A General Background of Higher Order Geometry and Induced Objects on Subspaces

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Abstract

The higher order bundles defined by an affine bundle E and a vector pseudo-field on E are investigated this paper. The acceleration bundles become non-trivial examples which motivate the above extension. Moreover, the main ideas of our construction can be used as well in other cases.

Using affine bundles, a dual theory between Lagrangians and Hamiltonians (via Legendre transformations) is considered. A canonical way to induce a Hamiltonian on an affine subbundle is given, too.

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Using the bundles of accelerations, a theory of higher order Finsler and Lagrange spaces was developped in [15, 10], but a dual theory of higher order Hamilton spaces was recentely studied [12, 16]. In this paper we investigate the possibility to use these ideas in a more general setting. We perform a *recursive definition of higher order bundles* defined by an affine bundle E and a vector pseudo-field on E. The acceleration bundles are particular cases, but the ideas can be used in many other cases (for example in the case of the non-holonomic spaces, which will be done in a subsequent paper). Using affine bundles, a dual theory between lagrangians and hamiltonians (via *Legendre transformations*) is considered. A *canonical way* to induce a hamiltonian on an affine subbundle is also given, solving a problem from [17, 11] concerning the possibility to induce an hamiltonian on a submanifold *in an intrinsic way.*

1 Basic constructions on affine bundles

A surjective submersion $E \stackrel{\pi}{\to} M$ is usually called a *fibered manifold*. A *morphism* of the fibered manifolds $E' \stackrel{\pi}{\to} M'$ and $E \stackrel{\pi}{\to} M$ is a couple (f_0, f), where $M' \stackrel{f_0}{\to} M$ and $E' \stackrel{f}{\to} E$ satisfy the condition $\pi \circ f = f_0 \circ \pi'$ (i.e. f sends fibers to fibers); it is also said that f is an f_0-*morphism* of fibered manifolds. Using local calculus, the change

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rules of the local coordinates are: $\bar{x}^i = \bar{x}^i(x^j)$ on M and $\bar{x}^i = \bar{x}^i(x^j)$, $\bar{y}^\alpha = \bar{y}^\alpha(x^i, y^\alpha)$ on the total space E.

An *affine bundle* $E \xrightarrow{\pi} M$ is a fibered manifold in which the change rules of the local coordinates on E have the form

$$\bar{x}^i = \bar{x}^i(x^j), \quad \bar{y}^\alpha = g^\alpha_\beta(x^j)y^\beta + v^\alpha(x^j). \tag{1}$$

An *affine section* in the bundle E is a differentiable map $M \xrightarrow{s} E$ such that $\pi \circ s = id_M$ and its local components change according to the rule $\bar{s}^\alpha(\bar{x}^i) = g^\alpha_\beta(x^j)\bar{s}^\beta(x^j) + v^\alpha(x^j)$. The set of affine sections is denoted by $\Gamma(E)$ and it is an affine module over $\mathcal{F}(M)$, i.e. for every $f_1, \dots, f_p \in \mathcal{F}(M)$ such that $f_1 + \cdots + f_p = 1$ and $s_1, \dots, s_p \in \Gamma(E)$, then $f_1 s_1 + \cdots + f_p s_p \in \Gamma(E)$, where the affine combination is taken at every point of the base. Using a partition of unity on the base M it can be easily proved that every affine bundle allows an affine section.

A vector bundle $\bar{E} \xrightarrow{\bar{\pi}} M$ can be canonically associated with the affine bundle $E \xrightarrow{\pi} M$. More precisely, using local coordinates, the coordinates change on $\bar{E}$ following the rules $\bar{x}^i = \bar{x}^i(x^j)$, $\bar{z}^\alpha = g^\alpha_\beta(x^j)z^\beta$, when the coordinates on E change according to the formulas (1).

Some examples of affine bundles are given below.

1) Every vector bundle is an affine bundle, called a *central affine bundle*. In this case $v^\alpha(x^j) = 0$.

2) Consider an affine bundle E. If an affine section is given in $\Gamma(E)$, then one can consider a central affine structure on E. Thus the manifolds E and $\bar{E}$ are diffeomorphic and the diffeomorphism depends only on the affine section.

3) The second acceleration bundle $T^{(2)}M$ of the manifold M is an affine bundle over $T^{(1)}M = TM$, having as local coordinates $(x^i, y^{(1)j}, y^{(2)})$ which change according to the rules

$$\bar{x}^i = \bar{x}^i(x^j),\ \bar{y}^j = \frac{\partial \bar{x}^j}{\partial x^p} y^{(1)p},\ \bar{y}^{(2)k} = \frac{\partial \bar{x}^j}{\partial x^p} y^{(2)p} + \frac{1}{2}\frac{\partial^2 \bar{x}^j}{\partial x^p \partial x^q} y^{(1)p} y^{(1)q}.$$

4) The acceleration bundle of order k of a manifold M, denoted by $T^{(k)}M$, can be defined inductively as an affine bundle over $T^{(k-1)}M$. The construction is quite technical and we do not need it here in the sequel. Notice that in general the manifold $T^{(k)}M$ is considered as a bundle over M [10, 12], but here we regard $T^{(k)}M$ as an affine bundle over $T^{(k-1)}M$.

Let $\ker \pi_* = VE \to E$ be the vertical vector bundle of E and $\Gamma(VE)$ be the module of vertical sections. The local coordinates on VE have the form (x^i, y^α, Y^β) and change according to the rules $\bar{x}^i = \bar{x}^i(x^j)$, $\bar{y}^\alpha = g^\alpha_\beta(x^j)y^\beta + v^\alpha(x^j)$ and $\bar{Y}^\alpha = g^\alpha_\beta(x^j)Y^\beta$.

