On the Geometric Meaning of the Classical Equation of Gauss

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

Let M be a submanifold of a Riemannian manifold $\tilde{M}$. Some geometrical relations, expressing the difference of the sectional (and bisectional) curvatures of $\tilde{M}$ and of M are obtained. These relations result to be equivalent to the classical Equation of Gauss.

The special case when M is λ-isotropic at a point x is also examined.

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1 Introduction

The aim of the present paper is to obtain formulas, involving only geometrical elements, that result to be equivalent to the classical Equation of Gauss.

Let $\tilde{M}$ be a Riemannian manifold and M a submanifold of M. Consider a point x of $M \subset \tilde{M}$ and a pair p, q of oriented planes of $T_x(M) \subset T_x(\tilde{M})$.

It is known that, starting from the Equation of Gauss, we can derive a formula expressing the difference $\tilde{\chi}_{pq} - \chi_{pq}$ of the bisectional curvatures of $\tilde{M}$ and of M. This relation, however, cannot be considered as completely satisfactory from a geometrical point of view (Sec.3).

In Sec. 4-7 we show how the above formula can be rewritten, in two different ways, in terms of the planes p, q only, by introducing convenient *means* on p, q (Theorem 1, Sec. 4; Theorem 2, Sec.6). In particular, when $q = p$ we obtain two relations for the difference $\tilde{K}_p - K_p$ of the sectional curvatures.

The research ends by showing that, if at any point x of M and for any plane p of $T_x(M)$ one of the mentioned relations expressing $\tilde{K}_p - K_p$ is satisfied, then we are able to derive the classical Equation of Gauss (Theorem 3, Sec.8).

In Sec. 9-13 we apply the general results of Sec. 4,6 to the special case, when the submanifold M is assumed to be λ-isotropic at the point x in the sense of B. O'Neill (Sec.11). Using the geometric notions of *related bases* and of *canonical isometries*

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(Sec.10), we obtain interesting formulas concerning sectional and bisectional curvatures of M and $\tilde{M}$ (Theorem 4, Sec.12). In particular, if M is umbilical at x, we are led to known relations.

2 Preliminaries

Let V be an n-dimensional real vector space and g an inner product on V. In the sequel the 2-dimensional subspaces of V are called planes. Let p, q be two *oriented planes* of V.

We denote by ρ_p, ρ_q the *rotations of* $\frac{\pi}{2}$ on p, q, respectively. Let $i : p \longrightarrow q$ be an isometry. We say that i *preserves the orientation* if and only if i maps an oriented orthonormal basis of p onto an oriented orthonormal basis of q.

More explicitly, let X, Y be an oriented orthonormal basis of p, then ρ_p is the isomorphism defined by $\rho_p X = Y, \rho_p Y = -X$. Similarly for ρ_q. An isometry i preserves the orientation, if and only if iX, iY is an oriented orthonormal basis of q.

It is worth remarking that the geometrical notion of *rotation of* $\frac{\pi}{2}$ in an oriented plane and of *isometry preserving the orientation* for a pair of oriented planes are *intrinsic notions.* In effect, we can easily prove that the definitions do not depend on the oriented orthonormal basis X, Y of the plane p.

3 The Equation of Gauss

Let $\tilde{M} = \tilde{M}(g)$ be an $\tilde{m}$-dimensional Riemannian manifold and M an m-dimensional submanifold ($m \geq 2$), with induced metric still denoted by g.

We refer to $[4]_{\mathrm{II}}$ Ch.7, to [1] Ch.2 and to [10] Ch.2 for the basic facts about the geometry of the submanifolds. In the sequel B denotes the *second fundamental form* and $H = \frac{1}{m}$ trace B the *mean curvature vector field* of M.

Let x be a point of $M \subset \tilde{M}$ and $R, \tilde{R}$ be the Riemann curvature tensor of $M, \tilde{M}$ at x, respectively. Then, for any vectors X, Y, Z, W of $T_x(M) \subset T_x(\tilde{M})$, we have the well known *Equation of Gauss*

$$(1) \quad \tilde{R}(X,Y,Z,W) - R(X,Y,Z,W) = g(B(X,W), B(Y,Z)) - g(B(X,Z), B(Y,W))$$

where the metric tensor g and the form B must be considered at x.

