Solutions of DEs and PDEs as Potential Maps Using First Order Lagrangians

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Abstract

Recently we have solved a problem rised for the first time by Poincaré (find a suitable geometric structure that describes the trajectories of a given vector field like geodesics), showing that the trajectories of a given vector field are pregeodesics in a suitable Riemann-Jacobi, Riemann-Jacobi-Lagrange or Finsler-Jacobi structure [6]-[10]. Continuing to develop similar ideas, the present paper and [11] show that solutions of DEs or PDEs are potentials maps via first order Lagrangians or via generalized Lorentz world-force laws.

This paper is organized as follows. Section 1 and Section 3 review the notion of (single-time respectively multi-time) jet bundle of order one, and establish second-order derivative operator along a local section (suitable decomposition of the Laplacian), adapted dual bases, a Sasaki-like metric, a generalized Lorentz World-Force Law, a suitable second-order prolongation of a first-order (DEs respectively PDEs) system, first-order Lagrangians producing the prolongation, and Lorentz-Udrişte World-Force Law. Section 2 and Section 4 give the Hamiltonian and the non-degenerate distinguished symplectic relative 2-form which permit the changing of the Lagrangian dynamics into covariant Hamilton equations.

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1 Solutions of DEs as potential maps

Unless specifically denied, all manifolds, all objects on them, and all maps from one manifold into another will be C^∞; however, we sometimes redundantly write "a C^∞ manifold", and so on, for emphasis.

Let $(T = R, h)$ and (M, g) be semi-Riemann manifolds of dimensions 1 and n. Hereafter we shall assume that the manifold T is oriented. Latin letters will be used for indexing the components of geometrical objects attached to the manifold M.

Local coordinates will be written

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$$t = t^1, \quad x = (x^i), \quad i = 1, \ldots, n,$$

and the components of the corresponding metric tensors and Christoffel symbols will be denoted by h_{11}, g_{ij}, H^1_{11}, G^i_{jk}. Indices of distinguished objects will be rised and lowered in the usual fashion.

Let $C^\infty(T, M) = \{\varphi : T \to M \mid \varphi$ of class $C^\infty\}$. For any $\varphi, \psi \in C^\infty(T, M)$, we define the equivalence relation $\varphi \sim \psi$ at $(t_0, x_0) \in T \times M$ by

$$x^i(t_0) = y^i(t_0) = x^i_0, \quad \frac{dx^i}{dt}(t_0) = \frac{dy^i}{dt}(t_0).$$

Using the factorization

$$J^1_{(t_0,x_0)}(T, M) = C^\infty(T, M)/\sim$$

we introduce the jet bundle of order one

$$J^1(T, M) = \bigcup_{(t_0,x_0)\in T\times M} J^1_{(t_0,x_0)}(T, M).$$

Denoting by $[\varphi]_{(t_0,x_0)}$ the equivalence class of the map φ, we define the projection

$$\pi : J^1(T, M) \to T \times M, \quad \pi[\varphi]_{(t_0,x_0)} = (t_0, \varphi(t_0)).$$

Suppose that the base $T \times M$ is covered by a system of coordinate neighborhoods $(U \times V, t^\alpha, x^i)$. Then we can define the diffeomorphism

$$F_{U\times V} : \pi^{-1}(U \times V) \to U \times V \times R^{1\cdot n}$$

$$F_{UV}[\varphi]_{(t_0,x_0)} = \left(t_0, x^i_0, \frac{dx^i}{dt}(t_0)\right).$$

Consequently $J^1(T, M)$ is a differentiable manifold of dimension $1 + n + 1 \cdot n = 2n + 1$. The coordinates on $\pi^{-1}(U \times V) \subset J^1(T, M)$ will be

$$\left(t^1 = t, x^i, y^i = \frac{dx^i}{dt}\right),$$

where

$$t^1([\varphi]_{(t_0,x_0)}) = t^1(t_0), x^i([\varphi]_{(t_0,x_0)}) = x^i(x_0),\ y^i([\varphi]_{(t_0,x_0)}) = \frac{dx^i}{dt}(t_0).$$

A local changing of coordinates $(t, x^i, y^i) \to (\bar{t}, \bar{x}^i, \bar{y}^i)$ is given by

$$\bar{t} = \bar{t}(t),\ \bar{x}^i = \bar{x}^i(x^j),\ \bar{y}^i = \frac{\partial \bar{x}^i}{\partial x^j}\frac{dt}{d\bar{t}}\, y^j, \tag{1}$$

where

$$\frac{d\bar{t}}{dt} > 0, \quad \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) \neq 0.$$

The expression of the Jacobian matrix of the local diffeomorphism (1) shows that the jet bundle of order one $J^1(T, M)$ is always orientable.