A *Liouville type* section is a vertical section $S \in \Gamma(VE)$ which has the local form $S^\alpha(x^i, y^\alpha) = y^\alpha + t^\alpha(x^i)$. The change rules of the local functions S^α are $\bar{S}^\alpha(\bar{x}^j, \bar{y}^\beta) = g^\alpha_\beta(x^i)S^\beta(x^j, y^\gamma)$. Taking into account the forms of $\bar{S}^\alpha$ and S^β, it follows that $\bar{y}^\alpha + \bar{t}^\alpha(\bar{x}^j) = g^\alpha_\beta(x^i)(y^\alpha + t^\alpha(x^j))$. Since $\bar{y}^\alpha = g^\alpha_\beta(x^i)y^\beta + v^\alpha(x^i)$, it follows that $-\bar{t}^\alpha(\bar{x}^j) = -g^\alpha_\beta(x^i)t^\beta(x^j) + v^\alpha(x^j)$, thus the local functions $(-t^\beta(x^j))$ are the local components of a global section from $\Gamma(E)$. Conversely, for a global section $s \in \Gamma(E)$ having the local components $(s^\alpha(x^j))$, the local functions $(y^\alpha - s^\alpha(x^j))$ on E are the local components of a Liouville type section. Thus we have proved the following result.

Proposition 1.1 *Every Liouville type section in* $\Gamma(VE)$ *defines an affine section in* $\Gamma(E)$ *and conversely. A one to one correspondence between the Liouville type sections in* $\Gamma(VE)$ *and the affine sections in* $\Gamma(E)$ *follows.*

The set of Liouville type sections in $\Gamma(VE)$ is non-void since every affine bundle allows affine sections. It is easy to see that the set of Liouville type sections is an affine module over $\mathcal{F}(M)$.

An example of an affine section which defines a Liouville type section can be constructed using the affine bundle $T^{(2)}M \to T^{(1)}M$ defined by the second order acceleration bundle $T^{(2)}M$ of a manifold M and the local coefficients $\{N^i_j(x^k, y^l)\}$ of a non-linear connection on $T^{(1)}M$. The local functions $s^{(2)i}(x^i, y^{(1)j}) = -N^i_k(x^i, y^{(1)j})y^{(1)k}$ are the local components of an affine section on $T^{(2)}M$. Indeed, since the local coefficients of the non-linear connection change according the rule

$$\bar{N}^i_k(\bar{x}^p, \bar{y}^q)\frac{\partial \bar{x}^k}{\partial x^j} = \frac{\partial \bar{x}^i}{\partial x^k} N^k_j(x^p, y^q) - \frac{1}{2} y^p \frac{\partial^2 \bar{x}^i}{\partial x^p \partial x^j},$$

the assertion follows easily. This affine section defines a Liouville type section on the vertical bundle of the affine bundle $T^{(2)}M \to T^{(1)}M$.

More generally, a non-linear connection on $T^{(k-1)}M$ defines a section on the affine bundle $T^{(k)}M$ using the dual coefficients, as in [12]. This affine section defines a Liouville type section on the vertical bundle of the affine bundle $T^{(k-1)}M \to T^{(k)}M$.

2 Vector pseudo-fields and iterated affine bundles

Consider a fibered manifold $E \xrightarrow{\pi} M$ and an atlas on E, which corresponds to an atlas on M, i.e. if (x^i) are coordinates on M on an open domain $W \subset M$, then there are coordinates of the form (x^i, y^α) on an open domain $U \subset E$, $\pi(U) = W$. We say that the atlas on E is *adapted*. In the case when the fibered manifold is locally trivial, then $U = \pi^{-1}(W)$ (for example it is the case of an affine bundle or of a vector bundle). A *vector pseudo-field* on E is an association of a local vector field $\Gamma_U \in \mathcal{X}(U)$ with every domain U of the given atlas on E, such that $\Gamma_U(y^\alpha) = 0$ and for every two domains U and $\bar{U}$, which have the coordinates (x^i, y^α) and $(\bar{x}^j, \bar{y}^\beta)$ respectively, then on the intersection $U \cap \bar{U}$ we have $\Gamma_U(x^i) = \Gamma_{\bar{U}}(x^i)$ and $\Gamma_U(\bar{x}^i) = \Gamma_{\bar{U}}(\bar{x}^i)$. It is easy to see that the change rule on the interesection $U \cap \bar{U}$ is

$$\Gamma_{\bar{U}} = \Gamma_U - \Gamma_U(\bar{y}^\alpha)\frac{\partial}{\partial \bar{y}^\alpha}. \tag{2}$$

Indeed, $\Gamma_{\bar{U}} = \Gamma_{\bar{U}}(\bar{x}^i)\frac{\partial}{\partial \bar{x}^i} = \Gamma_U(\bar{x}^i)\frac{\partial}{\partial \bar{x}^i} = \Gamma_U - \Gamma_U(\bar{y}^\alpha)\frac{\partial}{\partial \bar{y}^\alpha}$.

Conversely, it can be proved that the association of a local vector field $\Gamma_U \in \mathcal{X}(U)$ with the domain U, such that the condition (2) holds on the intersection $U \cap \bar{U}$, then a vector pseudofield is obtained.

Some examples of vector pseudo-fields are given below.