Now, let p, q be two oriented planes of $T_x(M) \subset T_x(\tilde{M})$. Denote by χ_{pq} and by $\tilde{\chi}_{pq}$ the *bisectional curvatures* of M and of $\tilde{M}$ with respect to the pair p, q. Let X, Y and Z, W be oriented orthonormal bases of p and of q, respectively. Since we have

$$\chi_{pq} = R(X,Y,Z,W) \qquad \tilde{\chi}_{pq} = \tilde{R}(X,Y,Z,W)$$

(see for example [7],(1) p.148), from relation (1) we derive relation

$$(2) \qquad \tilde{\chi}_{pq} - \chi_{pq} = g(B(X,W), B(Y,Z)) - g(B(X,Z), B(Y,W))$$

In particular, when $q = p$, we get the relation

$$(2') \qquad \tilde{K}_p - K_p = g(B(X,Y),B(X,Y)) - g(B(X,X),B(Y,Y))$$

concerning the *sectional curvatures.*

It is worth remarking that equation (2), (2′) already evidence the strict connection of the Equation of Gauss with the geometrical notions of bisectional and sectional creature. However, these equations cannot be considered as completely satisfactory from a geometrical point of view. As a matter of fact, the first members of (2),(2′) depend only on the oriented planes p, q; at the same time we see that in their second members oriented orthonormal bases occur.

In the sections 4-7, *our problem will be that of rewriting the second members of* (2), (2′) *in terms of the oriented planes p,q only.*

4 A first result

The problem posed at the end of Sec. 3 can be solved in more ways. The basic idea is that of making use of *the notion of mean.*

Let X, Y and Z, W be oriented orthonormal bases of the oriented planes p and q, respectively. Consider the sets

$$S_p = \{P \in p \,|\, g(P,P) = 1\} \qquad S_q = \{Q \in q | g(Q,Q) = 1\}.$$

We can write

$$P = X\cos\phi + Y\sin\phi \qquad Q = Z\cos\psi + W\sin\psi$$

and consequently

$$\rho_p P = Y\cos\phi - X\sin\phi \qquad \rho_q Q = W\cos\psi - Z\sin\psi$$

where ρ_p, ρ_q are the rotations of $\frac{\pi}{2}$ on the oriented planes p, q, respectively (Sec. 2).

Last we introduce the *mean*

$$(3) \qquad \mathbf{m}_\rho = \frac{1}{4\pi^2}\int_{S_p\times S_q} g(B(P,Q),\, B(\rho_p P, \rho_q Q))\, d\phi\, d\psi.$$

In particular, if $q = p$, we can choose $Z = X, W = Y$. Putting $P = P_1$, $\phi = \phi_1, Q = P_2, \psi = \phi_2$, the mean $\mathbf{m}_\rho$ reduces to the *mean*

$$(3^*) \qquad \mathbf{m}^*_\rho = \frac{1}{4\pi^2}\int_{S_p\times S_p} g(B(P_1,P_2),\, B(\rho_p P_1, \rho_p P_2))\, d\phi_1\, d\phi_2.$$

We are now able to state

Theorem l. *The relations* (2), (2′) *of Sec.* 3, *derived from the Equation of Gauss, can be written in the form*

(4) $$\tilde{\chi}_{pq} - \chi_{pq} = -2\ \mathbf{m}_\rho$$

(4′) $$\tilde{K}_p - K_p = -2\ \mathbf{m}^*_\rho$$

where $\mathbf{m}_\rho, \mathbf{m}^*_\rho$ *are given by* (3), (3*).

Since the notion of rotation of $\frac{\pi}{2}$ in an oriented plane is an intrinsic notion (Sec.2), Theorem 1 gives an exaustive answer to our problem.

5 Proof of Theorem 1

Since we have

$$\begin{aligned} B(P,Q) &= \cos\phi\cos\psi\ B(X,Z) + \cos\phi\sin\psi\ B(X,W) \\ &+ \sin\phi\cos\psi\ B(Y,Z) + \sin\phi\sin\psi\ B(Y,W) \end{aligned}$$

$$\begin{aligned} B(\rho_p P,\rho_p Q) &= \cos\phi\cos\psi\ B(Y,W) - \cos\phi\sin\psi\ B(Y,Z) \\ &- \sin\phi\cos\psi\ B(X,W) + \sin\phi\sin\psi\ B(X,Z) \end{aligned}$$

we derive

$$\begin{aligned} g(B(P,Q),B(\rho_p P,\rho_q Q)) &= g(B(X,Z),B(Y,W))[\cos^2\phi\cos^2\psi + \sin^2\phi\sin^2\psi] \\ &- g(B(X,W),B(Y,Z))[\cos^2\phi\sin^2\psi + \sin^2\phi\cos^2\psi] + ... \end{aligned}$$

Here and in the sequel, the dots stand for terms, that will give zero by integration on S_p or on S_q.

