On Normal Sections of Stiefel Submanifold

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Abstract

In this study we consider Stiefel submanifold $\mathbf{V}_{n,k}$ which is a projective m-space $\mathbf{P}^m$ $(n = mk)$ isometrically imbedded in $\mathbf{R}^{m+\frac{1}{2}m(m+1)}$ by its first standard imbedding. We also consider the normal sections of $\mathbf{V}_{n,k}$ and show that $\mathbf{V}_{n,k}$ has $P2-PNS$ property.

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1 Introduction

Let M be an n-dimensional submanifold in $(m+d)$- dimensional Euclidean space $\mathbf{R}^{n+d}$. Let ∇, $\overline{\nabla}$, and $\widetilde{\nabla}$ denote the covariant derivatives in $T(M)$, $N(M)$ and $\mathbf{R}^{n+d}$ respectively. Thus $\widetilde{\nabla}_X$ is just the directional derivative in the direction X in $\mathbf{R}^{n+d}$. Then for tangent vector fields X, Y and Z and normal vector field v over M we have $\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$ and $\widetilde{\nabla}_X v = -A_v X + \overline{\nabla}_X v$, where h is the second fundamental form and A_v is the shape operator of M [6]. For tangent vector fields X, Y, Z over M we define $\overline{\nabla} h$ as usual by

$$\overline{\nabla}_X h(Y,Z) = (\overline{\nabla}_X h)(Y,Z) + h(\nabla_X Y, Z) + h(Y, \nabla_X Z).$$

Let M be a smooth n-dimensional submanifold in $(n+d)$-dimensional Euclidean space $\mathbf{R}^{n+d}$. For a point x in M and a non-zero tangent vector $X \in T_x M$, we define the $(d+1)$-dimensional affine subspace $E(x,X)$ of $\mathbf{R}^{n+d}$ by $E(x,X) = x + span\{X, T_x^{\perp} M\}$. In a neighborhood of x the intersection $M \cap E(x,X)$ is a regular curve $\gamma : (-\varepsilon, \varepsilon) \to M$. We suppose the parameter $t \in (-\varepsilon, \varepsilon)$ is a multiple of the arc-length such that $\gamma(0) = x$ and $\gamma'(0) = X$. Each choise of $X \in T(M)$ yields a different curve which is called the *normal section of* M at x in the direction of X,where $X \in T_x(M)$ [8]. For such a normal section we can write

$$\gamma(t) = x + \lambda(t) X + N(t) \tag{1}$$

where $N(t) \in T_x^{\perp} M$ and $\lambda(t) \in \mathbf{R}$.

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Definition 1.1. The submanifold M is said to have pointwise k-planar normal sections $(Pk - PNS)$ if for each normal section γ, the higher order derivatives $\gamma'(0)$, $\gamma''(0)$,..., $\gamma^{(k+1)}(0)$ are linearly dependent as vectors in $\mathbf{R}^{n+d}$.

If $k = 1$ then M is totally geodesic. Taking $k = 2$, we note that submanifolds with pointwise 2-planar normal sections have been classified. They have parallel second fundamental form (i. e. $\overline{\nabla}h = 0$) or hypersurfaces [1], see also[10] and [2].

Theorem 1.1. [1] *Let M be a submanifold of $\mathbf{R}^{n+d}$ then M has pointwise 2-planar normal sections $(P2 - PNS)$ if and only if*

$$\|h(X,X)\|^2 (\overline{\nabla}_X h)(X,X) = \langle(\overline{\nabla}_X h)(X,X), h(X,X)\rangle h(X,X). \tag{2}$$

2 Stiefel Submanifolds $\mathbf{V}_{n,k}$

For each n, k $(k \leq n)$ the Stiefel manifold $\mathbf{V}_{n,k}$ has as its points all orthonormal frames $x = (e_1,...,e_k)$ of k vectors in Euclidean $n-$space (i.e. ordered sequences of k orthonormal vectors in $\mathbf{R}^n$). Any orthogonal matrix A of degree n sends any such orthonormal frame x to another, namely $Ax = (Ae_1, ..., Ae_k)$, this defines an action of $O(n)$ on $\mathbf{V}_{n,k}$ which is transitive (see [9]).

