Monotone Point-to-Set Vector Fields

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Dedicated to Prof.Dr. Constantin UDRIŞTE on the occasion of his sixtieth birthday

Abstract

We introduce the concept of monotone point-to-set field in Riemannian manifold and give a characterization, that make clear in this definition the occult geometric meaning. We will show that the sub-differential operator of a Riemannian convex function is a monotone point-to-set field. The concept of directional derivative, which appears already in other publications, plays an important role in the proof of the result above. We study some of its properties, in particular, we obtain the chain rule, which is fundamental in our work. Some topological consequences of the existence of strictly monotone point-to-set fields are presented.

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1 Introduction

A large class of non-convex constrained minimization problems can be seen as convex minimization problems in Riemannian manifolds. The study of the extension of known optimization methods to solve minimization problems over Riemannian manifolds was the subject of various works-see [2], [8], [13] and their references.

A generalization of convex minimization problem is the variational inequality problem. In the study of variational inequality problems and convergence properties of iterative methods to solve them, several classes of monotone operator were introduced.

The concepts of monotonicity and strict monotonicity of fields defined on a Riemannian manifold were introduced in [6]. The concept of strong monotonicity of such fields was introduced in [3]. We introduce the concept of point-to-set monotone vector field and will show that the subdifferential operator of a Riemannian convex function is a monotone point-to-set field.

In Section 3, we study some properties of the directional derivative, in particular, we obtain the chain rule. The concept of Riemannian directional derivative was introduced by C.Udriste in [12].

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In section 4, it is defined the concept of monotone point-to-set vector field, and it is given a characterization of these vector fields. It will show that the subdifferential operator of a convex function is monotone.

In Section 5, we study some topological consequences of the existence of strictly monotone fields. If there exists a strictly monotone field, then there is no closed geodesic in the manifold. If, moreover, the manifold is non compact and has nonnegative sectional curvature, then its soul has dimension 0 and therefore the manifold is diffeomorphic to R^n.

2 Basics concepts

In this section are announced some frequent used notations, basic definitions and important properties of Riemannian manifolds . They can be found in any introductory book on Riemannian Geometry, for example [1] and [9]. Throughout this paper, all manifolds are smooth and connected and all functions and vector fields are smooth.

Given a manifold M, denote by T_pM the tangent space of M at p. Let M be endowed with a Riemannian metric $\langle , \rangle$, with corresponding norm denoted by $\| \ \|$, so that M is now a *Riemannian* manifold. Recall that the metric can be used to define the length of piecewise C^1 curve $c : [a, b] \to M$ joining p to q, p, $q \in M$, i.e., such that $c(a) = p$ and $c(b) = q$, by $l(c) = \int_a^b \|c'(t)\|dt$. Minimizing this length functional over the set of all such curves we obtain a distance $d(p, q)$ which induces the original topology on M. Let ∇ be the Levi-Civita connection associated to $(M, \langle , \rangle)$. If c is a curve joining points p and q in M , then, for each $t \in [a, b]$, ∇ induces an isometry, relative to $\langle , \rangle$, $P(c)_t^a : T_{c(a)}M \to T_{c(t)}M$, the so-called *parallel transport* along c from $c(a)$ to $c(t)$. The inverse map of $P(c)_t^a$ is denoted by $P(c^{-1})_t^a := T_{c(t)}M \to T_{c(a)}M$. A vector field V along c is said to be *parallel* if $\nabla_{c'}V = 0$. If c' itself is parallel we say that c is a *geodesic*. The geodesic equation $\nabla_{\gamma'}\gamma' = 0$ is a second order nonlinear ordinary differential equation, and consequently γ is determined by a point and the velocity at this point. It is easy to check that $\|\gamma'\|$ is constant. We say that γ is *normalized* if $\|\gamma'\| = 1$. The restriction of a geodesic to a closed bounded interval is called a *geodesic segment*. A geodesic segment joining p to q in M is said to be *minimal* if its length equals $d(p, q)$.