It is elementary to check that the second member of relation (2) of Sec.3 can be replaced by $-2\ \mathbf{m}_\rho$ and this proves Theorem l.

6 A second result

In the present section, by using the intrinsic notion of isometry preserving the orientation (Sec.2), we will be able to give a second answer to the problem, we posed at the end of Sec.3.

We begin by remarking that to any oriented plane p of $T_x(M) \subset T_x(\tilde{M})$ we can intrinsecally associate the *vector*

(5) $$B_p = \frac{1}{2\pi}\int_{S_p} B(P,P)d\phi$$

where S_p is the set of the unit vectors of p. In other words, we consider the mean of the *normal curvature vectors* ([5], p.149) of the unit vectors of p.

Now, let P_1, P_2 be two vectors of S_p and let $i : p \longrightarrow q$ be any isometry, that preserves the orientation (Sec.2). We introduce the *means*

(6) $$\mathbf{m}_C(i) = \frac{1}{4\pi^2}\int_{S_p\times S_p} g(B(P_1, iP_2), B(P_2, iP_1))d\phi_1 d\phi_2$$

(7) $$\mathbf{m}_S(i) = \frac{1}{4\pi^2}\int_{S_p\times S_p} g(B(P_1, iP_1), B(P_2, iP_2))d\phi_1 d\phi_2$$

Remark that in (7) the vectors P_1, P_2 act separately, while in (6) they act in a crossed way. Hence the notations $\mathbf{m}_S(i), \mathbf{m}_C(i)$, respectively.

In particular, if $q = p$, we can choose i = identity. Then $\mathbf{m}_C(i), \mathbf{m}_S(i)$ reduce to

(6*) $$\mathbf{m}_N = \frac{1}{4\pi^2}\int_{S_p\times S_p} g(B(P_1, P_2), B(P_1, P_2))d\phi_1 d\phi_2$$

(7*) $$g(B_p, B_p) = \frac{1}{4\pi^2}\int_{S_p\times S_p} g(B(P_1, P_1), B(P_2, P_2))d\phi_1 d\phi_2$$

respectively.

Note that the second member of (6*) is, essentially, the integral of a norm. Hence the notation $\mathbf{m}_N$. To prove (7*), just remark that the second member can be written in the form

$$g(\frac{1}{2\pi}\int_{S_p} B(P_1, P_1)d\phi_1, \frac{1}{2\pi}\int_{S_p} B(P_2, P_2)d\phi_2)$$

and that B_p is defined by (5).

We are now able to state

Theorem 2. *The relations* (2),(2′) *of Sec.* 3, *derived from the Equation of Gauss, can be written in the form*

(8) $$\tilde{\chi}_{pq} - \chi_{pq} = 2(\mathbf{m}_C(i) - \mathbf{m}_S(i))$$

(8′) $$\tilde{K}_p - K_p = 2(\mathbf{m}_N - g(B_p, B_p))$$

where $i : p \longrightarrow q$ *is any isometry preserving the orientation and* $\mathbf{m}_C(i), \mathbf{m}_S(i), \mathbf{m}_N, B_p$ *are defined by* (6),(7),(6*),(5), *respectively.*

Since the notions of isometry preserving the orientation, that occurs in (6),(7), as well as the definitions of $\mathbf{m}_N$ and of B_p, are intrinsic, so Theorem 2 represents a completely satisfactory solution of the problem at the end of Sec.3.

7 Proof of Theorem 2

Let X, Y be an oriented orthonormal basis of the oriented plane p. Consequently, since $i : p \longrightarrow q$ is assumed to be an isometry that preserves the orientation, we can choose as oriented orthonormal basis of q the pair $Z = iX$, $W = iY$ (Sec.2).