Each Stiefel manifold $\mathbf{V}_{n,k}$ can be realized as a non-singular surface in the Euclidean space $\mathbf{R}^{nk}$ in the following way. Fix on an orthonormal basis for $\mathbf{R}^n$ (e.g. the standard basis), and introduce the following notation for the components with respect to this basis of any k-frame $(e_1,...,e_k)$ (i.e. point of $\mathbf{V}_{n,k}$):

$$e_i = (x_{i1}, ..., x_{in}) \quad , \quad i = 1, ..., k.$$

The nk quantities x_{ij} , $i = 1, ..., k$; $j = 1, ..., n$, (in lexicographic order, say) are now to be regarded as the co-ordinates of a point in nk- dimensional Euclidean space $\mathbf{R}^{nk}$, related by the following $k(k+1)/2$ equations:

$$\langle e_i, e_j\rangle = \delta_{ij} \Longleftrightarrow \sum_{s=1}^{n} x_{is}x_{js} = \delta_{ij} \quad , \quad i, j = 1, ..., k,\ i \leq j$$

We now investigate the isotropy group of this homogenous space. Take any orthonormal k-frame e_1,...,e_k and enlarge it to an orthonormal basis e_1,...,e_n for the whole of Euclidean n-space. Any orthogonal transformation fixing the vectors e_1,...,e_k must (relative to the above basis for $\mathbf{R}^n$) have the form

$$\begin{bmatrix} 1 & & 0 & \\ & \ddots & & 0 \\ & & 1 & \\ & 0 & & A \end{bmatrix} \quad A \in O(n-k)$$

whence the isotropy group is isomorphic to $O(n, k)$, and $\mathbf{V}_{n,k}$ can be identified with $O(n)/O(n-k)$. In fact $\mathbf{V}_{n,k} \cong O(n)/O(n-k)$.

The Stiefel manifolds $\mathbf{V}_{n,k}$ for $k < n$ are also homogeneous spaces for the group $SO(n)$ (see [9]). From this point of view the isotropy group is clearly (isomorphic to) $SO(n-k)$, and therefore also

$$\mathbf{V}_{n,k} \cong SO(n)/SO(n-k).$$

In particular, we have

$$\mathbf{V}_{n,n} \cong O(n) \quad , \quad \mathbf{V}_{n,n-1} \cong SO(n) \quad , \quad \mathbf{V}_{n,1} \cong S^{n-1}.$$

In [11] Jimenez shown that; i) $\mathbf{V}_{n,k}$ belongs to the class of reduced Riemannian $\sum$-spaces, ii) if $n = 2k > 30$ and $k = n-2$, $\mathbf{V}_{n,k}$ belongs to the class of all pointwise Riemannian $\mathbf{k}$ -symmetric spaces ($\mathbf{k} \geq 2$) and furthermore, it is not diffeomorphic to the underlying manifold of any symmetric in the class of all regular Riemannian $\mathbf{k}$-symmetric spaces ($\mathbf{k} \geq 2$), and iii) if n and k are appropriately chosen $\mathbf{V}_{n,k}$ is not diffeomorphic to the underlying manifold of a space in class of all pointwise Riemannian r-symmetric spaces.

3 The First Standard Imbedding Of The Real Projective Space $\mathbf{P}^m$

In this section we will consider the Stiefel submanifold as the first standard imbedding of the real projective space $\mathbf{P}^m$ $(m = nk)$ in $\mathbf{R}^{m+\frac{1}{2}m(m+1)}$. The notation here is essentially the same as in [7].

Let $M(m+1;\mathbf{R})$ be the space of $(m+1)\times(m+1)$ matrices over $\mathbf{R}$. It is considered as an $(m+1)^2$-dimensional Euclidean space with the inner product $\langle A, B\rangle = \dfrac{1}{2} trace AB^T$, where B^T is the transpose of the matrix B.

Someone consider $\mathbf{P}^m$ as the quotient space of the hypersphere

$$\mathbf{S}^m = \{\zeta \in \mathbf{R}^{n+1} : \zeta^T\zeta = I_k\};\ m = nk,$$

obtained by identifying ζ with $\zeta\lambda$, where ζ is a column vector and $\lambda \in \mathbf{R}$ such that $|\lambda| = 1$.