1) Let $E \xrightarrow{\pi} M$ be a fibered manifold and $X \in \mathcal{X}(M)$ be a vector field on the base M. If the vector field X has the local form $X = X^i(x^j)\frac{\partial}{\partial x^i} \in \mathcal{X}(W)$ (X^i are real

functions on W and $\left\{\dfrac{\partial}{\partial x^i}\right\}$ are vector fields on W), then $\Gamma_U = X^i(x^j)\dfrac{\partial}{\partial x^i} \in \mathcal{X}(W)$ is a vector pseudo-field (X^i are real functions on $U \subset \pi^{-1}(W)$ and $\left\{\dfrac{\partial}{\partial x^i}\right\}$ are vector fields on U).

2) Let $E = T^{(1)}M \to M$ be the tangent bundle. Considering some local coordinates $(x^i, y^{(1)j})$ on an open set $U \subset T^{(1)}M$, then $\Gamma_U = y^{(1)j}\dfrac{\partial}{\partial x^j} \in \mathcal{X}(U)$ defines a vector pseudo-field.

3) The above example 2) can be extended. Let $T^{(k)}M \to T^{(k-1)}M$ be the affine bundle defined by the total space of the acceleration bundle of order k. Then the local vector field $\Gamma = y^{(1)j}\dfrac{\partial}{\partial x^j} + 2y^{(2)j}\dfrac{\partial}{\partial y^{(1)j}} + \cdots + ky^{(k)j}\dfrac{\partial}{\partial y^{(k-1)j}}$ defines a vector pseudo-field on $T^{(k)}M$ considered in [10].

4) Let $E \xrightarrow{\pi} M$ be a fibered manifold, $D : VE \to \tau M$ be a π-morphism of vector bundles and $Y \in \mathcal{X}(VE)$ be a vertical vector field. Using local coordinates, $Y = Y^\alpha(x^j, y^\alpha)\dfrac{\partial}{\partial y^\alpha} \in \mathcal{X}(VE)$ and

$$Y^\alpha(x^j, y^\beta)\frac{\partial}{\partial y^\alpha} \xrightarrow{D} D^i_\alpha(x^j, y^\beta)Y^\alpha(x^j, y^\beta)\frac{\partial}{\partial x^i}$$

(X^i are local real local functions on VE and $\left\{\dfrac{\partial}{\partial x^i}\right\}$ are local vector fields on M). Then $\Gamma_U = D^i_\alpha(x^j, y^\beta)Y^\alpha(x^j, y^\beta)\dfrac{\partial}{\partial x^i} \in \mathcal{X}(U)$ defines a vector pseudo-field (here $\left\{\dfrac{\partial}{\partial x^i}\right\}$ are local vector fields on E).

5) If $E \xrightarrow{\pi} M$ is a vector bundle, then a particular case of the previous example can be considered using the vertical vector field $Y = y^\alpha\dfrac{\partial}{\partial y^\alpha}$ (called the *Lioville vector field*). Then the vector pseudo-field has the form $\Gamma_U = D^i_\alpha(x^j, y^\beta)y^\alpha\dfrac{\partial}{\partial x^i} \in \mathcal{X}(U)$.

6) The previous examples can be extended considering on a fibered manifold $E \xrightarrow{\pi} M$ an horizontal d-vector field X, i.e. X is defined by its local components $\{X^i(x^j, y^\alpha)\}$ which change on the intersection of two domains of coordinates following the rule $\bar{X}^i(\bar{x}^j, \bar{y}^\alpha) = \dfrac{\partial \bar{x}^i}{\partial x^j}X^j(x^j, y^\alpha)$. For every connection in the fibered manifold (i.e. a left splitting C of the inclusion $VE \to TE$) which has the local coefficients $\{N^\alpha_i(x^j, y^\beta)\}$ (i.e. C has the local form $(x^i, y^\alpha, X_j, Y^\beta) \xrightarrow{C} (x^i, y^\alpha, Y^\beta + X^j N^\beta_j(x^i, y^\alpha)))$, then there is a global vector field $\tilde{X}$ on E which is locally defined by $\tilde{X} = X^i(x^i, y^\alpha)\dfrac{\delta}{\delta x^i}$, where $\dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} - N^\beta_j(x^i, y^\alpha)\dfrac{\partial}{\partial y^\beta}$ ($\tilde{X}$ is a horizontal vector field, which is a section in the horizontal vector bundle of C). Then $\Gamma_U = X^i(x^j, y^\beta)\dfrac{\partial}{\partial x^i} \in \mathcal{X}(U)$ defines a vector pseudo-field on E.

Proposition 2.1 *If $E \xrightarrow{\pi} M$ is an affine bundle and Γ is a vector pseudo-field on E, then there is an affine bundle $E' \xrightarrow{\pi} E$ and a vector pseudo-field Γ' on E' such that the couple (E', Γ') is canonically associated with the couple (E, Γ).*

Proof. We assume that the local coordinates change on E according to the formulas (1) and we define the change rule of the coordinates on E' by

$$\bar{z}^\beta(x^i, y^\alpha, z^\beta) = g^\beta_\gamma(x^i) z^\gamma + \Gamma(\bar{y}^\beta).$$