Consequently, if P_1, P_2 are vectors of S_p, i.e.

$$P_1 = X\cos\phi_1 + Y\sin\phi_1 \qquad P_2 = X\cos\phi_2 + Y\sin\phi_2$$
$$iP_1 = Z\cos\phi_1 + W\sin\phi_1 \qquad iP_2 = Z\cos\phi_2 + W\sin\phi_2 .$$

It follows

$$\begin{aligned}
B(P_1, iP_2) &= B(X,Z)\cos\phi_1\cos\phi_2 + B(X,W)\cos\phi_1\sin\phi_2 \\
&+ B(Y,\,Z)\,\sin\phi_1\cos\phi_2 + B(Y,W)\,\sin\phi_1\sin\phi_2 \\
B(P_2, iP_1) &= B(X,Z)\cos\phi_1\cos\phi_2 + B(X,W)\sin\phi_1\cos\phi_2 \\
&+ B(Y,\,Z)\,\cos\phi_1\sin\phi_2 + B(Y,W)\,\sin\phi_1\sin\phi_2 \\
B(P_1, iP_1) &= B(X,Z)\cos^2\phi_1 + B(X,W)\cos\phi_1\sin\phi_1 \\
&+ B(Y,\,Z)\,\sin\phi_1\cos\phi_1 + B(Y,W)\,\sin^2\phi_1 \\
B(P_2, iP_2) &= B(X,Z)\cos^2\phi_2 + B(X,W)\cos\phi_2\sin\phi_2 \\
&+ B(Y,\,Z)\,\sin\phi_2\cos\phi_2 + B(Y,W)\,\sin^2\phi_2 .
\end{aligned}$$

Therefore we can write

$$\begin{gathered}
g(B(P_1, iP_2), B(P_2, iP_1)) = \\
g(B(X,Z),B(X,Z))\cos^2\phi_1\cos^2\phi_2 + g(B(Y,W),B(Y,W))\sin^2\phi_1\sin^2\phi_2 \\
+g(B(X,W),B(Y,Z))[\sin^2\phi_1\cos^2\phi_2 + \cos^2\phi_1\sin^2\phi_2] + ...
\end{gathered}$$

$$\begin{gathered}
g(B(P_1, iP_1), B(P_2, iP_2)) = \\
g(B(X,Z),B(X,Z))\cos^2\phi_1\cos^2\phi_2 + g(B(Y,W),B(Y,W))\sin^2\phi_1\sin^2\phi_2 \\
+g(B(X,Z),B(Y,W))[\sin^2\phi_1\cos^2\phi_2 + \cos^2\phi_1\sin^2\phi_2] + ...
\end{gathered}$$

where the dots stand for terms, that will give zero by integration on $S_p \times S_p$

It is not difficult now to see that the second member of relation (2) of Sec.3 can be replaced by $2(\mathbf{m}_C(i) - \mathbf{m}_S(i))$ and this proves Theorem 2.

8 The main result

The aim of the present section is to prove

Theorem 3. *At any point x of the submanifold M the classical Equation of Gauss results to be equivalent to relation* $(4')$ *as well as to relation* $(8')$ *for any plane p of $T_x(M)$.*

Corollary 1. *Each one of the relations* $(4')$ *of Sec.*4, $(8')$ *of Sec.*6, *both concerning the sectional curvature, summarizes the whole geometrical content of the Equation of Gauss.*

We begin with

Corollary 2. *For the means* $\mathbf{m}_\rho, \mathbf{m}^*_\rho$ *and the means* $\mathbf{m}_C(i), \mathbf{m}_S(i), \mathbf{m}_N, B_p$, *introduced in Sec.* 4, 6, *we have*

$$\mathbf{m}_\rho = \mathbf{m}_S(i) - \mathbf{m}_C(i) \qquad \mathbf{m}^*_\rho = g(B_p, B_p) - \mathbf{m}_N.$$

This fact follows immediately by comparing $(4), (4')$ with $(8), (8')$, respectively.

As a consequence, we see that relations $(4')$ and $(8')$, occurring in Theorem 3, are equivalent. On the other hand, the last sentence of Sec.4 implies that (4) and (2) are equivalent. In particular $(4')$ results to be equivalent to relation $(2')$.

In conclusion, since (2),$(2')$ have been derived from the Equation of Gauss, to prove Theorem 3 we have only to prove (1) for any $\overline{X}, \overline{Y}, \overline{Z}, \overline{W}$ of $T_x(M)$, starting from $(2')$ for any oriented plane p of $T_x(M)$, being X, Y any oriented orthonormal basis of p.