A Riemannian manifold is *complete* if geodesics are defined for any values of t. Hopf-Rinow's theorem asserts that if this is the case, then any pair of points, say p and q, in M can be joined by a (not necessarily unique) minimal geodesic segment. Moreover, (M, d) is a complete metric space and bounded and closed subsets are compact. In this paper, all manifolds are assumed to be complete. The *exponential map* $exp_p : T_pM \to M$ is defined by $exp_pv = \gamma_v(1, p)$, where $\gamma(\cdot) = \gamma_v(\cdot, p)$ is the geodesic defined by its position p and velocity v at p. We can prove that, $exp_ptv = \gamma_v(t, p)$ for any values of t.

Denote by K the sectional curvature of M. Some interesting results are obtained when the sign of curvature is constant. If $K \leq 0$, then the manifold is refereed as *manifold with nonpositive curvature*, in the other case, the manifold is refereed as *manifold with nonnegative curvature*. When the sectional curvature is nonnegative at each point of M, then the two next important results are valid. The following Theorem is due to J. Cheeger and D. Gromoll.

Theorem 2.1 *Let M be a complete noncompact Riemannian manifold of nonnegative curvature. Then M contains a compact totally geodesic submanifold S with $dimS < dimM$, which is totally convex. Furthermore, M is diffeomorphic to the normal bundle of S.*

Proof. See [9], Theorem 3.4, page 215. Beginning at any point of M a such S, called *soul of M*, can be built. G. Perelman [6] proved the following result.

Theorem 2.2 *Let M be a complete non compact Riemannian manifold of nonnegative sectional curvature. If there exists a point of M at which the sectional curvature is positive, then the soul S of M consists of one point, which is called a simple point, and M is diffeomorphic to R^n.*

Proof. See [7].

Let M be a Riemannian manifold. A function $f : M \to R$ is said to be *convex* (respectively, strictly convex) if the composition $f \circ \gamma : R \to R$ is convex (respectively, strictly convex) for any geodesic γ of M. This definition implies that all convex functions are continuous. A vector $s \in T_pM$ is said to be a *subgradient* of f at p if for any geodesic γ of M with $\gamma(0) = p$,

$$(f \circ \gamma)(t) \geq f(x) + t\langle s, \gamma'(0)\rangle$$

for any $t \geq 0$. The set of all subgradients of f at p, denoted by $\partial f(p)$, is called the *subdifferential* of f at p - see [12], [13].

3 Directional derivatives

C.Udriste introduced the concept of Riemannian directional derivatives in [12]. In this section we will study some of its properties. In particular, we will show that the directional derivative depends only of the direction and not of the curve. An important property, which we will show, is the chain rule. Another reference about directional derivative in Riemannian manifold is [13]. Several result, related with directional derivative in Riemannian manifold, are similar to results in R^n. We use [4] and [14] as references of convex analysis in R^n.

Let M be a complete Riemannian manifold and $f\colon M \to R$ a convex function. Take $p \in M$ and $v \in T_pM$ and let $c : (-\varepsilon, \varepsilon) \longrightarrow M$ be a C^1 curve such that $c(0) = p$ and $c'(0) = v$. Consider the quotient

$$q_c(t) = \frac{f(c(t)) - f(p)}{t}. \tag{1}$$

If $\gamma_v : R \to M$ is a geodesic such that $\gamma_v(0) = p$, then $f \circ \gamma : R \to R$ is a real convex function. Therefore $q_{\gamma_v} : R \to R$ is nondecreasing, and since that f is locally Lipschitzian, it follows that q_{γ_v} is bounded near zero. Then the following definition makes sense.

Definition 3.1 *(see [12]) Let M be a complete Riemannian manifold and let $f : M \to R$ be a convex function. Then the directional derivative of f at p in the direction of $v \in T_pM$ is defined by*

$$f'(p, v) = \lim_{t \to 0^+} q_{\gamma_v}(t) = \inf_{t>0} q_{\gamma_v}(t),$$

where $\gamma_v : R \to M$ is the geodesic such that $\gamma_v(0) = p$ and $\gamma_v'(0) = v$.

Next we show that the directional derivative of f at p in direction $v \in T_pM$, depends only of the direction and not of the curve, i.e., in the Definition 3.1 we can take any curve c, non necessary the geodesic, such that $c(0) = p$ and $c'(0) = v$, and still obtain that $\lim_{t \to 0^+} q_c(t) = f'(p, v)$. We need some auxiliary results.