Let

$$H^1_{11} = \frac{1}{2}h^{11}\frac{dh_{11}}{dt^1} = \frac{1}{2}h^{-1}_{11}\frac{dh_{11}}{dt^1} = \frac{1}{2}\frac{d}{dt^1}\sqrt{|h_{11}|},$$

G^i_{jk} be the components of the connections induced by h and g respectively. If $\left(t = t^1, x^i, y^i = \dfrac{dx^i}{dt}\right)$ are the coordinates of a point in $J^1(T, M)$, then

$$\frac{\delta}{dt}\frac{dx^i}{dt} = \frac{d^2x^i}{dt^2} - H^1_{11}\frac{dx^i}{dt} + G^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}$$

are the components of a distinguished tensor on $T \times M$. Also

$$\left(\frac{\delta}{\delta t} = \frac{d}{dt} + H^1_{11}y^i\frac{\partial}{\partial y^i}, \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G^h_{ik}y^k\frac{\partial}{\partial y^h}, \ \frac{\partial}{\partial y^i}\right)$$

$$(dt, \ dx^j, \ \delta y^j = dy^j - H^1_{11}y^j dt + G^j_{hk}y^h dx^k)$$

are dual frames on $J^1(T, M)$, i.e.,

$$dt\left(\frac{\delta}{\delta t}\right) = 1, \ dt\left(\frac{\delta}{\delta x^i}\right) = 0, \ dt\left(\frac{\partial}{\partial y^i}\right) = 0$$

$$dx^j\left(\frac{\delta}{\delta t}\right) = 0, \ dx^j\left(\frac{\delta}{\delta x^i}\right) = \delta^j_i, \ dx^j\left(\frac{\partial}{\partial y^i}\right) = 0$$

$$\delta y^j\left(\frac{\delta}{\delta t}\right) = 0, \ \delta y^j\left(\frac{\delta}{\delta x^i}\right) = 0, \ \delta y^j\left(\frac{\partial}{\partial y^i}\right) = \delta^j_i.$$

Using these frames, we define on $J^1(T, M)$ the induced Sasaki-like metric

$$S_1 = h_{11}dt \otimes dt + g_{ij}dx^i \otimes dx^j + h^{11}g_{ij}\delta y^i \otimes \delta y^j.$$

The semi-Riemann geometry of the manifold $(J^1(T, M), S_1)$ was developed recently in [4].

Now we shall generalize the Lorentz world-force law which was initially stated [5] for particles in nonquantum relativity.

Definition. Let $F = (F_j{}^i)$ and $U = (U^i)$ be C^∞ distinguished tensors on $T \times M$, where $\omega_{ji} = g_{hi}F_j{}^h$ is skew-symmetric with respect to j and i. Let $c(t, x)$ be a C^∞ real function on $T \times M$. A map $\varphi : T \to M$ obeys a *Generalized Lorentz World-Force Law* with respect to F, U, c iff

$$h^{11}\frac{\delta}{dt}\frac{dx^i}{dt} = h^{11}\left(g^{ij}\frac{\partial c}{\partial x^j} + F_j{}^i\frac{dx^j}{dt} + U^i\right).$$

Now we remark that a C^∞ distinguished tensor field $X^i(t, x)$, $i = 1, \ldots, n$ on $T \times M$ defines a family of trajectories as solutions of DEs system of order one

$$\frac{dx^i}{dt} = X^i(t, x(t)). \tag{2}$$

The distinguished tensor field $X^i(t,x)$ and semi-Riemann metrics h and g determine the *potential energy density*

$$f : T\times M \to R, \quad f = \frac{1}{2}h^{11}g_{ij}X^iX^j.$$

The distinguished tensor field (family of trajectories) X^i on $(T\times M, h_{11}+g)$ is called:
1) *timelike*, if $f<0$;
2) *nonspacelike* or *causal*, if $f\le 0$;
3) *null* or *lightlike*, if $f=0$;
4) *spacelike*, if $f>0$.

Let X^i be a distinguished tensor field of everywhere constant energy. If X^i (the system (2)) has no critical point on M, then upon rescaling, it may be supposed that $f\in\{-1,0,1\}$. Generally, $\mathcal{E}=\{x_0\in M|X^i(t,x_0)=0,\forall t\in T\}$ is the set of critical points of the distinguished tensor field, and this rescaling is possible only on $T\times(M\setminus\mathcal{E})$.

Using the operator (derivative along a solution of (2) via the decomposition $\frac{\delta}{dt}\cdot\frac{d}{dt}$)

$$\frac{\delta}{dt}\frac{dx^i}{dt} = \frac{d^2x^i}{dt^2} - H^1_{11}\frac{dx^i}{dt} + G^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt},$$

the Levy-Civita connection D of (R,h) and the Levy-Civita connection ∇ of (M,g), we obtain the prolongation (DEs system of order two)

$$\frac{d^2x^i}{dt^2} - H^1_{11}\frac{dx^i}{dt} + G^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = DX^i + (\nabla_jX^i)\frac{dx^j}{dt}, \tag{3}$$

where

$$\nabla_jX^i = \frac{\partial X^i}{\partial x^j} + G^i_{jk}X^k, \quad DX^i = \frac{\partial X^i}{\partial t} - H^1_{11}X^i.$$

The distinguished tensor field X^i, the metric g, and the connection ∇ determine the external distinguished tensor field

$$F_j{}^i = \nabla_jX^i - g^{ih}g_{kj}\nabla_hX^k,$$

which characterizes the *helicity* of the distinguished tensor field X^i.

The DEs system (3) can be written in the equivalent form

$$\frac{d^2x^i}{dt^2} - H^1_{11}\frac{dx^i}{dt} + G^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = g^{ih}g_{kj}(\nabla_hX^k)\frac{dx^j}{dt} + F_j{}^i\frac{dx^j}{dt} + DX^i. \tag{4}$$

Now we modify this DEs system into

$$\frac{d^2x^i}{dt^2} - H^1_{11}\frac{dx^i}{dt} + G^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = g^{ih}g_{kj}(\nabla_hX^k)X^j + F_j{}^i\frac{dx^j}{dt} + DX^i. \tag{5}$$

The system (5) is still a prolongation of the DEs system (2).