Define a mapping $\widetilde{\varphi} : \mathbf{S}^m \to H(m+1;\mathbf{R}) = \{A \in M(m+1;\mathbf{R}) : A^T = A\}$ as follows

$$\widetilde{\varphi}(\zeta) = \zeta\zeta^T = \begin{pmatrix} |\zeta_0|^2 & \zeta_0\zeta_1 & - & \zeta_0\zeta_n \\ - & - & - & - \\ \zeta_n\zeta_0 & \zeta_n\zeta_1 & - & |\zeta_n|^2 \end{pmatrix}$$

for $\zeta = (\zeta_i) \in \mathbf{S}^m \subset \mathbf{R}^{m+1}$, $0 \leq i \leq m$. Then it is easy to verify that $\widetilde{\varphi}$ induces a mapping φ of $\mathbf{P}^m$ into $H(m+1;\mathbf{R})$:

$$\varphi((\pi(\zeta)) = \widetilde{\varphi}(\zeta) = \zeta\zeta^T. \tag{3}$$

where $\pi : \mathbf{S}^m \to \mathbf{P}^m$ is a Riemannian submersion [7]. We simply denote $\varphi((\pi(\zeta))$ by $\varphi(\zeta)$.

From (3) the image of $\mathbf{P}^m$ under φ is given by

$$\varphi(\mathbf{P}^m) = \{A \in H(m+1;\mathbf{R}) : A^2 = A \text{ and trace } A = 1\},\ m = nk,.$$

Let $A = \zeta\zeta^T$ be a point in $\varphi(\mathbf{P}^m)$. Consider the curve

$$A(t) = \zeta\zeta^T;\ \zeta \in \mathbf{R}^{m+1} \tag{4}$$

in $\varphi(\mathbf{P}^m)$ with $A(0) = A$ and $A^{'}(0) = X \in T_A(\mathbf{P}^m)$. From $A^2(t) = A(t)$ one gets $XA^T + AX^T = X$. So we have

$$T_A(\mathbf{P}^m) = \{X \in H(m+1;\mathbf{R}) : XA^T + AX^T = X\}.$$

Proposition 3.1 [**7**]. *Let Y be a vector field tangent to $\boldsymbol{P}^m$ and $X \in T_A(\boldsymbol{P}^n)$. Consider a curve $A(t)$ in $\varphi(\boldsymbol{P}^m)$ so that $A(0) = A$ and $A'(0) = X$. Denote by $Y(t)$ the restriction of Y to $A(t)$. Then*

$$A(t)Y(t) + Y(t)A(t) = Y(t).$$

Corollary 3.2. *Let $A(t) = \zeta(t)\zeta(t)^T$; $\zeta(t) \in \boldsymbol{R}^{m+1}$ be a curve in $\varphi(\boldsymbol{P}^m)$ then*

$$A(0) = X = \xi a^T + a\xi^T \tag{5}$$

where

$$\zeta(0) = \xi, \zeta^{'}(0) = a.$$

Proof. Let $A(t) = \zeta\zeta^T$ be a curve in $\varphi(\mathbf{P}^m)$ with $A(0) = A$ and $A'(0) = X \in T_A(\mathbf{P}^m)$. Differentiating (5) we get

$$A'(t) = \zeta'\zeta^T + \zeta\zeta'^T.$$

At point $t = 0$ since $\zeta(0) = \xi, \zeta^{'}(0) = a$ we get the result.

A vector ν in $H(m+1;\mathbf{R})$ is normal to $\mathbf{P}^m$ at A if and only if $< X, \nu >= 0$ for all X in $T_A(\mathbf{P}^m)$. Thus, ν is in $T_A^{\perp}(\mathbf{P}^m)$ if and only if $trace(X\nu) = 0$ for all X in $T_A(\mathbf{P}^m)$. Therefore by (5) we obtain

$$T_A^{\perp}(\mathbf{P}^m) = \{\nu \in H(m+1;\mathbf{R}) : A\nu = \nu A\}.$$

Theorem 3.3 [**7**]. *The isometric imbedding $\varphi : \boldsymbol{S}^m \to H(m+1;\boldsymbol{R})$, determined by (3), is the first standard imbedding of $\boldsymbol{P}^n$ into $H(n+1,\boldsymbol{R})$ and $\boldsymbol{P}^m$ lies in a hypersphere $\boldsymbol{S}(r)$ of $H(m+1;\boldsymbol{R})$ centered at $\frac{1}{m+1}I$ and with radius $r = \sqrt{\frac{m}{2}(n+1)}$.*

Definition 3.1. A real projective m-space $\mathbf{P}^m$ $(m = nk)$ isometrically imbedded in $\mathbf{R}^{m+\frac{1}{2}m(m+1)}$ by its first standard imbedding φ determined by (3) is called the Stiefel submanifold $\mathbf{V}_{n,k}$.