Recalling the definition of sectional curvature ($[4]_{\text{I}}$, p.202), we first rewrite $(2')$ as

$$(1') \quad \tilde{R}(X, Y, X, Y) - R(X, Y, X, Y) = g(B(X, Y), B(X, Y)) - g(B(X, X), B(Y, Y))$$

where X, Y is any pair of orthonormal vector of $T_x(M)$. Then, it is elementary to check that $(1')$ *holds true for any pair* $\overline{X}, \overline{Y}$ *of vectors of* $T_x(M)$.

Consider now the *quadrilinear forms*

$$\begin{aligned} Q_1(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) &= \tilde{R}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) - R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) \\ Q_2(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) &= g(B(\overline{X}, \overline{W}), B(\overline{Y}, \overline{Z})) - g(B(\overline{X}, \overline{Z}), B(\overline{Y}, \overline{W})) \end{aligned}$$

It is an easy matter to check that Q_1 and Q_2 satisfy the conditions a, b, c of p.198 of $[4]_{\text{I}}$,

Finally, since by a preceeding remark we have $Q_1(\overline{X}, \overline{Y}, \overline{X}, \overline{Y}) = Q_2(\overline{X}, \overline{Y}, \overline{X}, \overline{Y})$, by Proposition 1.2 of $[4]_{\text{I}}$, p.198 we find $Q_1 = Q_2$, that is the Equation of Gauss.

Therefore the proof of Theorem 3 is complete.

9 The means $\overline{\mathbf{m}}_C(i), \overline{\mathbf{m}}_S(i)$

In order to treat the special case of Sec.12 we need some premises.

First of all, we introduce the *means*

$$(\overline{6}) \qquad \overline{\mathbf{m}}_C(i) = \frac{1}{4\pi^2} \int_{S_p \times S_p} g(B(P_1, P_2), B(iP_1, iP_2)) d\phi_1 d\phi_2$$

$$(\overline{7}) \qquad \overline{\mathbf{m}}_S(i) = \frac{1}{4\pi^2}\int_{S_p\times S_p} g(B(P_1,P_1),B(iP_2,iP_2))d\phi_1 d\phi_2$$

that are analogous to the means $\mathbf{m}_C(i), \mathbf{m}_S(i)$ defined by (6),(7) in Sec. 6.

Proceeding as in Sec.7, we get

$$g(B(P_1,P_2),B(iP_1,iP_2)) =$$

$$g(B(X,X),B(Z,Z))\cos^2\phi_1\cos^2\phi_2 + g(B(Y,Y),B(W,W))\sin^2\phi_1\sin^2\phi_2$$

$$+g(B(X,Y),B(Z,W))[\cos^2\phi_1\sin^2\phi_2 + \sin^2\phi_1\cos^2\phi_2] + ...$$

$$g(B(P_1,P_1),B(iP_2,iP_2)) =$$

$$g(B(X,X),B(Z,Z))\cos^2\phi_1\cos^2\phi_2 + g(B(Y,Y),B(W,W))\sin^2\phi_1\sin^2\phi_2$$

$$+g(B(X,X),B(W,W))\cos^2\phi_1\sin^2\phi_2 + g(B(Y,Y),B(Z,Z))\sin^2\phi_1\cos^2\phi_2 + ...$$

where the dots stand for terms, that will give zero by integration on $S_p \times S_p$.

Now, taking into account these relations and the analogous ones of Sec.7 concerning $g(B(P_1,iP_2),B(P_2,iP_1))$ and $g(B(P_1,iP_1),B(P_2,iP_2))$ we can write the *relations*

$$(10) \qquad 2(\mathbf{m}_C(i) + \overline{\mathbf{m}}_C(i) + \mathbf{m}_S(i)) = E(i)$$

$$(11) \qquad \overline{\mathbf{m}}_S = g(B_p, B_q)$$

where

$$\begin{array}{rcl} 2E(i) & = & g(B(X,X),B(Z,Z)) + 2g(B(X,Z),B(X,Z)) \\ (12) & + & g(B(Y,Y),B(W,W)) + 2g(B(Y,W),B(Y,W)) \\ & + & 2[g(B(X,Y),B(Z,W)) + g(B(X,Z),B(W,Y)) + g(B(X,W),B(Y,Z))]. \end{array}$$

To prove (11) it is worth remarking that from definition (5) of Sec.6 we can derive

$$(13) \qquad B_p = \frac{1}{2}(B(X,X)+B(Y,Y)) \qquad B_q = \frac{1}{2}(B(Z,Z)+B(W,W)).$$

10 Canonical isometries

We recall first that two oriented orthonormal bases X, Y and Z, W for the oriented planes p, q, respectively, are said to be *related bases,* if we have

$$g(X, W) = g(Y, Z) = 0 \tag{14}$$

The geometric notion of related bases plays an essential role in the papers [8], [3]. We refer to [9] for notations and details.