Theorem. *The kinematic system (2) can be prolonged to the second order dynamical system (5).*

Corollary. *Choosing the metrics h and g such that $f \in \{-1, 0, 1\}$, then the kinematic system (2) can be prolonged to the second order dynamical system*

$$\frac{d^2x^i}{dt^2} - H^1_{11}\frac{dx^i}{dt} + G^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = F_j{}^i\frac{dx^j}{dt} + DX^i.$$

We shall show that the dynamical system (5) has a variational structure, being in fact an Euler-Lagrange system. We identify $J^1(T \times M)$ with its dual via the semi-Riemann metrics h and g.

Theorem. 1) *The solutions of the DEs system (5) are the extremals of the Lagrangian*

$$L = \frac{1}{2}h^{11}g_{ij}\left(\frac{dx^i}{dt} - X^i\right)\left(\frac{dx^j}{dt} - X^j\right)\sqrt{|h_{11}|} =$$

$$= \left(\frac{1}{2}h^{11}g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} - h^{11}g_{ij}\frac{dx^i}{dt}X^j + f\right)\sqrt{|h_{11}|}.$$

2) *If $F_j{}^i = 0$, then the solutions of the DEs system (5) are the extremals of the Lagrangian*

$$L = \left(\frac{1}{2}h^{11}g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} + f\right)\sqrt{|h_{11}|}.$$

3) *Both Lagrangians produce the same Hamiltonian*

$$H = \left(\frac{1}{2}h^{11}g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} - f\right)\sqrt{|h_{11}|}.$$

Theorem (Lorentz-Udrişte World-Force Law) see also [6]-[10].

1) *Every solution of DEs system*

$$\frac{d^2x^i}{dt^2} - H^1_{11}\frac{dx^i}{dt} + G^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = g^{ih}g_{kj}(\nabla_h X^k)X^j + DX^i$$

is a potential map on the semi-Riemann manifold $(T \times M, h + g)$.

2) *Every solution of DEs system (5) is a horizontal potential map on the semi-Riemann-Lagrange manifold*

$$(T \times M, h + g,\ N\binom{i}{1}_j = G^i_{jk}y^k - F_j{}^i,\quad M\binom{i}{1}_1 = -H^1_{11}y^i).$$

Corollary. *Every DE generates a Lagrangian of order one via the associated first order DEs system and suitable metrics on the manifold of independent variable and on the manifold of functions. In this sense the solutions of the initial DE are potential maps produced by a suitable Lagrangian.*

Proof. Let $t \in R$ denote a real variable, usually referred to as the time. It may be pointed out that the DE

$$\frac{d^nx}{dt^n} = f\left(t, x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}\right), \tag{6}$$

where x is the unknown function, is equivalent to a system (2). For if we set $x = x^1$, then (6) is equivalent to

$$\frac{dx^1}{dt} = x^2, \quad \frac{dx^2}{dt} = x^3, \dots, \frac{dx^{n-1}}{dt} = x^n$$

$$\frac{dx^n}{dt} = f(t, x^1, x^2, \dots, x^n),$$

which is of type (2). Therefore, the preceding theory applies.

2 Hamiltonian approach

Let (Q, Ω) be a symplectic manifold (of even dimension). The Hamiltonian vector field X_H of the function $H \in \mathcal{F}(Q)$ is defined by

$$X_H \lrcorner \Omega = dH.$$

We generalize this relation as

$$X^1_H \lrcorner \Omega_1 = \sqrt{|h_{11}|} dH,$$

using the distinguished objects

$$X^1_H,\ \Omega_1, H$$

and the manifold $J^1(T, M)$. For another point of view, see also [11] and compare with [2], [3].

Theorem. *The DEs system*

$$\frac{d^2x^i}{dt^2} - H^1_{11}\frac{dx^i}{dt} + G^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = g^{ih}g_{kj}(\nabla_h X^k)X^j$$

transfers in $J^1(T, M)$ as a Hamilton DEs system with respect to the Hamiltonian

$$H = \frac{1}{2}h^{11}g_{ij}y^iy^j - f$$

and the non-degenerate distinguished symplectic relative 2-form

$$\Omega = \Omega_1 \otimes dt^1, \quad \Omega_1 = g_{ij}dx^i \wedge \delta y^j \sqrt{|h_{11}|}.$$

Proof. Let

$$\theta = \theta_1 \otimes dt^1, \quad \theta_1 = g_{ij}y^i dx^j \sqrt{|h_{11}|}$$

be the distinguished Liouville relative 1-form on $J^1(T, M)$. We find

$$\Omega_1 = -d\theta_1.$$

We introduce

$$X_H = X^1_H \frac{\delta}{\delta t},\ X^1_H = u^{1l}\frac{\delta}{\delta x^l} + \frac{\delta u^{1l}}{dt}\frac{\partial}{\partial y^l}$$

as the distinguished Hamiltonian object associated to the function H.