4 Normal Sections Of $\mathbf{V}_{n,k}$

An element $A \in \mathbf{V}_{n,k}$ can be written as $A = \zeta\zeta^T$ where $\zeta^T\zeta = I_{k\times k}$ and $\zeta = \xi m + \eta$, $\xi^T\eta = 0$, $m \in M_{k\times k}(\mathbf{R})$. Since ζ is not unique we can replace ζ by ζu where $u \in O(k)$. Put $m = \xi^T\zeta$ then we can find $u \in O(k)$ so that mu is lower triangular and the diagonal element of mu are ≥ 0 by using the Gram-Schmidt orthonormalization process. Further mu is uniquely defined by m and $\xi^T(\zeta - \xi m) = \xi^T\zeta - m = 0$. Thus we can write $A = \zeta\zeta^T$ where $\zeta = \xi m + \eta$ where m is lower triangular $k \times k$ with diagonal term ≥ 0 and $\xi^T\eta = 0$, $m \in M_{k\times k}(\mathbf{R})$ where m and η are unique. So $A = \xi mm^T\xi^T + \xi m\eta^T + \eta m^T\xi^T + \eta\eta^T$. Interchanging A with the curve $A(t)$ in $\varphi(\mathbf{P}^m)$ we get

$$A(t) = \zeta(t)\zeta(t)^T = \xi m(t)m(t)^T\xi^T + \xi m(t)\eta(t)^T + \eta(t)m(t)^T\xi^T + \eta(t)\eta(t)^T. \tag{6}$$

Lemma 4.1. *Let $A(t)$ be a normal section of $\boldsymbol{V}_{n,k}$ at point $A(0)$ in the direction of $A^{'}(0) = X$. Then*

$$\lambda(t)X = \xi m(t)\eta(t)^T + \eta(t)m(t)^T\xi^T \tag{7}$$

$$N(t) = \xi m(t)m(t)^T\xi^T + \eta(t)\eta(t)^T \tag{8}$$

where $a\xi^T + \xi a^T = X$, $\xi^T\xi = I_k$ and $a^T\xi + \xi^T a = 0$, $a = \eta^{'}(0) \in M_{n\times k}$.
Proof. Since $\xi^T\eta(t) = 0$ then $m(t)\eta(t)^T\xi + \xi^T\eta(t)m(t)^T = 0$ therefore $\xi m(t)\eta(t)^T + \eta(t)m(t)^T\xi^T \in T_x(\mathbf{V}_{n,k})$. Also $\eta(t)\eta(t)^T \in N_x(\mathbf{V}_{n,k})$ since $trace(a^T\eta(t)\eta(t)^T\xi) = 0$ (whatever $a \in M_{n\times k}$). Also

$$\begin{aligned} a^T(\xi m(t)m(t)^T\xi^T)\xi &= trace(a^T\xi m(t)m(t)^T) = trace(m(t)m(t)^T\xi^T a) \\ &= \frac{1}{2}trace(a^T\xi + \xi^T a)m(t)m(t)^T = 0 \end{aligned}$$

if $a^T\xi + \xi^T a = 0$. Therefore $\xi m(t)m(t)^T\xi^T \in N_x(\mathbf{V}_{n,k})$. Hence comparing (1) with (6) we get the result.
Lemma 4.2. *Let $A(t) = \zeta(t)\zeta(t)^T$ be a normal section of $\boldsymbol{V}_{n,k}$ at point $A(0)$ in the direction of $A^{'}(0) = X$. Then*

$$m(t)^Tm(t) + \eta(t)^T\eta(t) = I_k. \tag{9}$$

Proof. Since $\zeta(t) = \xi m(t) + \eta(t)$,

$$\zeta(t)^T\zeta(t) = I_k, \tag{10}$$

and

$$\xi^T\eta(t) = 0, \eta \in M_{n\times k} \tag{11}$$

we get the result.
Theorem 4.3. *Stiefel submanifold $\boldsymbol{V}_{n,k}$ (considered as a projective m-space $\boldsymbol{P}^m$ ($n = mk$) isometrically imbedded in $\boldsymbol{R}^{m+\frac{1}{2}m(m+1)}$ by its first standard imbedding defined by (3)) has $P2-PNS$ property.*
Proof. The normal section $A(t)$ of $\mathbf{V}_{n,k}$ at point $A(0) = \xi\xi^T$ in the direction of $A^{'}(0) = X$ is given by $A(t) = \zeta(t)\zeta^T(t)$, where $\zeta(t) = \xi m(t) + \eta(t), \eta, m \in M_{n\times k}$,

$$m(t) = \xi^T\zeta(t) \tag{12}$$

and

$$\eta(t)m(t)^T = \lambda(t)a. \tag{13}$$

At point $t = 0$ we get

$$\left.\begin{aligned} \zeta(0) &= \xi \\ m(0) &= I_k \\ \eta(0) &= 0 \\ A(0) &= \xi\xi^T \end{aligned}\right\} \tag{14}$$

at least for sufficiently small $A(t)$. Thus we now differentiate (9)-(13) and so determine m', η', λ', m'', η'', λ''. Then using $N(t) = \xi m(t)m(t)^T\xi^T + \eta(t)\eta(t)^T$ we compute N', N'', etc.