Let us show that $E' \xrightarrow{\pi'} E$ is an affine bundle. Consider some new coordinates $(\bar{\bar{x}}^i, \bar{\bar{y}}^\alpha, \bar{\bar{z}}^\beta)$ which change according to the rules $\bar{\bar{x}}^i = \bar{\bar{x}}^i(\bar{x}^j)$, $\bar{\bar{y}}^\alpha = \bar{g}^\alpha_\beta(\bar{x}^j)\bar{y}^\beta + \bar{v}^\alpha(\bar{x}^j)$, thus $\bar{\bar{z}}^\beta = \bar{g}^\beta_\gamma(\bar{x}^i)\bar{z}^\gamma + \bar{\Gamma}(\bar{\bar{y}}^\beta)$. We have to express the link between the coordinates $(\bar{\bar{x}}^i, \bar{\bar{y}}^\alpha, \bar{\bar{z}}^\beta)$ and (x^i, y^α, z^β) respectively on the intersection of their domains. The link between the couples of coordinates (x^i, y^α) and $(\bar{\bar{x}}^i, \bar{\bar{y}}^\alpha)$ respectively is $\bar{\bar{x}}^i = \bar{\bar{x}}^i(x^j), \bar{\bar{y}}^\alpha = \bar{\bar{g}}^\alpha_\beta(x^j) y^\beta + \bar{\bar{v}}^\alpha(x^j)$, where $\bar{\bar{g}}^\alpha_\beta = \bar{g}^\alpha_\gamma g^\gamma_\beta$ and $\bar{\bar{v}}^\alpha = \bar{g}^\alpha_\beta v^\beta + \bar{v}^\alpha$. We denote by Γ and $\bar{\Gamma}$ the local form of the given vector pseudo-field on E on the domains of the coordinates (x^i, y^α) and $(\bar{x}^i, \bar{y}^\alpha)$ respectively. We have $\bar{\bar{z}}^\beta = \bar{g}^\beta_\gamma(\bar{x}^i)\bar{z}^\gamma + \bar{\Gamma}(\bar{\bar{y}}^\beta) = \bar{\bar{z}}^\beta = \bar{g}^\beta_\gamma(\bar{x}^i)(g^\gamma_\delta(x^i) z^\delta + \Gamma(\bar{y}^\gamma)) + \bar{\Gamma}(\bar{\bar{y}}^\beta) = \bar{\bar{g}}^\alpha_\delta z^\delta + \bar{g}^\beta_\gamma \Gamma(\bar{y}^\gamma) + \bar{\Gamma}(\bar{\bar{y}}^\beta)$. But $\bar{g}^\beta_\gamma \Gamma(\bar{y}^\gamma) + \bar{\Gamma}(\bar{\bar{y}}^\beta) = \bar{g}^\beta_\gamma \Gamma(\bar{y}^\gamma) + \Gamma(\bar{\bar{y}}^\beta) - \Gamma(\bar{y}^\gamma)\dfrac{\partial \bar{\bar{y}}^\beta}{\partial \bar{y}^a} = \Gamma(\bar{\bar{y}}^\beta)$. We have proved that $\bar{\bar{z}}^\beta = \bar{\bar{g}}^\alpha_\delta z^\delta + \Gamma(\bar{\bar{y}}^\beta)$, thus $E' \xrightarrow{\pi'} E$ is an affine bundle. We define $\Gamma' = \Gamma^i(x^i)\dfrac{\partial}{\partial x^i} + z^\alpha \dfrac{\partial}{\partial y^\alpha} = \Gamma + z^\alpha \dfrac{\partial}{\partial y^\alpha}$ on $\pi'^{-1}(U)$, where U is the domain of Γ (and also the domain of the coordinates (x^i, y^α)). We have to check that this assignment defines a vector pseudo-field on E'. Indeed, $\Gamma'(z^\alpha) = 0$ and on the intersection of two domains in E' we have: $\Gamma'(x^i) = \Gamma(x^i) = \bar{\Gamma}(x^i) = \bar{\Gamma}'(x^i)$; $\Gamma'(\bar{x}^i) = \Gamma(\bar{x}^i) + z^\alpha \dfrac{\partial \bar{x}^i}{\partial y^\alpha} = \Gamma(\bar{x}^i) = \bar{\Gamma}(\bar{x}^i) = \bar{\Gamma}'(\bar{x}^i)$; $\Gamma'(y^\alpha) = z^\alpha$ and $\bar{\Gamma}'(y^\alpha) = \bar{\Gamma}(y^\alpha) + \bar{z}^\beta \dfrac{\partial y^\alpha}{\partial \bar{y}^\beta} = 0 + z^\alpha = \Gamma'(y^\alpha)$; $\bar{\Gamma}'(\bar{y}^\alpha) = \bar{\Gamma}(\bar{y}^\alpha) + \bar{z}^\gamma \dfrac{\partial \bar{y}^\alpha}{\partial \bar{y}^\gamma} = \bar{z}^\alpha = \Gamma(\bar{y}^\alpha) + g^\alpha_\beta z^\beta = \Gamma(\bar{y}^\alpha) + z^\beta \dfrac{\partial \bar{y}^\alpha}{\partial y^\beta} = \Gamma'(\bar{y}^\alpha)$. □

The previous result allows to obtain the higher order manifold $E^{(n)}$, $n \geq 1$ and $E^{(1)} = E$, defined by a couple (E, Γ), where $E \xrightarrow{\pi} M$ is an affine bundle (particularly E can be a vector bundle) and Γ is a vector pseudo-field on E. The manifold $E^{(n)}$ is obtained inductively from $E^{(n-1)}$, using the above Proposition.

An important particular case is obtained when $E = TM$ and Γ is the vector pseudo-field defined by the Liouville d-vector field, viewed as a horizontal d-vector field (in local coordinates $\Gamma = y^i \dfrac{\partial}{\partial x^i}$). In this case $E^{(n)} = T^{(n)}M$ is the total space of the acceleration bundle of order k, studied for example in [14, 15, 10, 16].