The relation

$$X^1_H \lrcorner \Omega_1 = \sqrt{|h_{11}|} dH,$$

where

$$dH = h^{11}g_{ij}y^j\delta y^i - h^{11}g_{ij}(DX^i)X^j dt - h^{11}g_{ij}X^j\nabla_k X^i dx^k,$$

implies

$$g_{ij}u^{1i}\delta y^j - g_{ij}\frac{\delta u^{1j}}{dt}dx^i = dH.$$

Consequently, it appears the PDEs system of Hamilton type

$$\begin{cases} u^{1i} = h^{11}y^i \\ \dfrac{\delta u^{1i}}{dt} = g^{hi}h^{11}g_{jk}X^j(\nabla_h X^k) \end{cases}$$

together the condition

$$h^{11}g_{ij}(DX^i)X^j = 0.$$

Theorem. *The DEs system*

$$\frac{d^2x^i}{dt^2} - H^1_{11}\frac{dx^i}{dt} + G^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = g^{ih}g_{kj}(\nabla_h X^k)X^j + F_j{}^i\frac{dx^j}{dt} + DX^i$$

transfers in $J^1(T, M)$ *as a Hamilton DEs system with respect to the Hamiltonian*

$$H = \frac{1}{2}h^{11}g_{ij}y^i y^j - f$$

and the non-degenerate distinguished symplectic relative 2-form

$$\Omega = \Omega_1 \otimes dt,\ \Omega_1 = (g_{ij}dx^i \wedge \delta y^j + \omega_{ij}dx^i \wedge dx^j + g_{ij}(DX^i)dt \wedge dx^j)\sqrt{|h_{11}|},$$

where

$$\omega_{ji} = g_{hi}F_j{}^h.$$

Proof. Let

$$\theta = \theta_1 \otimes dt^1,\ \theta_1 = (g_{ij}y^i dx^j - g_{ij}X^i dx^j)\sqrt{|h_{11}|}$$

be the distinguished Liouville relative 1-form on $J^1(R, M)$. We find

$$\Omega_1 = -d\theta_1.$$

We denote

$$X_H = X^1_H\frac{\delta}{\delta t},\ X^1_H = h^{11}\frac{\delta}{\delta t} + u^{1l}\frac{\delta}{\delta x^l} + \frac{\delta u^{1l}}{dt}\frac{\partial}{\partial y^l}$$

the distinguished Hamiltonian object of the function H. The relation

$$X^1_H \lrcorner \Omega_1 = \sqrt{|h_{11}|}dH$$

can be written

$$g_{ij}u^{1i}\delta y^j - g_{ij}\frac{\delta u^{1j}}{dt}dx^i + 2\omega_{ij}u^{1i}dx^j - g_{ij}(DX^i)u^{1j}dt + h^{11}g_{ij}(DX^i)dx^j = dH,$$

where

$$dH = -h^{11}g_{ij}(DX^i)X^j dt + h^{11}g_{ij}y^j\delta y^i - h^{11}g_{ij}X^j(\nabla_k X^i)dx^k.$$

Via these relations we identify a PDEs system of Hamilton type,

$$\begin{cases} u^{1i} = h^{11}y^i \\ \dfrac{\delta u^{1i}}{dt} = g^{hi}h^{11}g_{jk}X^j(\nabla_h X^k) + 2g^{hi}\omega_{jh}u^{1j} + h^{11}DX^i \end{cases}$$

together the condition

$$g_{ij}(DX^i)(u^{1j} - h^{11}X^j) = 0.$$

3 Solutions of PDEs as Potential Maps

All manifolds and maps are C^∞, unless otherwise stated.

Let (T,h) and (M,g) be semi-Riemann manifolds of dimensions p and n. Hereafter we shall assume that the manifold T is oriented. Greek (Latin) letters will be used for indexing the components of geometrical objects attached to the manifold T (manifold M).

Local coordinates will be written

$$t = (t^\alpha), \quad \alpha = 1, \dots, p$$

$$x = (x^i), \quad i = 1, \dots, n,$$

and the components of the corresponding metric tensors and Christoffel symbols will be denoted by $h_{\alpha\beta}, g_{ij}$, $H^\alpha_{\beta\gamma}$, G^i_{jk}. Indices of tensors or distinguished tensors will be rised and lowered in the usual fashion.

Let $C^\infty(T,M) = \{\varphi : T \to M |\, \varphi \text{ of class } C^\infty\}$. For any $\varphi, \psi \in C^\infty(T,M)$ we define the equivalence relation $\varphi \sim \psi$ at $(t_0, x_0) \in T \times M$, by

$$x^i(t_0) = y^i(t_0) = x^i_0,\ \frac{\partial x^i}{\partial t^\alpha}(t_0) = \frac{\partial y^i}{\partial t^\alpha}(t_0).$$

Using the factorization

$$J^1_{(t_0,x_0)}(T,M) = C^\infty(T,M)/_\sim,$$

we introduce the jet bundle of order one

$$J^1(T,M) = \bigcup_{(t_0,x_0)\in T\times M} J^1_{t_0,x_0}(T,M).$$

Denoting by $[\varphi]_{(t_0,x_0)}$ the equivalence class of the map φ, we define the projection

$$\pi : J^1(T,M) \to T \times M,\ \pi[\varphi]_{(t_0,x_0)} = (t_0, \varphi(t_0)).$$

