On Lagrange Epimorphisms and Lagrange Submersions

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Dedicated to Prof.Dr. Constantin UDRIŞTE on the occasion of his sixtieth birthday

Abstract

The aim of the paper is to define and study some aspects of Lagramge epimorphisms, which are generalizations for vector bundles of Lagrange submersions. We construct a Lagrangian on $f^*\xi''$ canonically associated with a Lagrange vector bundle ξ and an f_0-epimorphism of vector bundles $f : \xi \to \xi''$. The main result is that a Lagrangian L on ξ has a locally f-projectable metric on ξ'' iff it is a local Lagrange submersion. All the definitions and results in the paper can be stated in particular for Lagrange submersions.

Mathematics Subject Classification: 58F05
Key words: Lagrangian, Legendre transform epimorphism, submersion

A *Lagrangian* on the vector bundle $\xi = (E, \pi, M)$ is a real and differentiable function $L : E \to I\!R$. A *regular Lagrangian* on ξ is a Lagrangian L which has $Hess(L)$ non-degenerate.. All the Lagrangians considered in the sequel are regular. Notice that every Lagrangian on ξ define a (pseudo)metric on the fibres of the vertical bundle $V\xi$. In local coordinates, if (x^i) and (x^i, y^a) are coordinates on M and E respectively, then $Hess(L)$ is a bilinear form on the vertical bundle $V\xi$, which has the matrix:

$$g_{bc}(x^i, y^a) = \frac{\partial^2 L}{\partial y^b \partial y^c}.$$

The Legendre transform associated with the Lagrangian L on ξ is:

$$\mathcal{L} : E \to E^*,\ \mathcal{L}(x^i, y^a) = \left(x^i, \frac{\partial L}{\partial y^a}(x^i, y^a) \right).$$

If L is regular, then $\mathcal{L}$ is a local diffeomorphism. Since most of our constructions are local, we can suppose, without loss of generality, that $\mathcal{L}$ is a global diffeomorphism.

In the sequel $\xi = (E, \pi, M)$ and $\xi'' = (E'', \pi'', M'')$ are two vector bundles, $f_0 : M \to M''$ is a submersion and $f : \xi \to \xi''$ is an f_0-epimorphism (i.e. f is surjective on fibers). The restriction of the differential f_* to the vertical bundle $V\xi \subset \tau E$ defines an f-(epi)morphism of vector bundle

Balkan Journal of Geometry and Its Applications, Vol.5, No.2, 2000, pp. 97-101

$$F : V\xi \cong \pi^{*}\xi \rightarrow V\xi'' \cong (\pi'')^{*}\,\xi'' \tag{1}$$

If (ξ, L) is a Lagrange space, the *vertical distribution* and *horizontal distribution* of f are $Vf = \ker F$ and $Hf = (\ker F)^{\perp}$ respectively, where the orthogonal is taken according the Lagrange metric, assuming that it exists. Notice that there is a canonical Whitney sum decomposition $V\xi = Vf \oplus Hf$.

We say that the Lagrangian L on ξ has an f- *projectable* metric if there exists a metric on $V\xi''$such that the f -epimorphism $F : V\xi \rightarrow V\xi''$ is Riemannian, i.e. the fibers of Hf are isometric with the fibers of $V\xi''$.

We say that $f : \xi \rightarrow \xi'$ is a *Lagrange epimorphism* if there is a Lagrangian on ξ'' such that the induced f-epimorphism $F : V\xi| \rightarrow V\xi''$ is Riemannian on fibres.