More generally, any horizontal vector field on an affine bundle $E \xrightarrow{\pi} M$ defines a vector pseudo-field on E, thus it allows iterations. For example, if $E \xrightarrow{\pi} M$ is a vector bundle and $E \xrightarrow{\rho} TM$ is a vector bundle map (called an *anchor*), then a horizontal vector field X on E can be defined using the formula $X^i(x^j, y^a) = y^\alpha \rho^i_\alpha(x^j)$. Particularly, ρ can be the inclusion morphism of a vector subbundle of TM, defined by a non-necessarily integrable distribution, i.e. a non-holonomic space. This case is of particular interest and will be studied in a subsequent paper.

3 The Legendre and Legendre* transformations on affine bundles

Let E be an affine bundle. A *lagrangian* on E is a differentiable function $L : E \to \mathbf{R}$. In particular if E is a vector bundle, it can be viewed as a central affine bundle.

If the affine bundle E is a fibered manifold on an other base $E \to M_0$ such that a global section $s_0 : M_0 \to E$ is given, then a lagrangian $L : E \to \mathbf{R}$ is *admissible* if it is globally continuous and smooth on $\tilde{E} = E\setminus s_0(M_0)$. An (admissible) lagrangian is *non-degenerate* if the vertical Hessian $\left(\dfrac{\partial^2 L}{\partial y^\alpha y^\beta}\right)$ of L is non-degenerate. In this case the vertical hessian defines a (pseudo)metric structure on the fibers of the vertical bundle VE (or $V\tilde{E}$ if the lagrangian is admissible).

An interesting example follows when M is a manifold and $T^{(2)}M \to T^{(1)}M$ is the affine bundle on $T^{(2)}M$. A *lagrangian of second order* is defined by $L^{(2)} : T^{(2)}M \to \mathbf{R}$. Then $T^{(1)}M \to M$ is the tangent bundle and $T^{(2)}M \to M$ becomes the acceleration bundle of order two, which allows the "null" section $s_0 : M \to T^{(2)}M$. Thus an allowed lagrangian of order two is continuous on $T^{(2)}M$ and smooth on $T^{(2)}M\setminus s_0(M)$. For every $k \geq 1$, the "null" section $s_0 : M \to T^{(k)}M$ of the acceleration bundle of order k, $T^{(k)}M \to M$, is defined in an analogous way and the admissible lagrangians of order $k \geq 1$ are considered in [10, 12].

If $E \to M$ is an affine bundle, then we have denoted by $\bar{E} \to M$ the vector bundle canonically associated with E, which can be regarded as a central affine bundle. We denote by $\bar{E}^* \to M$ its dual vector bundle. A *hamiltonian* defined by E is a lagrangian $H : \bar{E}^* \to \mathbf{R}$ defined on the central affine bundle $\bar{E}^*$.

If $L : E \to \mathbf{R}$ is a lagrangian, then the *Legendre transformation* is the fibered manifold map $\mathcal{L} : E \to \bar{E}^*$ defined in local coordinates by $(x^i, y^\alpha) \xrightarrow{\mathcal{L}} (x^i, \dfrac{\partial L}{\partial y^\beta}(x^i, y^\alpha))$. It is easy to see that if L is a non-degenerate lagrangian, then $\mathcal{L}$ is a global diffeomorphism. Considering a non-degenerate and allowed lagrangian, then we can infer that $\mathcal{L}$ is a local diffeomorphism.

The Legendre transformation defines an $\mathcal{L}$-morphism of the vertical vector bundles $VE \to V\bar{E}^*$ (called the *vertical Legendre morphism*) and expressed in local coordinates by $(x^i, y^\alpha, Y^\beta) \to (x^i, \dfrac{\partial L}{\partial y^\beta}(x^i, y^\alpha), Y^\beta \dfrac{\partial^2 L}{\partial y^\beta y^\gamma}(x^i, y^\alpha))$.

Theorem 3.1 *Let $s : M \to E$ be an affine section and $L : E \to \boldsymbol{R}$ be a non-degenerate lagrangian.*

Then there is a hamiltonian $H : \bar{E}^* \to \boldsymbol{R}$ defined by L and s such that the vertical Legendre morphism is an isometry and the vertical hessian of H does not depend on the section s.

Proof. Let $(x^i) \xrightarrow{s} (s^\alpha(x^i))$ be the local form of the section s. According to Proposition 1.1 the section s defines a Liouville section $S : E \to VE$ given in local coordinates by $(x^i, y^\alpha) \xrightarrow{S} (x^i, y^\alpha, y^\alpha - s^\alpha(x^i))$. Since L is non-degenerate it means that $\mathcal{L}$ is a diffeomorphism, thus consider $\mathcal{H} = L^{-1} : \bar{E}^* \to E$ and denote by $\bar{S} = S\circ\mathcal{H}:\bar{E}^* \to VE$. Notice that $\mathcal{H}$ has the local form

$$(x^i, p_\alpha) \xrightarrow{\mathcal{H}} (x^i, H^\beta(x^i, p_\alpha)),$$

where

$$H^{\gamma}(x^i, \frac{\partial L}{\partial y^{\beta}}(x^i, y^{\alpha})) = y^{\gamma} \quad and \quad \frac{\partial L}{\partial y^{\gamma}}(x^i, H^{\beta}(x^i, p_{\alpha})) = p_{\gamma}.$$

Differentiating the first formula, we obtain:

$$\frac{\partial^2 L}{\partial y^{\alpha} \partial y^{\gamma}}(x^i, y^{\gamma}) \frac{\partial H^{\gamma}}{\partial p_{\beta}}(x^i, \frac{\partial L}{\partial y^{\beta}}(x^i, y^{\alpha})) = \delta_{\alpha\beta}. \tag{3}$$