Suppose that the base $T \times M$ is covered by a systems of coordinate neighborhood $(U \times V, t^\alpha, x^i)$. Then we can define the diffeomorphism

$$F_{U\times V} : \pi^{-1}(U \times V) \to U \times V \times R^{pn},$$

$$F_{UV}[\varphi]_{(t_0,x_0)} = \left(t_0^\alpha, x_0^i, \frac{\partial x^i}{\partial t^\alpha}(t_0)\right).$$

Consequently $J^1(T,M)$ is a differentiable manifold of dimension $p+n+pn$. The coordinates on $\pi^{-1}(U\times V)\subset J^1(T,M)$ will be

$$(t^\alpha, x^i, x^i_\alpha),$$

where

$$t^\alpha\left([\varphi]_{(t_0,x_0)}\right) = t^\alpha(t_0), x^i\left([\varphi]_{(t_0,x_0)}\right) = x^i(x_0), x^i_\alpha\left([\varphi]_{(t_0,x_0)}\right) = \frac{\partial x^i}{\partial t^\alpha}(t_0).$$

A local changing of coordinates $(t^\alpha, x^i, x^i_\alpha)\to(\bar t^\alpha, \bar x^i, \bar x^i_\alpha)$ is given by

$$\bar t^\alpha = \bar t^\alpha(t^\beta),\ \bar x^i = \bar x^i(x^j),\ \bar x^i_\alpha = \frac{\partial \bar x^i}{\partial x^j}\frac{\partial t^\beta}{\partial \bar t^\alpha}x^j_\beta, \tag{7}$$

where

$$\det\left(\frac{\partial \bar t^\alpha}{\partial t^\beta}\right) > 0, \quad \det\left(\frac{\partial \bar x^i}{\partial x^j}\right)\neq 0.$$

The expression of the Jacobian matrix of the local diffeomorphism (7) shows that the jet bundle of order one $J^1(T,M)$ is always orientable.

Let $H^\alpha_{\beta\gamma}, G^i_{jk}$ be the components of the connections induced by h and g respectively. If $(t^\alpha, x^i, x^i_\alpha)$ are the coordinates of a point in $J^1(T,M)$, then

$$x^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial t^\alpha\partial t^\beta} - H^\gamma_{\alpha\beta}x^i_\gamma + G^i_{jk}x^j_\alpha x^k_\beta$$

are the components of a distinguished tensor on $T\times M$. Also

$$\left(\frac{\delta}{\delta t^\alpha} = \frac{\partial}{\partial t^\alpha} + H^\gamma_{\alpha\beta}x^i_\gamma\frac{\partial}{\partial x^i_\beta},\ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G^h_{ik}x^k_\alpha\frac{\partial}{\partial x^h_\alpha},\ \frac{\partial}{\partial x^i_\alpha}\right),$$

$$\left(dt^\beta, dx^j,\ \delta x^j_\beta = dx^j_\beta - H^\gamma_{\beta\lambda}x^j_\gamma dt^\lambda + G^j_{hk}x^h_\beta dx^k\right)$$

are dual frames on $J^1(T,M)$, i.e.,

$$dt^\beta\left(\frac{\delta}{\delta t^\alpha}\right) = \delta^\beta_\alpha, \quad dt^\beta\left(\frac{\delta}{\delta x^i}\right) = 0, \quad dt^\beta\left(\frac{\partial}{\partial x^i_\alpha}\right) = 0$$

$$dx^j\left(\frac{\delta}{\delta t^\alpha}\right) = 0, \quad dx^j\left(\frac{\delta}{\delta x^i}\right) = \delta^j_i, \quad dx^j\left(\frac{\partial}{\partial x^i_\alpha}\right) = 0$$

$$\delta x^j_\beta\left(\frac{\delta}{\delta t^\alpha}\right) = 0, \quad \delta x^j_\beta\left(\frac{\delta}{\delta x^i}\right) = 0, \quad \delta x^j_\beta\left(\frac{\partial}{\partial x^i_\alpha}\right) = \delta^j_i\delta^\alpha_\beta.$$

Using these frames, we define on $J^1(T,M)$ the induced Sasaki-like metric

$$S_1 = h_{\alpha\beta}dt^\alpha\otimes dt^\beta + g_{ij}dx^i\otimes dx^j + h^{\alpha\beta}g_{ij}\delta x^i_\alpha\otimes\delta x^j_\beta.$$

The semi-Riemann geometry of the manifold $J^1(T,M)$ was developed recently in [4].

The Lorentz world-force law formulated usually for particles [5] can be generalized as follows:

Definition. Let $F_\alpha = (F_j{}^i{}_\alpha)$ and $U_{\alpha\beta} = (U^i_{\alpha\beta})$ be C^∞ distinguished tensors on $T \times M$, where $\omega_{ji\alpha} = g_{hi}F_j{}^h{}_\alpha$ is skew-symmetric with respect to j and i. Let $c(t,x)$ be a C^∞ real function on $T \times M$. Suppose (T,h) is a Riemannian manifold. A C^∞ map $\varphi : T \to M$ obeys a *Generalized Lorentz World-Force Law* with respect to F_α, $U_{\alpha\beta}$, c iff

$$h^{\alpha\beta}x^i_{\alpha\beta} = g^{ij}\frac{\partial c}{\partial x^j} + h^{\alpha\beta}F_j{}^i{}_\alpha x^j_\beta + h^{\alpha\beta}U^i_{\alpha\beta},$$

i.e., iff it is a potential map of a suitable geometrical structure.