We use in the sequel local coordinates adapted to the submersion f_0 and the epimorphism f: $(x^i) = (x^u, x^{\bar u})$ on M, $(x^{\bar u})$ on M'', $(x^u, x^{\bar u}, y^a, y^{\bar a})$ on E and $(x^{\bar u}, y^{\bar a})$ on E'', such that f_0 and f has the local forms $(x^u, x^{\bar u}) \rightarrow (x^{\bar u})$ and $(x^u, x^{\bar u}, y^a, y^{\bar a}) \rightarrow (x^{\bar u}, y^{\bar a})$ respectively. We denote as $\{s_u, s_{\bar u}\}$ and $\{s''_{\bar u}\}$ the local bases of sections in the vector bundles $V\xi$ and $V\xi''$ respectively. If g is a (pseudo)metric tensor on $V\xi$, then we denote as $\{g_{ij}\} = \{g_{uv},\, g_{\bar u v} = g_{v\bar u},\, g_{\bar u\bar v}\}$ its components using the above base. Notice that $\{s_u\}$ is a local base of sections in Vf and $\{\bar s_{\bar u} = s_{\bar u} - \tilde g^{uv} g_{v\bar u} s_u\}$ is a local base of sections in Hf, where $(\tilde g^{uv}) = (g_{uv})^{-1}$ as matrices. It is easy to see that we have $g(\bar s_{\bar u}, \bar s_{\bar v}) = g_{\bar u\bar v} - g_{\bar u u}\tilde g^{uv} g_{v\bar v}$.

The annihilator of the vector subbundle $Vf \subset \xi$ is a vector subbundle $Vf^{*} \subset \xi^{*}$, defined by the linear forms in ξ^{*} which are null on vectors in $V\xi$. If $(x^u,\, x^{\bar u},\, p_a,\, p_{\bar a})$ and $(x^{\bar u},\, p_{\bar a})$ are local coordinates on E^{*} and $\acute{E}''^{*}$ respectively, then the local coordinates on Vf^{*} are $(x^u,\, x^{\bar u},\, p_{\bar a})$ and the inclusion $Vf^{*} \subset \xi^{*}$ has the local form $(x^u,\, x^{\bar u},\, p_{\bar a}) \rightarrow (x^u,\, x^{\bar u},\, p_{\bar a},\, 0)$. Since $\mathcal{L}$ is a diffeomorphism, then $WE = \mathcal{L}^{-1}(Vf^{*}) \subset E$ is a submanifold. Notice the manifold WE has as coordinates $(x^u,\, x^{\bar u},\, Q^u(x^u,\, x^{\bar u},\, y^{\bar a}),\, y^{\bar a})$, such that

$$\frac{\partial L}{\partial y^a}(x^u, x^{\bar u}, Q^a(x^u, x^{\bar u}, y^{\bar a}), y^{\bar a}) = 0. \tag{2}$$

Let us observe that $f_{|WE} : WE \rightarrow f_0^{*}E''$ is a diffeomorphism, where $f_0^{*}E''$ is the total space of the induced bundle:

$$\begin{array}{ccc} M & \stackrel{f_0}{\rightarrow} & M'' \\ & & \uparrow \pi'' \\ & & E'' \end{array}$$

Indeed, the local form of $f_{|WE}$ is

$$(x^u, x^{\bar u}, Q^a(x^u, x^{\bar u}, y^{\bar a}), y^{\bar a}) \rightarrow (x^u, x^{\bar u}, y^{\bar a}). \tag{3}$$

Let us define

$$S = f_{|WE}^{-1} : f_0^{*}E'' \rightarrow WE$$

and

$$\bar L'' = L \circ S : \pi^{*}E'' \rightarrow I\!R.$$

In local coordinates

$$\bar L''(x^u, x^{\bar u}, y^{\bar a}) = L(x^u, x^{\bar u}, Q^u(x^u, x^{\bar u}, y^{\bar a}), y^{\bar a}).$$

Notice that

The map S is a section of the fibered manifold defined by the epimorphism $E \overset{f_0^* f}{\to} f_0^* E''$ of vector bundles over M.

The couple $(f_0^*\xi'', \bar{L}'')$ is a Lagrange vector bundle and we call $\bar{L}''$ the *canonical Lagrangian* on $f_0^*\xi''$ (induced by L and f).

Consider again the f-epimorphism (1), denote as $I : WE \to E$ the inclusion and consider the vector bundle $I^* Hf$ (i.e. the fibres of Hf along the submanifold WE).