We begin with some preliminaries. Take $p \in M$, let c_1 and c_2 two C^1 curves in M, such that $c_1(0) = c_2(0) = p$ and $c_1'(0) = v$, $c_2'(0) = w$. Consider $\alpha : [0,1] \times (-\varepsilon, \varepsilon) \to M$ a variation of geodesics given by

$$\alpha(t, s) = \exp_{c_1(s)}(t \exp^{-1}_{c_1(s)} c_2(s)), \tag{2}$$

where $\varepsilon > 0$ such that $B_\varepsilon(p)$ is a totally normal neighborhood.

Note that $\alpha(0, s) = c_1(s)$, $\alpha(1, s) = c_2(s)$ and that for each fixed $s \in (-\varepsilon, \varepsilon)$, the curve $\alpha_s : [0,1] \to M$ given by $\alpha_s(t) = \alpha(t, s)$ is geodesic. In particular, when $s = 0$, it is the constant geodesic $\alpha_0(t) = \alpha(t, 0) = p$. Now, consider the fields

$$T(\cdot, s) = \frac{\partial \alpha}{\partial t}(\cdot, s), \tag{3}$$

and

$$J(\cdot, s) = \frac{\partial \alpha}{\partial s}(\cdot, s). \tag{4}$$

The vector field $T(\cdot, s)$ is tangent to the geodesic α_s. The vector field $J(\cdot, s)$ is called *Jacobi vector field* through α_s and it satisfies the differential equation

$$\frac{D^2 J}{\partial t^2}(t, s) + R(J(t, s), T(t, s))T(t, s) = 0, \tag{5}$$

where R is the curvature tensor field.

Lemma 3.1 *Let c_1 and c_2 be C^1 curves in M, such that $c_1(0) = c_2(0) = p$, $c_1'(0) = v$ and $c_2'(0) = w$. If $T(\cdot, s)$ is defined by (3) and $J(\cdot, s)$ is defined by (4), then*

i) $J(t, 0) = v + t(w - v)$ *is the Jacobi vector field along the constant geodesic* $\alpha_0(t) = \alpha(t, 0) = p$.

Moreover, by symmetry,

ii) $\dfrac{DT}{\partial s}(t, 0) = \dfrac{DJ}{\partial t}(t, 0) = w - v.$

Proof. Consider α the variation of a geodesic defined by (2). Making $s = 0$ in (5) we have

$$\frac{D^2 J}{\partial t^2}(t,0) = 0,$$

because $\alpha_0(s) = p'$ and $T(t,0) = 0$. The boundary value problem

$$\frac{D^2}{dt^2}J(t,0) = 0, \quad J(0,0) = v, \quad J(1,0) = w,$$

implies that $J(t,0) = v + t(w - v)$ what proves (i). By Symmetry's Lemma - see [1] - it is valid that

$$\frac{DT}{\partial s}(t,0) = \frac{D}{\partial s}\frac{\partial}{\partial t}\alpha(t,0) = \frac{D}{\partial t}\frac{\partial}{\partial s}\alpha(t,0) = \frac{D}{\partial t}J(t,0) = w - v.$$

The proof of the Lemma is completed.

Lemma 3.2 *Let c_1 and c_2 be C^2 curves in M, such that $c_1(0) = c_2(0) = p$, $c_1'(0) = v$ and $c_2'(0) = w$. If $\psi(s) = d(c_1(s), c_2(s))$, then*

i) $\frac{d}{ds}(\psi^2(s))|_{s=0} = 0$;

ii) $\frac{d^2}{ds^2}(\psi^2(s))|_{s=0} = 2\|w - v\|^2$.

Furthermore, the Taylor's Formula for ψ^2 in some neighborhood of $s = 0$ is given by

$$\psi^2(s) = \|w - v\|^2 s^2 + \mathcal{O}(s^2), \tag{6}$$

where $\lim_{s\to 0^+} \frac{\mathcal{O}(s^2)}{s^2} = 0$.