Let us show that the solutions of a system of PDEs of order one are potential maps in a suitable geometrical structure of the jet bundle of order one. For that we remark that any C^∞ distinguished tensor field $X^i_\alpha(t,x)$ on $T \times M$ defines a family of p-dimensional sheets as solutions of the PDEs system of order one

$$x^i_\alpha = X^i_\alpha(t, x(t)), \tag{8}$$

if the complete integrability conditions

$$\frac{\partial X^i_\alpha}{\partial t^\beta} + \frac{\partial X^i_\alpha}{\partial x^j}X^j_\beta = \frac{\partial X^i_\beta}{\partial t^\alpha} + \frac{\partial X^i_\beta}{\partial x^j}X^j_\alpha$$

are satisfied.

To any distinguished tensor field $X^i_\alpha(t,x)$ and semi-Riemann metrics h and g we associate the *potential energy density*

$$f : T \times M \to R, \quad f = \frac{1}{2}h^{\alpha\beta}g_{ij}X^i_\alpha X^j_\beta.$$

The distinguished tensor field X^i_α (family of p-dimensional sheets) on $(T \times M, h+g)$ is called:

1) *timelike*, if $f < 0$;
2) *nonspacelike or causal*, if $f \leq 0$;
3) *null or lightlike*, if $f = 0$;
4) *spacelike*, if $f > 0$.

Let $\mathcal{E} = \{x_0 \in M |\ X^i_\alpha(t,x_0) = 0,\ \forall t \in T\}$ be the set of critical points of the system (8). If $f =$ constant, upon rescaling on $T \times (M \setminus \mathcal{E})$, it may be supposed that $f \in \{-1, 0, 1\}$.

The derivative along a solution of (8),

$$\frac{\delta}{\partial t^\beta}x^i_\alpha = x^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} - H^\gamma_{\alpha\beta}x^i_\gamma + G^i_{jk}x^j_\alpha x^k_\beta,$$

produces the decomposition $\dfrac{\delta}{\partial t^\beta} \cdot \dfrac{\partial}{\partial t^\alpha}$. This operator, the Levy-Civita connection D of (T,h) and the Levy-Civita connection ∇ of (M,g) produce the prolongation (PDEs system of order two)

$$x^i_{\alpha\beta} = D_\beta X^i_\alpha + (\nabla_j X^i_\alpha)x^j_\beta, \tag{9}$$

which can be converted into the prolongation

$$h^{\alpha\beta}x^i_{\alpha\beta} = g^{ih}h^{\alpha\beta}g_{ij}(\nabla_h X^k_\alpha)X^j_\beta + h^{\alpha\beta}F_j{}^i{}_\alpha x^j_\beta + h^{\alpha\beta}D_\beta X^i_\alpha, \tag{10}$$

where

$$F_j{}^i{}_\alpha = \nabla_j X^i_\alpha - g^{ih}g_{kj}\nabla_h X^k_\alpha$$

is the external distinguished tensor field which characterizes the *helicity* of the distinguished tensor field X^i_α.

Theorem. *Any solution of PDEs system (8) is a solution of the PDEs system (10).*

The first term in the second hand member of the PDEs system (10) is the force $(grad f)^i$. Therefore, choosing the metrics h and g such that $f \in \{-1, 0, 1\}$, the system (10) reduces to

$$h^{\alpha\beta}x^i_{\alpha\beta} = g^{ih}F_j{}^i{}_\alpha x^j_\beta + h^{\alpha\beta}D_\beta X^i_\alpha. \tag{10'}$$

Now, let us describe the variational structure of the PDEs system (10).

Theorem. *The solutions of PDEs system (10) are the extremals of the Lagrangian*

$$\begin{aligned} L &= \frac{1}{2}h^{\alpha\beta}g_{ij}(x^i_\alpha - X^i_\alpha)(x^j_\beta - X^j_\beta)\sqrt{|h|} = \\ &= \left(\frac{1}{2}h^{\alpha\beta}g_{ij}x^i_\alpha x^j_\beta - h^{\alpha\beta}g_{ij}x^i_\alpha X^j_\beta + f\right)\sqrt{|h|}. \end{aligned}$$

If $F^i_{j\alpha} = 0$, *then this Lagrangian can be replaced by*

$$L = \left(\frac{1}{2}h^{\alpha\beta}g_{ij}x^i_\alpha x^j_\beta + f\right)\sqrt{|h|}.$$

2) *Both Lagrangians produce the same Hamiltonian*

$$H = \left(\frac{1}{2}h^{\alpha\beta}g_{ij}x^i_\alpha x^j_\beta - f\right)\sqrt{|h|}.$$

Theorem (Lorentz-Udrişte World-Force Law). *Suppose* (T, h) *is a Riemannian manifold. Every solution of the PDEs system (8) is a horizontal potential map of the semi-Riemann-Lagrange manifold*

$$(T \times M, h + g,\ N(^i_\alpha)_j = G^i_{jk}x^k_\alpha - F_j{}^i{}_\alpha,\ M(^i_\alpha)_\beta = -H^\gamma_{\alpha\beta}x^i_\gamma).$$

Corollary. *Every PDE generates a Lagrangian of order one via the associated first order PDEs system and suitable metrics on the manifold of independent variables and on the manifold of functions. In this sense the solutions of the initial PDE are potential maps produced by a suitable Lagrangian.*

Proof. Let

$$\frac{\partial^r x}{\partial(t^p)^r} = F(t^\alpha, x, \bar{x}^{(r)})$$

be a PDE of order r, where $\bar{x}^{(r)}$ represent the partial derivatives of x with respect to t^α, till the order r inclusively, excepting the partial derivative $\dfrac{\partial^r x}{\partial(t^p)^r}$. This equation is equivalent to a system (8).