Substituting $y^{\gamma} = H^{\gamma}(x^i, p_{\theta})$ we also have

$$\frac{\partial^2 L}{\partial y^{\alpha} \partial y^{\gamma}}(x^i, H^{\gamma}(x^i, p_{\theta})) \frac{\partial H^{\gamma}}{\partial p_{\beta}}(x^i, p_{\theta}) = \delta_{\alpha\beta}. \tag{4}$$

Then $\bar{S}$ has the form $(x^i, p_{\alpha}) \xrightarrow{\bar{S}} (x^i, H^{\beta}(x^i, p_{\alpha}), H^{\gamma}(x^i, p_{\alpha}) - s^{\gamma}(x^i))$. We define $H : \bar{E}^* \to \mathbf{R}$ using the formula

$$H(x^i, p_{\alpha}) = p_{\alpha} \left(H^{\alpha}(x^i, p_{\alpha}) - s^{\alpha}(x^i) \right) - L(x^i, H^{\gamma}(x^i, p_{\alpha})). \tag{5}$$

It is easy to see that H is globally defined on $\bar{E}^*$. In order to prove that the vertical hessian of H is non-degenerate and also that the vertical bundle morphism is an isometry, it suffices to prove that

$$\left(\frac{\partial H^2}{\partial p_{\alpha} \partial p_{\beta}} \left(x^i, \frac{\partial L}{\partial y^{\gamma}} \left(x^i, y^{\delta} \right) \right) \right) = \left(\frac{\partial L^2}{\partial y^{\alpha} \partial y^{\beta}} \left(x^i, y^{\delta} \right) \right)^{-1}.$$

This can be obtained by a straightforward computation, as follows. Using formula (3), we obtain $\frac{\partial H}{\partial p_{\alpha}}(x^i, p_{\beta}) = H^{\alpha}(x^i, p_{\alpha})$, then using the relations (4) and (3), the above formula follows. It is easy to see that the vertical hessian of the hamiltonian does not depend on the section s. □

If the lagrangian L is admissible and non-degenerate, then it is possible that the differentiability of L misses on $s_0(M_0)$. Then H is differentiable on $\mathcal{L}(E \backslash s_0(M_0))$.

An inverse construction is performed in the sequel. Starting from a hamiltonian (i.e. a lagrangian on $\tilde{E}^*$), a lagrangian on E can be constructed.

Given a hamiltonian $H : \tilde{E}^* \to \mathbf{R}$ and a section s of E, the *Legendre* transformation* is the fibered manifold morphism $\mathcal{H} : \tilde{E}^* \to E$ defined by the local formula $(x^i, p_{\alpha}) \xrightarrow{\mathcal{H}} (x^i, \frac{\partial H}{\partial p_{\beta}}(x^i, p_{\alpha}) + s^{\beta}(x^i))$. If the hamiltonian is regular, then the Legendre* transformation is a diffeomorphism.

The Legendre* transformation defines an $\mathcal{H}$-morphism of the vertical vector bundles $V\bar{E}^* \to VE$ (called the *vertical Legendre* morphism*) and expressed in local coordinates by $(x^i, p_{\alpha}, P_{\beta}) \to (x^i, \frac{\partial H}{\partial p_{\beta}}(x^i, p_{\alpha}) + s^{\beta}(x^i), Y^{\beta} \frac{\partial^2 H}{\partial p_{\beta} \partial p_{\gamma}}(x^i, p_{\alpha}))$.

Theorem 3.2 *Let $s : M \to E$ be an affine section and $H : \bar{E}^* \to \boldsymbol{R}$ be a non-degenerate hamiltonian.*

Then there is a lagrangian $L : E \to \boldsymbol{R}$ on E such that the vertical Legendre* morphism is an isometry and the vertical hessian of L does not depend on the section s.

Proof. The proof is analogous to the proof of Theorem 3.1. In fact we reverse the order of H and L in the construction of H in the formula (5). We denote by $\mathcal{L} = \mathcal{H}^{-1} : E \to \bar{E}^*$ the inverse of the Legendre* transformation. It has the local form $(x^i, y^\alpha) \xrightarrow{\mathcal{L}} (x^i, L_\beta(x^i, y^\alpha))$, where

$$L_\gamma(x^i, \frac{\partial H}{\partial p_\beta}(x^i, p_\alpha) + s^\beta(x^i)) = p_\gamma \ and \ \frac{\partial H}{\partial p_\gamma}(x^i, L_\beta(x^i, y^\alpha)) + s^\gamma(x^i) = y^\gamma.$$

We define $H : \bar{E}^* \to \mathbf{R}$ using the formula

$$L(x^i, y^\alpha) = L_\alpha(x^i, y^\alpha)\left(y^\alpha - s^\alpha(x^i)\right) - H(x^i, L_\gamma(x^i, y^\alpha)). \tag{6}$$

The proof follows in the same manner as the proof of Theorem 3.1. □

4 Induced hamiltonians on affine subbundles

Besides the theory of Lagrange and Finsler submanifolds, which is studied by many authors, (see the Bibliography), an attempt to study the Hamilton submanifolds is performed in [17, 11], using an arbitrary section of the natural projection of the cotangent bundles. Here we show that there is a distinguished section, which depends only on the Hamiltonian. It solves a problem from [17, 11], concerning the possibility to induce in an intrinsic way a hamiltonian on a submanifold.