Let X, Y and Z, W be a pair of related bases of p, q. Then the isometry $i_* : p \longrightarrow q$ defined by $i_*X = Z$, $i_*Y = W$ is said to be a *canonical isometry.* By definition, canonical isometries preserve the orientation. Denote by $-i_*$ the isometry defined by $(-i_*)P = -(i_*P)$ for any vector P of p. Since X, Y and $-Z, -W$ is a pair of related bases of p, q, also $-i_*$ is a canonical isometry. Consequently Proposition 1 of [9] ensures the existence of two canonical isometries for any pair of oriented planes.

Let's denote by α_m, α_M the minimum, maximum value of the angle that a line (1-dimensional subspace) of p forms with the plane q. Referring to Remarks 1,2,3 of [9], we are now able to state

Remark 1. If p, q are not isoclinic planes, i.e. $\alpha_m \neq \alpha_M$, there exist only two canonical isometries. If p, q are isoclinic not strictly orthogonal planes, that is $\alpha_m = \alpha_M \neq \frac{\pi}{2}$, we have two cases, according to the fact that we act on the bases X, Y and Z, W by equal or opposite rotations. Correspondingly, we have two or ∞^1 canonical isometries. Finally, if p, q are strictly orthogonal, that is $\alpha_m = \alpha_M = \frac{\pi}{2}$, any isometry preserving the orientation is a canonical isometry; so we have ∞^2 canonical isometries.

The proof of Remark 1 is elementary.

We end the section by recalling that by virtue of (14) we have ([7],(4))

$$\cos pq = g(X, Z)\ g(Y, W). \tag{15}$$

11 λ-isotropy

In 1965 B. O'Neill introduced and studied the λ-isotropy of the submanifolds ([5][6]). A submanifold M of $\tilde{M}$ is said to be λ-isotropic at $x (\lambda \geq 0)$, if we have

$$g(B(X, X), B(X, X)) = \lambda^2 (g(X, X))^2 \tag{16}$$

for any vector X of $T_x(M)$.

It is immediate to check that the above definition is equivalent to the original one of B. O'Neill. Moreover we have

Proposition l. *The submanifold M of $\tilde{M}$ results to be λ-isotropic $(\lambda \geq 0)$ at x, if and only if we have*

$$\begin{aligned} &g(B(X, Y), B(Z, W)) + g(B(X, Z), B(W, Y)) + g(B(X, W), B(Y, Z)) = \\ &= \lambda^2 [g(X, Y) g(Z, W) + g(X, Z) g(W, Y) + g(X, W) g(Y, Z)] \end{aligned} \tag{17}$$

for any X, Y, Z, W *of* $T_x(M)$.

It is worth remarking that the relation of Lemma 1 as well as the relations (1),(2),(3) of Lemma 2 of [6] are special cases of (17).

Since relation (16) follows immediately from (17), to prove Proposition 1 we have only to show that, starting from (16), we can obtain relation (17). In effect, this can be done by using iterated polarizations and by remarking that at any step of the proof you can replace a vector variable, say X, by $kX (k \in \mathbb{R})$ and then use the identity principe of polynomials.

Finally, an immediate consequence of (17) is relation

$$g(B(X,X),B(Y,Y)) + 2g(B(X,Y),B(X,Y)) = \\ = \lambda^2[g(X,X)g(Y,Y) + 2g(X,Y)g(X,Y)] \tag{18}$$

for any X, Y of $T_x(M)$.

12 Special cases

We consider the case when *the submanifold* M *of* $\tilde{M}$ *is* λ*-isotropic* ($\lambda \geq 0$) *at the point* x.