For item (i) consider α the variation of a geodesic defined by (2). Then $\psi(s) = \|\alpha_s'(t)\|^2 = \|T(t,s)\|^2$. Since $T(t,0) = 0$, we have

$$\frac{d}{ds}(\psi^2(s))|_{s=0} = 2\langle\frac{DT}{\partial s}(t,0), T(t,0)\rangle 0.$$

For item (ii), observe that

$$\frac{d^2}{ds^2}(\psi^2(s))|_{s=0} = 2\Big(\langle\frac{D^2 J}{\partial s^2}(t,0), T(t,0)\rangle + \langle\frac{DT}{\partial s}(t,0), \frac{DT}{\partial s}(t,0)\rangle\Big).$$

Then, the statement of the item (ii) follows from the fact that $T(t,0) = 0$ and from Lemma 3.1, item ii).

Corollary 3.1 *Let c_1 and c_2 be a C^1 curves in M, such that $c_1(0) = c_2(0) = p$, $c_1'(0) = v$ and $c_2'(0) = w$. Then*

$$\lim_{s\to 0^+} \frac{d(c_1(s), c_2(s))}{s} = \|w - v\|,$$

where d is the Riemannian distance.

Immediately from (6).

Theorem 3.1 *Let M be a complete Riemannian manifold and $f : M \to R$ a convex function. If $c : (-\varepsilon, \varepsilon) \to M$ is C^1 curve such that $c(0) = p$, $c'(0) = v$, then*

$$f'(p, v) = \lim_{s \to 0^+} q_c(s),$$

where q_c is defined as in (1).

Let γ_v be geodesic with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. By definition, $f'(p, v) = \lim_{s \to 0^+} q_{\gamma_v}(s)$. Since f is locally Lipschitzian, then there exists $L(p) \geq 0$ such that

$$\begin{aligned} |f'(p,v) - \lim_{s\to 0^+} q_c(s)| &= \lim_{s\to 0^+} |q_{\gamma_v}(s) - q_c(s)| = \\ &= \lim_{s\to 0^+} \frac{|f(\gamma_v(s)) - f(c(s))|}{s} \leq \\ &\leq L(p) \lim_{s\to 0^+} \frac{d(\gamma_v(s), c(s))}{s}. \end{aligned}$$

From Corollary 3.1 we have $\lim_{s \to 0^+} \dfrac{d(\gamma_v(s), c(s))}{s} = 0$. Then the preceding inequality implies $f'(p, v) = \lim_{s \to 0^+} q_c(s)$. This fact completes the proof.

Theorem 3.2 *Let M be a complete Riemannian manifold and let $f : M \to R$ be a convex function. Then, for each fixed $p \in M$, the directional derivative map*

$$f'(p, \cdot) : T_pM \to R$$

is convex. Furthermore,

i) $f'(p, \lambda v) = \lambda f'(p, v)$ for all $\lambda > 0$ and $v \in T_pM$, i.e., $f'(p, \cdot)$ is positive homogeneous;

ii) $-f'(p, -v) \leq f'(p, v)$ for all $v \in T_pM$;

See [12].

Remark 3.1 *First part of Theorem 3.2 and item i) imply that the directional derivative map $f'(p, .) : T_pM \to R$ is a sublinear map.*

Proposition 3.1 *Let M be a complete Riemannian manifold and let $f : M \to R$ a convex function. Then, for each fixed $p \in M$, $|f'(p, v)| \leq L(p)\|v\|$ for all $v \in T_pM$, where $L(p) \geq 0$ is the Lipschitz constant of f in p.*

This fact is proved in the same way that its similar in R^n, see [4].

Theorem 3.3 *Let M be complete Riemannian manifold and let $f : M \to R$ be a convex function. Then, for each fixed $p \in M$, the subdifferential $\partial f(p)$ is non-empty, convex and compact.*

See [12] or [13].

Remark 3.2 *The proof of $\partial f(p) \subset B(0, L)$, where $L = L(p)$ is the Lipschitz constant of f at p, is similar as the one in R^n.*

Proposition 3.2 *Let M be complete Riemannian manifold and let $f : M \to R$ a convex function. Then, for each fixed $p \in M$, is true that*

i) $f'(p, v) = \max_{s \in \partial f(p)} \langle s, v \rangle$, *for all* $v \in T_pM$;

ii) $\partial f(p) = \{s \in T_pM : f'(p, v) \geq \langle s, v \rangle, v \in T_pM\}$.