If $E \xrightarrow{\pi} M$ is an affine bundle then an *affine subbundle* of E is a affine bundle $E' \xrightarrow{\pi'} M'$ such that $E' \subset E$ and $M' \subset M$ are submanifolds, π' is the restriction of π and the affine structure on the fibers of E' is induced by the affine structure on the fibers of E. We denote by $i : M' \to M$ and $I : E' \to E$ the submanifold inclusions. We consider also a section $s : M \to E$ which restricts to a section $s' : M' \to E'$. The existence of the section s is assured by the fact that $M' \subset M$ is closed and every section $s' : M' \to E'$ can be extended to a section s using a suitable partition of unity on M.

There are some local coordinates (x^u) on M' and (x^u, y^a) on E' which extend to local coordinates $(x^i) = (x^u, x^{\bar{u}})$ on M and $(x^i, y^\alpha) = (x^u, x^{\bar{u}}, y^a, y^{\bar{\alpha}})$ on E' respectively, such that the points in M' and in E' are characterized by the conditions $x^{\bar{u}} = 0$ and $x^{\bar{u}} = y^{\bar{\alpha}} = 0$ respectively. ($i, j, k, \ldots = \overline{1, m}$, $m = \dim M$, $u, v, \ldots = \overline{1, m'}$, $\bar{u}, \bar{v}, \ldots \in \overline{m'+1, m}$, $m' = \dim M'$, $\alpha, \beta, \gamma, \ldots = \overline{1, n}$, n is the dimension of fibers of E, $a, b, \ldots = \overline{1, n'}$, $\bar{a}, \bar{b}, \ldots \in \overline{n'+1, n}$, n' is the dimension of fibers of E')

We consider also local coordinates (x^u, p_a) on E' and $(x^i, p_\alpha) = (x^u,\, x^{\bar{u}},\, p_a,\, p_{\bar{a}})$ on $\tilde{E}^*$, which are adapted to the vector bundle structures and to the submanifolds structures. The local form of the sections s' and s are $(x^u) \xrightarrow{s'} (x^u, s^a(x^u, 0))$ and $(x^u, x^{\bar{u}}, y^a, y^{\bar{a}}) \xrightarrow{s'} (x^u, x^{\bar{u}}, s^a(x^u, x^{\bar{u}}), s^{\bar{a}}(x^u, x^{\bar{u}}))$, where $s^{\bar{a}}(x^u, 0) = 0$.

The local form of the Legendre* transformation $\mathcal{H}$ is $(x^i, p_\alpha) \to (x^i, \dfrac{\partial H}{\partial p_\alpha}(x^k, p_\beta) + s^\alpha(x^i))$, and we denote $\dfrac{\partial H}{\partial p_\alpha}(x^k, p_\beta) = H^\alpha(x^k, p_\beta)$. The local forms of the inclusions $i : M' \to M$, $I : E' \to E$ and of the canonical projection $I^* : \bar{E}^* \to \bar{E}'^*$ are $(x^u) \xrightarrow{i} (x^u, 0)$, $(x^u, y^\alpha) \to (x^u, 0, y^\alpha, 0)$ and $(x^u, x^{\bar{u}}, p_a, p_{\bar{a}}) \to (x^u, p_a)$ respectively.

Let $H : \tilde{E}^* \to \mathbf{R}$ be a regular hamiltonian, thus the Legendre* transformation $\mathcal{H} : \tilde{E}^* \to E$ is a diffeomorphism. We denote by $(x^i, y^\alpha) \to (x^i, L_\beta(x^i, y^\alpha))$ the local form of $\mathcal{L} = \mathcal{H}^{-1} : E \to \tilde{E}^*$, the inverse of the Legendre* transformation.

We have that $WE' = \mathcal{L} \circ I(E')$ is a submanifold of $\bar{E}^*$.

Proposition 4.1 *The restriction of I^* to WM', $I^*_{|WM'} : WM' \to \bar{E}'^*$ is a diffeomorphism.*

Proof. We have: $\mathcal{L}$ is a diffeomorphism, I^* is a surjective submersion and I is an injective immersion. The local form of $I^* \circ \mathcal{L} \circ I$ is $(x^u, y^a) \to (x^u, K_b(x^u, 0, y^a, 0))$, thus it is a local diffeomorphism. In fact $I^* \circ \mathcal{L} \circ I$ is a diffeomorphism, since it sends the fibre $\bar{E}'_x$ in the fibre E'_x for every $x \in M'$ and $\mathcal{L}$ is a diffeomorphism when it is restricted to the fiber, thus $I^*_{|WE'}$ is also a diffeomorphism. □

Taking into account of the local form of the Legendre* transformation and of the local coordinates, it follows that the points of the submanifold WE' have as coordinates $(x^u, 0, p_a, Q_{\bar{a}}(x^u, p_a))$ in $\bar{E}^*$, where

$$\frac{\partial H}{\partial p_{\bar{a}}}(x^u, 0, p_a, Q_{\bar{b}}(x^u, p_a)) = 0. \tag{7}$$

Differentiating this equation with respect to p_a, we get:

$$\frac{\partial^2 H}{\partial p_a \partial p_{\bar{a}}} + \frac{\partial^2 H}{\partial p_{\bar{b}} \partial p_{\bar{a}}} \cdot \frac{\partial Q_{\bar{b}}}{\partial p_a} = 0.$$

Denoting by $h^{\alpha\beta} = \dfrac{\partial^2 H}{\partial p_\alpha \partial p_\beta}$, we suppose that the matrix $\tilde{h} = (h^{\bar{a}\bar{b}})_{\bar{a},\bar{b}=\overline{n'+1,n}}$ is non-degenerate; if this condition holds, we say that the Hamiltonian is *non-degenerate* along the affine subbundle E' (notice that this condition automatically holds when the vertical hessian of the Hamiltonian defines a positive quadratic form). Considering the inverse $\tilde{h}^{-1} = (\tilde{h}_{\bar{a}\bar{b}})_{\bar{a},\bar{b}=\overline{n'+1,n}}$, it follows that

$$\frac{\partial Q_{\bar{b}}}{\partial p_a} = -h^{a\bar{a}} \tilde{h}_{\bar{a}\bar{b}}. \tag{8}$$

Denote by $\bar{I} = I^{*-1}_{|WE'} : \bar{E}'^* \to WE' \subset \bar{E}^*$. Using the above constructions, we obtain the following result.