For the sake of simplicity, we take $r = 2$. We denote $\dfrac{\partial x}{\partial t^\alpha} = x_\alpha = u^\alpha$ and we find the partial derivatives of the functions (x, u^α) using the system

$$\begin{cases} x_\alpha = u^\alpha \\ u^\alpha_\beta = u^\beta_\alpha,\ \alpha \neq \beta \\ u^2_2 = f(t^\alpha, x, u^\lambda_\mu),\ \text{excepting } \lambda = \mu = 2. \end{cases}$$

We shall find a PDEs system of order one with $p(1+p)$ equations, which is of type (8). Therefore, the preceding theory applies.

Example. Consider the Monge-Ampère equation $\det(Hess\, u) = F$, where $u : T \to R$ is the unknown function, Hess means the Hessian with respect to the semi-Riemann structure of the manifold T, and F is a given function of t, u, du. This equation is clasely related to the study of the curvature of a manifold [14]. The Monge-Ampère equation is equivalent to the first-order system

$$\frac{\partial u}{\partial t^\alpha} = \omega_\alpha, \quad D_\beta \omega_\alpha = \eta_{\beta\alpha}$$

with the restriction

$$\det(\eta_{\alpha\beta}) = F(t, u, \omega).$$

It appears the Lagrangian

$$L = \frac{1}{2} h^{\alpha\beta} \left(\frac{\partial u}{\partial t^\alpha} - \omega_\alpha \right) \left(\frac{\partial u}{\partial t^\beta} - \omega_\beta \right) + \frac{1}{2} h^{\alpha\beta} h^{\gamma\delta} (D_\gamma \omega_\alpha - \eta_{\gamma\alpha})(D_\delta \omega_\beta - \eta_{\delta\beta})$$

subject to

$$\det(\eta_{\alpha\beta}) = F(t, u, \omega),$$

and the preceding theory applies.

Particularly, the Monge-Amperè equation

$$u_{t^1 t^1} u_{t^2 t^2} - u^2_{t^1 t^2} = F(t^1, t^2, u, u_{t^1}, u_{t^2})$$

is equivalent to the first order PDEs system

$$\begin{cases} u_{t^1} = v \\ u_{t^2} = w \\ v_{t^1} = \sqrt{F} \cos h\zeta \\ v_{t^2} = \sqrt{F} \sin h\zeta \\ w_{t^1} = \sqrt{F} \sin h\zeta \\ w_{t^2} = \sqrt{F} \cos h\zeta \end{cases}$$

where ζ is an arbitrary function of (t^1, t^2, u, v, w). Let $(M, g_{ij} = \delta_{ij})$ be the Riemannian manifold of coordinates (u, v, w), $(T, h_{\alpha\beta} = \delta_{\alpha\beta})$ be the Riemannian manifold of coordinates (t^1, t^2) and $J^1(T \times M)$ be the jet bundle of order one. In this sense, the solutions of the previous system are extremals (potential maps) of the Lagrangian

$$\begin{aligned} L \;=\; & (u_{t^1} - v)^2 + (u_{t^2} - w)^2 + (v_{t^1} - \sqrt{F} \cos h\zeta)^2 + \\ +\; & (v_{t^2} - \sqrt{F} \sin h\zeta)^2 + (w_{t^1} - \sqrt{F} \sin h\zeta)^2 + (w_{t^2} - \sqrt{F} \cos h\zeta)^2 \end{aligned}$$

4 Covariant Hamilton Equations

Recall that on a symplectic manifold (Q, Ω) of even dimension q, the Hamiltonian vector field X_H of a function $H \in \mathcal{F}(Q)$ is defined by

$$X_H \lrcorner \Omega = dH.$$

This relation can be generalized as

$$X_H^\alpha \lrcorner \Omega_\alpha = \sqrt{|h|} dH,$$

using the distinguished objects X_H, Ω, H on $J^1(T, M)$. For another point of view, see also [11] and compare with [2], [3].