For item i), take $v \in T_pM$, a γ_v geodesic such that $\gamma_v(0) = p$. The definition of subgradient implies that

$$\frac{f(\gamma_v(t)) - f(p)}{t} \geq \langle s, v \rangle \tag{7}$$

for all $t > 0$ and all $s \in \partial f(p)$. Taking limit in (7) we obtain that $f'(p, v) \geq \max_{s \in \partial f(p)} \langle s, v \rangle$. We derive a contradiction on assuming that there exists $v_1 \in T_pM$ such that $f'(p, v_1) > \max_{s \in \partial f(p)} \langle s, v_1 \rangle$. Since $f'(v, \cdot)$ is a sublinear map, by Hahn-Banach Theorem in T_pM – see [5], it follows that, for all $v \in T_pM$, there exits $\bar{s} \in T_pM$ satisfying

$$f'(p, v) \geq \langle \bar{s}, v \rangle \quad \text{and} \quad f'(p, v_1) = \langle \bar{s}, v_1 \rangle. \tag{8}$$

Definition 3.1 implies that, for all $v \in T_pM$ and $t \geq 0$, we have $f(\gamma_v(t)) - f(p) \geq tf'(p, v) \geq t\langle \bar{s}, v \rangle$, from which one obtains $\bar{s} \in \partial f(p)$. Therefore, by (8)

$$f'(p, v_1) > \max_{s \in \partial f(p)} \langle s, v_1 \rangle \geq \langle \bar{s}, v_1 \rangle = f'(p, v_1).$$

This is our final contradiction and the proof of the item i) is complete.

For item ii). Define $\Gamma = \{s \in T_pM : f'(p, v) \geq \langle s, v \rangle, v \in T_pM\}$ and take $s \in \Gamma$. Fix $v \in T_pM$ and $t > 0$, set γ_v as the geodesic with $\gamma_v(0) = p$ and $\gamma_v' = (0) = v$. From item i) and convexity of f it follows that

$$t\langle s, v \rangle \leq f'(p, tv) = \lim_{\lambda \to 0^+} \frac{f(\gamma_{tv}(\lambda)) - f(p)}{\lambda} \leq$$
$$\leq \lim_{\lambda \to 0^+} \frac{(1 - \lambda)f(\gamma_{tv}(0)) + \lambda f(\gamma_{tv}(1)) - f(p)}{\lambda} =$$
$$\lim_{\lambda \to 0^+} \frac{(1 - \lambda)f(p) + \lambda f(\gamma_v(t)) - f(p)}{\lambda} = f(\gamma_v(t)) - f(p).$$

Then $s \in \partial f(p)$ and consequently $\Gamma \subset \partial f(p)$.

Now take $s \in \partial f(p)$. Fix $v \in T_pM$, set γ_v as the geodesic such that $\gamma_v(0) = p$; then

$$f'(p, v) = \lim_{t \to 0^+} \frac{f(\gamma_v(t)) - f(p)}{t} \geq \lim_{t \to 0^+} \frac{t\langle s, v \rangle}{t} = \langle s, v \rangle,$$

which implies that $\partial f(p) \subset \Gamma$ and the proof of the item ii) is complete. Therefore $\Gamma = \partial f(p)$ and the proof of the Proposition is complete.

Let M be a complete Riemannian manifold and $f : M \to R$ a convex function. Given a geodesic $\gamma : R \to M$, consider the real function $\varphi : R \to R$ defined by $\varphi(t) = f(\gamma(t))$.

Now we will calculate $\partial\varphi$.

