we have to get an equation or a system of differential equations satisfied by the coordinates of an arbitrary element of a figure and thus of all transformed figures". By this method Tzitzeica already studied curves, surfaces, webs of curves and congruences and obtained important centro-affine, affine, projective and conformal properties of them. For this reason Tzitzeica should be considered as a founder of centro/affine, affine, projective and conformal geometries. Further we shall say something about the research work of Romanian mathematicians

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on centro-affine geometry and their international response.

A centro-affine space can be considered as a projective space in which we have a fixed point O called center and a hyperplane Ω called hyperplane at infinity. When 0 belongs to Ω we say that the centro-affine space is parabolic. For the contrary case we have the hyperbolic centro-affine space. The transformations preserving the point 0 and hyperplane Ω are called centro-affine transformations. By Klein's viewpoint the main purpose of centro-affine geometry is to study those properties of figures invariant to all transformations of the centro-affine group. The first who investigated centro-affine geometry of curves and surfaces was Gh. Tzitzeica. Starting from a problem concerning the deformations of Euclidean surfaces he finds in 1907 a class of surfaces (S1) characterized by the property that the metric invariant $I_1 = Kd^{-4}$, where K is the total curvature in a point M and d is the distance from the tangent space in M to a fixed point, is a constant. Later he remarks that the asymptotic lines of a surface (S1) are the curves (C1) characterized by the property that the metric invariant $J_1 = \tau d^{-2}$, where τ is the torsion of the curve in a point M and d is the distance from the osculating plane in M to a fixed point, is also a constant. Also he points out that I_1 and J_1 are relative centro-affine invariants, that is, curves (C1) and surfaces (S1) are carried by centro-affine transformations in curves and respectively surfaces of the same type. Later, he finds another class of surfaces (S2) for which the metric invariant $I^2 = K(cos\theta)^{-4}$, where θ is the angle between the normal to the surface in a point M with a fixed direction, is a constant. The asymptotic lines of these surfaces are curves (C2) with property that the Euclidean invariant $J_2 = \tau(cos\theta)^{-2}$, where θ is the angle between the binormal in M and a fixed direction, is also a constant. Tzitzeica shows that both, the surfaces (S2) and curves (C2) are invariant with respect to the transformations of the parabolic centro-affine group. These curves and surfaces (C1 , C2 , S1 , S2) have been taken under research by other geometers and they have been called Tzitzeica curves and respectively Tzeitzeica surfaces and denoted by T.

Tzitzeica obtained important properties of these surfaces and curves and even generalizations in affine spaces of arbitrary dimension. Later, taking into account their importance in affine geometry, Tzitzeica surfaces have been called by Blaschke and his school, affine spheres. We have to remark that in the fundamental equations obtained by Tzitzeica in 1907-1909 we find, for the first time, two conjugate linear connections which are not symbols of Christoffel for any covariant tensor of second order. Thus, Tzitzeica anticipates the idea of affine connections which has been introduced much later by H. Weyl in 1918. Since 1927, O. Mayer devoted a series of papers to the study of Tzitzeica curves and surfaces, especially in the case of ruled surfaces. Later, in 1932, in a joint paper with A. Myller he published a comprehensive study on curves of a centro-affine plane. In this paper, affine geometry appears as a separate branch of geometry and there are introduced specifical methods for its investigation. Taking into account the importance and beauty of their results, O. Mayer published in 1934 a memoir on theory of surfaces in a hyperbolic centro-affine space. By means of the theory of differential forms used successfully by Gauss, Blaschke and Fubini-Cech for the Euclidean, equiaffine and respectively projective case, O. Mayer associated to a surface a quadratic form and a cubic form which in fact determine the surface up to a centro-affinity. He constructed a complete system of invariants for a surfaces and found their geometrical meaning. Also he studied remarkable classes of surfaces in which we find again the Tzitzeica surfaces. In the same year, I. Popa constructed the hyperbolic centro-affine geometry of curves and the parabolic centro-affine geometry of curves and surfaces.

The first series of papers, elaborated in Iasi, is closed by a paper of O. Mayer published in 1937 and devoted to the centro-affine geometry of webs in a plane. These papers have been turn to the best account the ideas and results of Tzitzeica and have directed the lines of a new geometry. Thus the geometers A. Myller, O. Mayer and I. Popa appear as founders of a school in geometry at Iasi, which from that time had an important contribution to the progress of research in this field.

The Tzitzeica work and of the school from Iasi has influenced the work of other people from Romania and abroad. Thus Gh.Th. Gheorghiu pointed out new properties of Tzitzeica curves and surfaces and discovered a new class surfaces which in fact is a generalization of Tzitzeica surfaces. Also in a joint paper with I. Popescu, he extended some results of Tzitzeica to the n-dimensional case. Al. Nicolescu studied the parabolic centro-affine geometry of curves in a plane and the parabolic centro-affine geometry of curves in space. Also, A. Kahane pointed out new properties of curves in a centro-affine plane and space. Abroad, appeared in 1934 a paper of P. Delens (in Belgia) and the book on affine geometry by Salkowski (in Germany). The last one contains also some results on centro-affine geometry of curves and surfaces. Then, S. Golab has studied the rectifiability of curves in centro-affine geometry and J. Merza obtained new meanings for some centro-affine invariants.