Theorem. *The PDEs system*

$$h^{\alpha\beta} x^i_{\alpha\beta} = g^{ih} h^{\alpha\beta} g_{jk} (\nabla_h X^j_\alpha) X^k_\beta$$

transfers in $J^1(T, M)$ as a covariant Hamilton PDEs system with respect to the Hamiltonian

$$H = \frac{1}{2} h^{\alpha\beta} g_{ij} x^i_\alpha x^j_\beta - f$$

and the non-degenerate distinguished polysymplectic relative 2–form

$$\Omega = \Omega_\alpha \otimes dt^\alpha, \quad \Omega_\alpha = g_{ij} dx^i \wedge \delta x^j_\alpha \sqrt{|h|}.$$

Proof. Let

$$\theta = \theta_\alpha \otimes dt^\alpha, \quad \theta_\alpha = g_{ij} x^i_\alpha dx^j \sqrt{|h|}$$

be the distinguished Liouville relative 1–form on $J^1(T, M)$. It follows

$$\Omega_\alpha = -d\theta_\alpha.$$

We denote by

$$X_H = X_H^\beta \frac{\delta}{\delta t^\beta}, \quad X_H^\beta = u^{\beta l} \frac{\delta}{\delta x^l} + \frac{\delta u^{\beta l}}{\partial t^\alpha} \frac{\partial}{\partial x^l_\alpha}$$

the distinguished Hamiltonian object of the function H. Imposing

$$X_H^\alpha \lrcorner \Omega_\alpha = \sqrt{|h|} dH,$$

where

$$dH = h^{\alpha\beta} g_{ij} x^j_\beta \delta x^i_\alpha - h^{\alpha\beta} g_{ij} (D_\gamma X^i_\alpha) X^j_\beta dt^\gamma - h^{\alpha\beta} g_{ij} X^j_\beta \nabla_k X^i_\alpha dx^k$$

we find

$$g_{ij} u^{\alpha i} \delta x^j_\alpha - g_{ij} \frac{\delta u^{\alpha j}}{\partial t^\alpha} dx^i = dH.$$

Consequently, it appears the Hamilton PDEs system

$$\begin{cases} u^{\alpha i} = h^{\alpha\beta} x^i_\beta \\ \dfrac{\delta u^{\alpha i}}{\partial t^\alpha} = g^{hi} h^{\alpha\beta} g_{jk} X^j_\beta (\nabla_h X^k_\alpha) \end{cases}$$

together the condition

$$h^{\alpha\beta}g_{ij}(D_\gamma X^i_\alpha)X^j_\beta = 0.$$

Theorem. *The PDEs system*

$$h^{\alpha\beta}x^i_{\alpha\beta} = g^{ih}h^{\alpha\beta}g_{kj}(\nabla_h X^k_\alpha)X^j_\beta + h^{\alpha\beta}F_j{}^i{}_\alpha x^j_\beta + h^{\alpha\beta}D_\beta X^i_\alpha$$

transfers in $J^1(T,M)$ as a covariant Hamilton PDEs system with respect to the Hamiltonian

$$H = \frac{1}{2}h^{\alpha\beta}g_{ij}x^i_\alpha x^j_\beta - f$$

and the non-degenerate distinguished polysymplectic relative 2–form

$$\Omega = \Omega_\alpha \otimes dt^\alpha,$$

$$\Omega_\alpha = (g_{ij}dx^i \wedge \delta x^j_\alpha + \omega_{ij\alpha}dx^i \wedge dx^j + g_{ij}(D_\beta X^i_\alpha)dt^\beta \wedge dx^j)\sqrt{|h|}.$$

Proof. Let

$$\theta = \theta_\alpha \otimes dt^\alpha,$$

$$\theta_\alpha = (g_{ij}x^i_\alpha dx^j - g_{ij}X^i_\alpha dx^j)\sqrt{|h|}$$

be the distinguished Liouville relative 1–form on $J^1(T,M)$. It follows

$$\Omega_\alpha = -d\theta_\alpha.$$

We denote by

$$X_H = X^\beta_H \frac{\delta}{\delta t^\beta},$$

$$X^\beta_H = h^{\beta\gamma}\frac{\delta}{\delta t^\gamma} + u^{\beta l}\frac{\delta}{\delta x^l} + \frac{\delta u^{\beta l}}{\partial t^\alpha}\frac{\partial}{\partial x^l_\alpha}$$

the distinguished Hamiltonian object of the function H. Imposing

$$X^\alpha_H \lrcorner \Omega_\alpha = \sqrt{|h|}dH,$$

where

$$dH = -h^{\alpha\beta}g_{ij}(D_\gamma X^i_\alpha)X^j_\beta dt^\gamma + h^{\alpha\beta}g_{ij}x^j_\beta \delta x^i_\alpha - h^{\alpha\beta}g_{ij}X^j_\beta(\nabla_k X^i_\alpha)dx^k,$$

we find

$$\begin{aligned} g_{ij}u^{\alpha i}\delta x^j_\alpha \quad &- \quad g_{ij}\frac{\delta u^{\alpha j}}{\partial t^\alpha}dx^i + 2\omega_{ij\alpha}u^{\alpha i}dx^j - \\ &- \quad g_{ij}(D_\beta X^i_\alpha)u^{\alpha j}dt^\beta + h^{\alpha\beta}g_{ij}(D_\beta X^i_\alpha)dx^j = dH. \end{aligned}$$

Consequently, we obtain the Hamilton PDEs system

$$\begin{cases} u^{\alpha i} = h^{\alpha\beta}x^i_\beta \\ \dfrac{\delta u^{\alpha i}}{\partial t^\alpha} = g^{hi}h^{\alpha\beta}g_{jk}X^j_\beta(\nabla_h X^k_\alpha) + 2g^{hi}\omega_{jh\alpha}u^{\alpha j} + h^{\alpha\beta}D_\beta X^i_\alpha \end{cases}$$

together the condition

$$g_{ij}(D_\gamma X^i_\alpha)(u^{\alpha j} - h^{\alpha\beta}X^j_\beta) = 0.$$

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