Lemma 3.3 *(Chain rule). The subdifferential of φ is given by*

$$\partial\varphi(t) = \{\langle s, \gamma'(t)\rangle | s \ \in \partial = f(\gamma(t))\} = \langle \partial f(\gamma(t)), \gamma'(t)\rangle.$$

By Definition 3.1,

$$\varphi'(t, 1) = \lim_{\lambda \to 0^+} \frac{f(\gamma(t+\lambda)) - f(\gamma(t))}{\lambda} = f'(\gamma(t), \gamma'(t))$$

and

$$\varphi'(t, -1) = \lim_{\lambda \to 0^+} \frac{f(\gamma(t-\lambda)) - f(\gamma(t))}{\lambda} = f'(\gamma(t), -\gamma'(t)).$$

Then, $\partial\varphi(t) = [-\varphi'(t, -1), \varphi'(t, 1)]$. The Proposition 3.2 implies that

$$f'(\gamma(t), \gamma'(t)) = \max_{s \in \partial f(\gamma(t))} \langle s, \gamma'(t)\rangle$$

and

$$-f'(\gamma(t), -\gamma'(t)) = \min_{s \in \partial f(\gamma(t))} \langle s, \gamma'(t)\rangle.$$

Therefore, by convexity of $\partial f(\gamma(t))$, it follows that

$$\varphi(t) = \left[\min_{s \in \partial f(\gamma(t))} \langle s, \gamma'(t)\rangle, \max_{s \in \partial f(\gamma(t))} \langle s, \gamma'(t)\rangle\right] = \{\langle s, \gamma'(t)\rangle | s \in \partial f(\gamma(t))\},$$

the statement of the Lemma.

4 Monotone point-to-set vector field

A *point-to-set vector field on M* is a mapping X which associates to each $p \in M$ a subset $X(p)$ of T_pM. If $f : M \to R$ is convex, then the subdifferential map ∂f is a point-to-set vector field in M.

Definition 4.1 *A point-to-set vector field X on M is called monotone, if for all pair of points $p, q \in M$, $p \neq q$, and all geodesic γ linking p and q is true that*

$$\left\langle \gamma'(t_1), P(\gamma^{-1})_{t_2}^{t_1} v - u \right\rangle \geq 0, \tag{9}$$

whenever $t_1 < t_2$, $\gamma(t_1) = p$, $\gamma(t_2) = q$, $u \in X(p)$ and $v \in X(q)$.

Denote by $\mathcal{P}(R)$ the set of all subsets of R. Define the *point-to-set real function* $\varphi : R \to \mathcal{P}(R)$ by

$$\varphi_{(X,\gamma)}(t) = \Big\{\langle\gamma'(t), v\rangle : v \in X(\gamma(t))\Big\}, \tag{10}$$

where X is a point-to-set vector field on M and γ is a geodesic in M.

We recollect that a point-to-set real function φ is monotone iff $(t_2-t_1)(r_2-r_1) \geq 0$ for all $t_1 \in R$, $t_2 \in R$, $r_1 \in \varphi(t_1)$ and $r_2 \in \varphi(t_2)$. If α is a reparametrization of γ, then $\varphi_{(X,\gamma)}$ is monotone if and only if $\varphi_{(X,\alpha)}$ is monotone.

Proposition 4.1 *A point-to-set vector field X on M is monotone if and only if $\varphi_{(X,\gamma)}$ is monotone for all geodesic γ in M.*

Suppose that X is monotone. Take γ a geodesic in M, $t_1 \neq t_2$ such that $\gamma(t_1) \neq \gamma(t_2)$, $r_1 \in \varphi_{X,\gamma}(t_1)$, $r_2 \in \varphi_{X,\gamma}(t_2)$, $v_1 \in X(\gamma(t_1))$, $v_2 \in X(\gamma(t_2))$, such that $r_1 = \langle\gamma'(t_1), v_1\rangle$ and $r_2 = \langle\gamma'(t_2), v_2\rangle$. Then

$$\begin{aligned}
(t_2-t_1)(r_2-r_1) = \ & (t_2-t_1)\Big(\langle\gamma'(t_2), v_2\rangle - \langle\gamma'(t_1), v_1\rangle\Big) \\
= \ & (t_2-t_1)\Big(\langle P(\gamma^{-1})_{t_2}^{t_1}\gamma'(t_2), P(\gamma^{-1})_{t_2}^{t_1}v_2\rangle - \langle\gamma'(t_1), v_1\rangle\Big) \\
= \ & (t_2-t_1)\langle\gamma'(t_1), P(\gamma^{-1})_{t_2}^{t_1}v_2 - v_1\rangle \geq 0,
\end{aligned}$$

because X is monotone. Then $\varphi_{(X,\gamma)}$ is monotone for all geodesic γ.