Theorem 4.1 *The map $\bar{I}$ is a section of I^* which depends only on H.*

We define $H' = H \circ \bar{I} : \bar{E}'^* \to \mathbf{R}$ and we consider the vertical Hessian of H':

$$\left(\frac{\partial^2 H'}{\partial p_a \partial p_b}(x^t, p_c) \right)_{a,b=\overline{1,n'}}$$

in every point of $\bar{E}'^*$.

Proposition 4.2 *a) If the Hamiltonian H is non-degenerate along the affine subbundle E', then the vertical Hessian of H' is also non-degenerate at every point of $\bar{E}'^*$.*

b) If the Hamiltonian has a positive definite metric, then the vertical Hessian of H' is also positive definite.

Proof. We use local coordinates. We have $H'(x^u, p_a) = H(x^u,\ 0,\ p_a,\ Q_{\bar{a}}(x^u, p_a))$. Using formula (7) it follows that:

$$\frac{\partial H'}{\partial p_a}(x^u, p_b) = \frac{\partial H}{\partial p_a}(x^u, 0, p_b, Q_{\bar{b}}(x^u, p_b)).$$

Differentiating this formula with respect to p_b, then using formula (8), we get:

$$\frac{\partial^2 H'}{\partial p_a \partial p_b} = \frac{\partial^2 H}{\partial p_a \partial p_b} + \frac{\partial Q_{\bar{a}}}{\partial p_b}\frac{\partial^2 H}{\partial p_{\bar{a}} \partial p_a} = h^{ab} - h^{b\bar{b}}\tilde{h}_{\bar{a}\bar{b}}h^{\bar{a}a}.$$

We use now the following Lemma of linear algebra.

Lemma 4.1 *Let A be a symmetric matrix of dimension p, B a symmetric and non-degenerated matrix of dimension q and C a $p \times q$ matrix such that the symmetric matrix $\begin{pmatrix} A & C \\ C^t & B \end{pmatrix}$ of dimension $p+q$ is non-degenerate. Denote $\begin{pmatrix} A & C \\ C^t & B \end{pmatrix}^{-1} = \begin{pmatrix} X & Z \\ Z^t & Y \end{pmatrix}$, where X, Y and Z have the same dimensions as the matrices A, B and C respectively.*

Then the matrix $A - C \cdot B^{-1}C^t$ is invertible and its inverse is X.

Proof. We have $\begin{pmatrix} A & C \\ C^t & B \end{pmatrix} \cdot \begin{pmatrix} X & Z \\ Z^t & Y \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}$. Thus $A \cdot X + C \cdot Z^t = I_p$ and $C^t \cdot X + B \cdot Z^t = 0$. The second equality implies $Z^t = B^{-1} \cdot C^t \cdot X$, then introducing in the first equality we get $(A - C \cdot B^{-1} \cdot C^t) \cdot X = I_p$, thus the conclusion follows. □

Turning back to the proof of the Proposition 4.2, consider the matrix $h = (h^{ij}) = \begin{pmatrix} h^{uv} & h^{\bar{u}v} \\ h^{u\bar{v}} & h^{\bar{u}\bar{v}} \end{pmatrix}$. Using the Lemma 4.1, it follows that the matrix

$$\left(h^{uv} - h^{u\bar{u}}\tilde{h}_{\bar{u}\bar{v}}h^{\bar{v}u}\right)_{u,v=\overline{1,m'}}$$

is invertible and its inverse is (h_{uv}), where $\begin{pmatrix} h_{uv} & h_{\bar{u}v} \\ h_{u\bar{v}} & h_{\bar{u}\bar{v}} \end{pmatrix} = \begin{pmatrix} h^{uv} & h^{\bar{u}v} \\ h^{u\bar{v}} & h^{\bar{u}\bar{v}} \end{pmatrix}^{-1}$. □

We consider now the case of an admissible hamiltonian. Let $\bar{E}'^* \to M'_0$ be a fibered submanifold of the fibered manifold $\bar{E}^* \to M_0$ and a section $M_0 \stackrel{s_0}{\to} \bar{E}^*$ which extends a section $M'_0 \stackrel{s_0}{\to} \bar{E}'^*$. If H is an admissible hamiltonian then H' is also admissible.

We remark that in the case of $E = TM$ and $E' = TM$", where M' is a submanifold of M and H is a hamiltonian on M, then H' can be obtained as in [7], in the following way. Consider the Lagrangian $L : TM \to \mathbf{R}$ defined by the Hamiltonian H and the induced Lagrangian $L' : TM' \to \mathbf{R}$ on M'. Let $\mathcal{H}' : T^*M'^* \to TM'^*$ be the inverse of the Legendre transformation determined by L' and $\mathcal{L} : TM^* \to T^*M^*$ be the Legendre transformation determined by L. It can be shown that $\tilde{\imath} = \mathcal{L} \circ i_* \circ \mathcal{H}'$, thus $H' = H \circ \tilde{\imath}$ is the same as the induced Hamiltonian obtained in [7]. The condition on H to be non-degenerate along the submanifold M' reads to the condition that $\mathcal{H}'$ exists.

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