Taking account of the remarks of Sec. 9,10,11, we are now able to state

Theorem 4. *For any canonical isometry* $i_* : p \longrightarrow q$, *we have*

$$\tilde{\chi}_{pq} - \chi_{pq} = 4\ \mathbf{m}_C(i_*) + 2\ \overline{\mathbf{m}}_C(i_*) - E = -4\ \mathbf{m}_S(i_*) - 2\ \overline{\mathbf{m}}_C(i_*) + E \tag{19}$$

$$\tilde{K}_p - K_p = 2(3\ \mathbf{m}_N - 2\lambda^2) = 2\lambda^2 - 3g(B_p, B_p) \tag{20}$$

where

$$E = \lambda^2(1 + \cos^2\alpha_m + \cos^2\alpha_M + \cos pq) \tag{21}$$

Corollary 3. *For any plane* p *of* $T_x(M)$ *we have*

$$-4\lambda^2 \leq \tilde{K}_p - K_p \leq 2\lambda^2 \tag{22}$$

Few remarks complete the subject

Let's denote by p' the same plane as p with opposite orientation. Taking into account of (3) of [7], we get

$$\tilde{\chi}_{pp'} - \chi_{pp'} = 2(2\lambda^2 - 3\ \mathbf{m}_N) = 3g(B_p, B_p) - 2\lambda^2. \tag{23}$$

Moreover, it is immediate to check that we have

$$\mathbf{m}_C(-i^*) = \mathbf{m}_C(i^*) \qquad \mathbf{m}_S(-i^*) = \mathbf{m}_S(i^*) \qquad \overline{\mathbf{m}}_C(-i^*) = \overline{\mathbf{m}}_C(i^*).$$

So, by virtue of Remark 1 of Sec.10, if we have $\alpha_m \neq \alpha_M$ or $\alpha_m = \alpha_M \neq \frac{\pi}{2}$ (first case), then these three means are invariant under changements of the canonical isometries.

Last, the expression E depends only on the geometry of the pair of oriented planes p, q. When p and q are orthogonal or isoclinic or have a line in common, then E takes very simple forms.

We end the paper by considering the case when *the submanifold M is umbilical at the point x.*

Since we have $B(X,Y) = g(X,Y)H$ for any X, Y of $T_x(M)$, equation (16) is satisfied and M is λ-isotropic with $\lambda = |H|$. On the other hand, it is elementary to prove that, in the present case, we have

$$\begin{aligned} 4\,\mathbf{m}_C(i^*) &= g(H,H)[\cos^2\alpha_m + \cos^2\alpha_M], & 2\,\mathbf{m}_C(i^*) = g(H,H) \\ 4\,\mathbf{m}_S(i^*) &= g(H,H)[\cos^2\alpha_m + \cos^2\alpha_M + \cos pq], & \mathbf{m}_N = g(H,H) \\ E &= g(H,H)[1 + \cos^2\alpha_m + \cos^2\alpha_M + \cos pq], & B_p = H \end{aligned}$$

Consequently (19), (20) reduce to known relations (Cf. (8) of [2]).

13 Proofs

To prove Theorem 4 , we consider a pair X, Y and Z, W of related bases defining the canonical isometry i^*. Taking account of (17), (18) and of (14), from relation (12) we get

$$E(i^*) = \lambda^2(1 + (g(X,Z))^2 + (g(Y,W))^2 + g(X,Z)g(Y,W))$$

Then, using (11) of [9] and (15), we find that $E(i^*)$ coincide with the expression E defined by (21). Finally, starting from (8),(10) with $i = i^*$, we prove (19) by sum and difference.

When $q = p$ we can choose $Z = X, W = Y$. Remarking that X, Y and Z, W are related bases, we find that the corresponding canonical isometry is $i^* =$ identity. We know from Sec.6 that $\mathbf{m}_C(i), \mathbf{m}_S(i)$ reduce to $\mathbf{m}_N, g(B_p, B_p)$, respectively. It is immediate that also $\overline{\mathbf{m}}_C(i)$ reduces to M_N. On the other hand, since $q = p$ implies $\cos pq = 1$ and $\alpha_m = \alpha_M = 0$, the expression E reduces to $4\lambda^2$. Now, taking into account that relation (10) gives

$$2\mathbf{m}_N + g(B_p, B_p) = 2\lambda^2 \tag{24}$$

starting from (19) we prove (20).

This completes the proof of Theorem 4.

Finally, Corollary 3 is an immediate consequence of (20).

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