Now, suppose that $\varphi_{(X,\gamma)}$ is monotone. It is to prove that, taking p, $q \in M$, $u \in X(p)$, $v \in X(q)$ and γ geodesic with $\gamma(0) = p$ and $\gamma(1) = q$, it holds that $\langle\gamma'(0), P(\gamma^{-1})_1^0 v - u\rangle \geq 0$. Set $r_1 = \langle\gamma'(0), u\rangle \in \varphi(0)$ and $r_2 = \langle\gamma'(1), v\rangle \in \varphi(1)$. Then

$$\begin{aligned}
&\langle\gamma'(0), P(\alpha^{-1})_1^0 v - u\rangle = \langle P(\alpha^{-1})_1^0\gamma'(1), P(\alpha^{-1})_1^0 v\rangle - \langle\gamma'(0), u\rangle = \\
&= \langle\gamma'(1), v\rangle - \langle\gamma'(0), u\rangle = (1-0)(r_2-r_1) \geq 0,
\end{aligned}$$

because $\varphi_{X,\gamma}$ is monotone. Then $\langle\gamma'(0), P(\gamma^{-1})_1^0 v - u\rangle \geq 0$ which implies that X is monotone.

Proposition 4.2 *If $f : M \to R$ is convex, then ∂f is monotone.*

By Proposition 4.1, it is sufficient to prove that, for all geodesic γ, the mapping $\varphi_{(\partial f,\gamma)}$ is monotone. Take γ geodesic. Since f is convex, the real function $f \circ \gamma$ is convex and $\partial(f \circ \gamma)$ is monotone. By Lemma 3.3, it follows that

$$\partial(f \circ \gamma) = \{\langle\gamma'(t), v \in \partial f(\gamma(t))\} = \varphi_{(\partial f,\gamma)}(t).$$

Then ∂f is monotone.

5 Consequences of the existence of monotone point-to-set vector field

It is well known that the existence of convex function imposes some topological consequences on the Riemannian manifold M - see [10], [13]. The concept of monotonicity is, in certain sense, a generalization of the concept of convexity. Then it is to expect that the existence of monotone point-to-set vector field on M imposes topological consequences also on M.

Next we will prove that existence of strictly monotone point-to-set vector fields requires some topological properties of the manifold. First, observe that, if γ is closed geodesic then $\varphi_{(X,\gamma)}(t)$ is constant.

Proposition 5.1 *Let M be a complete Riemannian manifold. If there exists a strictly monotone point-to-set vector field X in M, then all compact totally geodesic submanifold of M are trivial, i.e., it consist of simple points.*

We derive a contradiction on assuming that there exists a nontrivial compact totally geodesic submanifold N of M. By Theorem 3.5 in page 299 of [9] the submanifold N has a closed geodesic γ. Observe that γ is geodesic in M. Then, by definition of $\varphi_{(X,\gamma)}$ follows that $\varphi_{(X,\gamma)}(t)$ is constant and X can't be strictly monotone.

Proposition 5.2 *Let M be a complete noncompact Riemannian manifold of nonnegative sectional curvature. If there exists a strictly monotone point-to-set vector field X in M, then the soul S of M consists of one point and M is diffeomorphic to R^n.*

Take $p \in M$ and build the soul S starting from p.

By Theorem 2.1, the soul S is a compact totally geodesic submanifold of M. Then, by Proposition 5.1 the submanifold S consists of one point. Again, by Theorem 2.1, M is diffeomorphic to normal bundle of S. Since that S is a simple point, it follows that the normal bundle of S is diffeomorphic to R^n. Therefore M is diffeomorphic to R^n.

Theorem 2.2 says that M, complete non compact Riemannian manifold of nonnegative sectional curvature, is diffeomorphic to R^n when exists a point of M at which the sectional curvature is positive. Proposition 5.2 says that M, complete non compact Riemannian manifold of nonnegative sectional curvature, is diffeomorphic to R^n when exists a strictly monotone vector field, i.e., we obtain the same result by mean of substitution of the existence of a point at which the sectional curvature is positive by the existence of strictly monotone vector field.

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