Lemma 1 *The restriction $F_{|WE} : I^*Hf \to f_0^*\xi''$ is an $f_{|WE}$-isomorphism which is an isometry on fibres according to the Lagrangians $L_{|I^*Hf}$ and $\bar{L}''$ respectively.*

Proof. We use local coordinates. The local form of $f_{|WE}$ is given by (3). The local correspondence of bases of sections by mean of $f_{|WE}$ are $\bar{s}_{\bar{a}} = s_{\bar{a}} - \tilde{g}^{ab} g_{b\bar{a}} s_a \to s''_{\bar{a}}$. We have

$$g(\bar{s}_{\bar{a}}, \bar{s}_{\bar{b}}) = g_{\bar{a}\bar{b}} - g_{\bar{a}a}\tilde{g}^{ab} g_{b\bar{b}} \tag{4}$$

and

$$\bar{g}''(s''_{\bar{a}}, s''_{\bar{b}}) = \frac{\partial^2 \bar{L}''}{\partial y^{\bar{a}} \partial y^{\bar{b}}},$$

where

$$\frac{\partial^2 \bar{L}''}{\partial y^{\bar{a}} \partial y^{\bar{b}}}(x^u, x^{\bar{u}}, y^{\bar{a}}) = \frac{\partial^2}{\partial y^{\bar{a}} \partial y^{\bar{b}}} L(x^u, x^{\bar{u}}, Q^a(x^u, x^{\bar{u}}, y^{\bar{a}}), y^{\bar{a}}).$$

But

$$\frac{\partial}{\partial y^{\bar{b}}} L(x^u, x^{\bar{u}}, Q^a(x^u, x^{\bar{u}}, y^{\bar{b}}), y^{\bar{b}}) = \frac{\partial L}{\partial y^{\bar{b}}}(x^u, x^{\bar{u}}, Q^b(x^u, x^{\bar{u}}, y^{\bar{b}}), y^{\bar{v}})$$

thus

$$\frac{\partial^2}{\partial y^{\bar{a}} \partial y^{\bar{b}}} L(x^u, x^{\bar{u}}, Q^a(x^u, x^{\bar{u}}, y^{\bar{a}}), y^{\bar{a}}) = \frac{\partial^2 L}{\partial y^{\bar{a}} \partial y^{\bar{b}}}(x^u, x^{\bar{u}}, Q^v(x^u, x^{\bar{u}}, y^{\bar{v}}), y^{\bar{v}}) +$$

$$+ \frac{\partial Q^c}{\partial y^{\bar{a}}}(x^u, x^{\bar{u}}, y^{\bar{a}}) \frac{\partial^2 L}{\partial y^{\bar{u}} \partial y^c}(x^u, x^{\bar{u}}, Q^a(x^u, x^{\bar{u}}, y^{\bar{a}}), y^{\bar{a}}) = g_{\bar{a}\bar{b}} + \frac{\partial Q^c}{\partial y^{\bar{a}}} g_{\bar{b}c}.$$

Differentiating partially the relation (2) with respect to $y^{\bar{a}}$ we obtain that

$$\frac{\partial Q^a}{\partial y^{\bar{a}}} = -\tilde{g}^{ab} g_{b\bar{a}},$$

where

$$(\tilde{g}^{ab}) = (g_{ab})^{-1}.$$

Thus

$$\bar{g}''(s''_{\bar{a}}, s''_{\bar{b}}) = g_{\bar{a}\bar{b}} - g_{\bar{a}a}\tilde{g}^{ab} g_{b\bar{b}}.$$

Comparing with relations (4), the conclusion follows. □

Notice that using a linear algebra computation we obtain the following equality of matrices:

$$\left(g_{\bar{a}\bar{b}} - g_{\bar{a}a}\tilde{g}^{ab} g_{b\bar{b}}\right) = \left(g^{\bar{a}\bar{b}}\right)^{-1}.$$

Using the Lemma 1 we obtain:

Proposition 1 *If f is a Lagrange epimorphism then $\xi \stackrel{f_0^* f}{\rightarrow} f_0^*\xi''$ is a Lagrange epimorphism, where the canonical Lagrangian $\bar{L}''$ is considered on $f_0^*\xi''$.*

Using also Lemma 1 we obtain:

Proposition 2 *Let $f : \xi \rightarrow \xi''$ be an epimorphism of vector bundles over the same base and L be a Lagrangian on ξ which is non-degenerate on Vf and has an f-projectable metric.*

Then f is a Lagrange submersion.