Differentiating (9), (12) and (11) (surprising the dependence of ζ on t to simplify the notation) we get

$$A^{'}(t) = \zeta^{'}(t)\zeta(t)^T + \zeta(t)\zeta^{'}(t)^T = \lambda^{'}(t)X + N^{'}(t), \tag{15}$$

$$\zeta^{'}(t)^T\zeta(t) + \zeta(t)^T\zeta^{'}(t) = 0, \tag{16}$$

$$m^{'}(t) = \xi^T\zeta^{'}(t) \tag{17}$$

Putting $t = 0$ we get

$$A^{'}(0) = X = \xi a^T + a\xi^T \tag{18}$$

$$\zeta(0) = \xi, \zeta^{'}(t) = \eta'(0) = a, \tag{19}$$

$$m^{'}(0) = \xi^T a = 0. \tag{20}$$

Differentiating (13) two times with respect to t we also get

$$\eta''(t)m(t)^T + 2\eta'(t)m'(t)^T + \eta(t)m''(t)^T = \lambda''(t)a.$$

At $t = 0$ by the use of (14) and (20) the above equation gives

$$\eta''(0) = 0. \tag{21}$$

Differentiating (9) and (8) we get

$$\begin{aligned} 0 &= m''(t)^Tm(t) + 2m'(t)^Tm'(t) + m(t)m''(t)^T + \\ &+ \eta''(t)^T\eta(t) + 2\eta'(t)^T\eta'(t) + \eta(t)\eta''(t)^T \end{aligned} \tag{22}$$

and

$$\begin{aligned} N''(t) &= \xi m''(t)m(t)^T\xi^T + 2\xi m'(t)m'(t)^T\xi^T + \xi m(t)m''(t)^T\xi^T \\ &+ \eta''(t)\eta(t)^T + 2\eta'(t)\eta'(t)^T + \eta(t)\eta''(t)^T. \end{aligned} \tag{23}$$

At $t = 0$ by (14), (20) and (21) we have

$$m''(0)^T + m''(0) = -2a^Ta$$

and

$$N''(0) = h(X, X) = -2\xi a^Ta\xi^T + 2aa^T. \tag{24}$$

Differentiating (22) and (23) with respect to t we get

$$
\begin{aligned}
0 &= m'''(t)^T m(t) + 3m''(t)^T m'(t) + 3m'(t)^T m''(t) + m^T(t)m'''(t) + \\
&+ \eta'''(t)^T \eta(t) + 3\eta''(t)^T \eta'(t) + 3\eta'(t)^T \eta''(t) + \eta^T(t)\eta'''(t)
\end{aligned} \tag{25}
$$

and

$$
\begin{aligned}
N'''(t) &= \xi m'''(t) m(t)^T \xi^T + 3\xi m''(t) m'(t)^T \xi^T + 3\xi m'(t) m''(t)^T \xi^T \\
&+ \xi m(t) m'''(t)^T \xi^T + \eta'''(t)\eta(t)^T + 3\eta''(t)\eta'(t)^T \\
&+ 3\eta'(t)\eta''(t)^T + \eta(t)\eta'''(t)^T.
\end{aligned} \tag{26}
$$

At $t = 0$, substituting (14), (20) and (21) into (25) and (26) one can get respectively

$$m'''(0)^T + m'''(0) = 0,$$

and

$$N'''(0) = (\overline{\nabla}_X h)(X, X) = \xi m'''(0)\xi^T + \xi m'''(0)^T \xi^T = 0.$$

This means that $\mathbf{V}_{n,k}$ has parallel second fundamental form. Therefore by Theorem 1.1 M has $P2$-PNS property.

Remark. For $k = 1$ the Stiefel submanifold $\mathbf{V}_{n,1}$ becomes a Veronese submanifold. In [3] we have shown that $\mathbf{V}_{n,1}$ has $P2 - PNS$ property.

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