Starting from Tzitzeica and Mayer papers and using the theory of normalized manifolds, elaborated by A. Norden in 1937, the Russian geometers Dubnov and Skrëdlov constructed a new theory of surfaces in a hyperbolic centro-affine space. They use the connection induced by the sheaf of central lines which was earlier considered by Tzitzeica and Mayer. In fact, later this connection has been called by A. Dobrescu and K. Yano the Tzitzeica connection. The results on 3-dimensional centro-affine geometry are extended by V. Vagner to the n-dimensional case. He used them for the construction of an affine and centro-projective geometry of hypersurfaces and for a new theory of Finsler spaces. A. Lopshitz constructed the theory of submanifolds of codimension two in an equi-centro-affine space.

Taking into account the successes of these investigations, O. Mayer came back to this topic. Thus in 1950 he published a study on congruences of lines in centro-affine geometry and in 1955 studied the deformations of centro-affine surfaces. Later, in 1965, he published a memoir on infinitezimal transformations of surfaces in a centro-affine space. The investigations of O. Mayer on centro-affine geometry of ruled surfaces were continued by Russian geometer L. Magazinicov and his disciples and some other Russian geometers.

In his lecture at the Conference on Geometry (Timisoara, 1955), O. Mayer said the following with respect to the future of geometries with fundamental group: "It is necessarily a general theory for completely reducible linear groups. Then the geometries of parabolic type whose absolutes are configurations of linear spaces connected with varieties of a projective space have also a special interest. Finally, the impulse for such researches could come from spaces endowed with connection".

Starting from this suggestion, D. Papuc and E. Munteanu sketched a theory of surfaces in a parabolic centro-axial-affine space and V. Cruceanu studied the theory of curves and surfaces in a general parabolic centro-affine space. By remarking an analogy between the parabolic center-affine geometry and parabolic affine-axial geometry, V. Cruceanu investigated curves and surfaces in the last case. His researches have been recently continued by O. Röschel from Austria who elaborated a wonderful theory of curves and surfaces in Galilei spaces. Gh. Gheorghiev and I. Popa studied the affine and centro-affine geometry of varieties of cones and descovered an important class of varieties of quadratic cones called "Tzitzeica varieties". D. Papuc has elaborated a series of papers on submanifolds in a Klein space with a complete reducible linear group and obtained, in particular, results on geometry of hypersurfaces in a centro-affine space. Starting from the same suggestion of Mayer and using the theory of fibre bundles, V. Cruceanu introduced in 1962 the concept of space with centro-affine connection. A centro-affine structure on a manifold M is defined by a vector field ξ called the fundamental field which determines in each tangent space $T_xM, x \in M$, a point O_x, called the center in x and organize T_xM as a centro-affine space. Then, one associates to M, the principal fibre bundle of centro-affine frames and a centro-affine connection on M is an infinitesimal connection on this bundle. One obtains then that a centro-affine connection is determined by a linear connection ∇ and a $(1,1)$ tensor field F on M, related by the condition $\nabla + F = 0$. A centro-affine connection is called normal if F is the Kronecker tensor field on M. There have been investigated some general properties of these spaces and in particular has been developed a theory of hypersurfaces which are transversal to ξ.Thus one remarked that many results obtained by Tzitzeica, Mayer and Vagner have their correspondent in theory of hypersurfaces in these spaces. We have also Tzitzeica hypersurfaces with interesting properties. There have been also investigated spaces with a metric, symplectic or complex centro-affine connection.

In a recent paper dealing with classical structures and connections on a differentiable manifold, V. Cruceanu gave a simple and natural definition for a projective structure and connection and pointed out a principle of duality in geometry of spaces with projective connection. One showed that affine structures and connections as well as those centro-projective and centro-affine appear in a natural way as being subordinated to projective structures and connections. Now, coming back to the centro-affine geometry we have to remark a paper of A. Dobrescu published in 1967 in which it is continued the study of the connection induced on a hypersurface of a Euclidean space of pencil of straight lines passing through a fixed point and which is called the Tzitzeica connection. By this study there are found again some centro-affine properties of these hypersurfaces and constructed some new centro-affine invariants by means of those Euclidean geometry. His investigations have been continued by K. Yano in 1969 who defined the notion of Tzitzeica connection for a hypersurface in a space with an affine connection and establishes its properties.

Taking into account the isomorphism between a centro-affine space and a vectorial space it is expected that researches of centro-affine geometry appear in a geometrization of functional analysis. The important investigations of D. Laugwitz dedicated to the geometry of vector spaces (finite or infinite - dimensional) based on results of Mayer, Salkowski, Vagner support our assertion and they are just the starting point for new researches. Also, the study of submanifolds in a complex centro-affine space or a quaternionic centro-affine space could be of interest, because they carries some interesting geometrical structures which are not connected with any Riemannian metric on the ambient space or on the submanifold.

Finally, many investigations on affine geometry of hypersurfaces are tight related with those centro-affine by means the affine indicatrix of normals. Therefore, people studying affine geometry are often dealing with centro-affine properties.

Vasile Cruceanu
University "Alexandru Ioan Cuza",
Faculty of Mathematics
700506, Iasi, Romania
email: cruceanu@uaic.ro