We say that a Lagrangian $\bar{L}''$ on $f_0^*\xi''$ *projects* on ξ'' if there is a Lagrangian L'' on ξ'' such that $\bar{L}'' = L'' \circ h$, where $h : f_0^*\xi'' \rightarrow \xi''$ is the canonical f_0-morphism which is the identity on fibers.

We are going to give sufficient conditions in order to induce a Lagrange submersion.

Proposition 3 *Let $f : \xi \rightarrow \xi''$ be an f_0-epimorphism and L a Lagrangian on ξ which is non-degenerate on Vf. Consider on $f_0^*\xi''$ the canonical Lagrangian.*

If $\xi \stackrel{f_0^* f}{\rightarrow} f_0^*\xi''$ is a Lagrange epimorphism and the canonical Lagrangian on $f_0^*\xi''$ is projectable on ξ'' then there is a Lagrangian L'' on ξ'' such that f is a Lagrange submersion.

Proof. Since $h \circ f_0^* f = f$, the conclusion follows using the composition of the Lagrange epimorphisms

$$\xi \stackrel{f_0^* f}{\rightarrow} f_0^*\xi'' \stackrel{h}{\rightarrow} \xi'',$$

which is a Lagrange epimorphism. □

Lemma 2 *Let $f_0 : M \rightarrow M''$ be a submersion, $\xi'' = (E'', \pi'', M'')$ be a vector bundle and $g : f_0^*\xi'' \rightarrow \xi''$ be the canonical f_0-morphism of vector bundles.*

Assuming that a Lagrangian $\bar{L}''$ on $f_0^*\xi''$ has a g-projectable metric on ξ'', then $\bar{L}''$ projects locally on ξ''.

Proof. Consider local coordinates $(x^u,\, x^{\bar{u}})$ on M, $(x^{\bar{u}})$ on M'' and $(x^{\bar{u}},\, y^{\bar{a}})$ on ξ'', thus g has the local form $(x^u,\, x^{\bar{u}}\ ,\, y^{\bar{a}}) \rightarrow (x^{\bar{u}}, y^{\bar{a}})$. The condition that $\bar{L}''$ has a g-projectable metric on ξ'' reads that the local functions $\dfrac{\partial^2 \bar{L}''}{\partial y^{\bar{a}} \partial y^{\bar{b}}}$ do not depend on x^u. It follows that there are local and real functions $L'' : \xi''_{U''} \rightarrow I\!R$, where $U'' \subset M''$ and $\xi''_{U''}$ is the restriction of the vector bundle ξ'' to U'', such that

$$\frac{\partial^2 \bar{L}''}{\partial y^{\bar{a}} \partial y^{\bar{b}}}(x^u, x^{\bar{u}}, y^{\bar{a}}) = \frac{\partial^2 L''}{\partial y^{\bar{a}} \partial y^{\bar{b}}}(x^{\bar{u}}, y^{\bar{a}}).$$

□

Using Proposition 3 and Lemma 2 we obtain the main result:

Theorem 1 *Let $f : \xi \rightarrow \xi''$ be an f_0-epimorphism and L a Lagrangian on ξ which is non-degenerate on Vf.*

Then L has a locally f-projectable metric on ξ'' iff it is a local Lagrange submersion.

Let $L : TM \to I\!R$ be a regular Lagrangian on M and $f : M \to M''$ be a surjective submersion. Notice that all the definitions and the results stated above for vector bundles apply in this case, for example:

f is a *Lagrange submersion* if f_* is a Lagrange submersion;

A Lagrangian L on M is f-projectable if it is f_*-projectable.

The case when f is *flat* (i.e. the horizontal distribution Hf_* is integrable) is need to Lagrange foliations (see [4]).

A theory of Lagrange submersion, in an analogous way of Riemannian submersion, will be done elsewhere.

Aknowledgements. A version of this paper was presented at the Third Conference of Balkan Society of Geometers, Workshop on Electromagentic Flows and Dynamics, July 31 - August 3, 2000, University POLITEHNICA of Bucharest, Romania

Supported by MEN Grant No 21815/28.09.1998, CNCSU-31.

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