

On 2 - Framed Riemannian Manifolds with Godbillon - Vey Structure Form

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*Dedicated to Prof.Dr. Constantin UDRIŞTE
on the occasion of his sixtieth birthday*

Abstract

In the last decade, contact, almost contact, paracontact cosymplectic, and conformal cosymplectic manifolds carrying $\kappa > 1$ structure vector fields ξ have been studied by many authors, e.g. [2], [7], [11], [15].

In the present paper we consider a $(2m + 2)$ -dimensional Riemannian manifold carrying two structure vector fields ξ^r ($r \in \{2m + 1, 2m + 2\}$), a $(1, 1)$ -tensor field Φ , and a structure 2 - form Ω of rank $2m$, such that for $\eta^r := (\xi_r)^\flat$

$$(0.1) \quad \begin{aligned} \Phi^2 &= -Id + \eta^r \otimes \xi_r & \Phi \xi_r &= 0, & \eta^r(\xi_s) &= \delta_s^r \\ \Omega(Z, Z') &= g(\Phi Z, Z'), & \Omega^m \wedge \eta^{2m+1} \wedge \eta^{2m+2} &\neq 0 \end{aligned}$$

holds. Here the $(2m)$ -dimensional subspace $Im\Phi$ of the tangent space is supposed to be Kählerian (see eq. (2.12) below). If the 3-forms

$$(0.2) \quad \gamma^r = \eta^r \wedge d\eta^r$$

satisfy

$$(0.3) \quad d\gamma^r = 0,$$

they are called *Godbillon-Vey forms* [6]. On the other hand, if

$$(0.4) \quad \begin{aligned} \nabla_X \xi_r &= f_r X \\ r &= 2m + 1, 2m + 2 \end{aligned}$$

holds for all X orthogonal to ξ_r and for some $f_r \in \Lambda^0 M$, the structure vector fields define a *concircular pairing* [1]. It will turn out that (0.3) follows from (0.1) and (0.4). Therefore we call such manifolds $M(\Phi, \Omega, \eta^r, \xi_r)$ *2-framed Godbillon-Vey manifold* (abbreviated *2FG-V*). We shall prove that they have the following properties:

Any 2FG-V manifold is equipped with a conformal symplectic structure $CSp(m + 1, \mathbb{R})$ with $\xi := \sum f_r \xi_r$ as vector of Lee, i.e.

$$(0.5) \quad d\Omega = 2\xi^\flat \wedge \Omega$$

and M is the local *Riemannian* product

$$M = M^\perp \times M^\top$$

such that

1. M^\perp is a flat surface tangent to the structure vector fields ξ_r ;
2. M^\top is a $2m$ -dimensional Kählerian submanifold, and the immersion $x : M^\top \rightarrow M$ has the following properties:
 - (a) The mean curvature vector field H associated with x is $-\xi$ and satisfies $\|H\|^2 = \text{const.}$
 - (b) The immersion x is umbilical. In section 3, the existence of a horizontal skew symmetric conformal (abbreviated SC) vector field C is proved by an exterior differential system in involution (in the sense of E. Cartan [3]). Denote by K and R the scalar curvature of M and the Ricci tensor field of ∇ , respectively. Then

$$\mathcal{L}_C K = -\rho K; \quad \mathcal{L}_C R(Z, Z') = 0; \quad \rho = \text{const.}; \quad Z, Z' \in \mathcal{X}M$$

and C is a module commuting vector field, i.e.

$$[C, \nabla \|C\|^2] = 0, \quad \nabla : \text{gradient of a scalar} .$$

- (c) C defines an infinitesimal homothety of all $(2q+1)$ -forms $(C^b)_q := C^b \wedge \Omega^q$, i.e.

$$\mathcal{L}_C (C^b)_q = (q+1)(C^b)_q,$$

and ΦC defines an infinitesimal automorphism of Ω :

$$\mathcal{L}_{\Phi C} \Omega = 0 .$$

Mathematics Subject Classification: 53C25

Key words: Riemannian manifold, 2-framed structure, concircular pairing, Godbillon - Vey form

1 Preliminaries

Let (M, g) be a Riemannian C^∞ -manifold and ∇ the covariant differential operator with respect to the metric g . We assume that M is oriented and ∇ is the Levi-Civita connection.

Define $\Gamma(TM) =: \mathcal{X}M$ and let $TM \overset{b}{\rightleftarrows} T^*M$ be the *musical isomorphism* defined

by g and

$$\Omega^b : TM \rightarrow T^*M; \quad Z \rightarrow -i_Z \Omega =: {}^b Z$$

the *symplectic isomorphism* defined by Ω . Following Poor [10], we set

$$A^q(M, TM) := \text{Hom}(\Lambda^q TM, TM)$$

and notice that the elements of $A^q(M, TM)$ are vector valued q -forms. The local field of orthonormal frames on an n -dimensional Riemannian manifold is denoted by

$$\mathcal{O} = \{e_A; A = 1, \dots, n\}$$

and the associated coframe by

$$\mathcal{O}^* = \{\omega^A; A = 1, \dots, n\} .$$

The soldering form dp is expressed by

$$(1.6) \quad dp = \omega^A \otimes e_A$$

and Cartan's structure equations in index-free notation are written as

$$(1.7) \quad \nabla e = \theta \otimes e$$

$$(1.8) \quad d\omega = -\theta \wedge \omega$$

$$(1.9) \quad d\theta = -\theta \wedge \theta + \Theta .$$

Here the 1-forms θ and the 2-form Θ are the connection forms in the tangent bundle TM and the curvature form, respectively.

Now let W be a conformal vector field, i.e. a vector field satisfying the conformal version of Killing's equation

$$(1.10) \quad \mathcal{L}_W g = \rho g ,$$

where the conformal scalar ρ is defined by

$$(1.11) \quad \rho = \frac{2}{\dim M} (\operatorname{div} W) .$$

We recall some basic formulas [14] which will be needed in the last section:

$$(1.12) \quad \mathcal{L}_W K = (n - 1) \Delta \rho - K \rho ; \quad n = \dim M$$

$$(1.13) \quad 2 \mathcal{L}_W R(Z, Z') = g(Z, Z') \Delta \rho - (n - 2) (\operatorname{Hess}_\nabla \rho)(Z, Z') ,$$

where

$$(\operatorname{Hess}_\nabla \rho)(Z, Z') = g(Z, \nabla_{Z'} \operatorname{grad} \rho) .$$

In these equations \mathcal{L}_W , K , Δ and R denote the Lie derivative with respect to W , the scalar curvature of M , the Laplacian and the Ricci tensor field of ∇ respectively.

2 2-Framed Godbillon - Vey manifolds

Let $M(\Phi, \Omega, \eta^r, \xi_r, g)$ be a $(2m + 2)$ - dimensional Riemannian manifold carrying two structure vector fields ξ_r ($r \in 2m + 1, 2m + 2$) and let η^r be their associated covectors. Suppose that the structure tensors $(\Phi, \Omega, \eta^r, \xi_r)$ satisfy (0.1). Then M carries a 2-framed structure in the sense of Yano and Kon [15]. We further assume that (0.4) holds. Defining $e_r := \xi_r$ and $\omega^r := \eta^r$, this yields

$$(2.1) \quad f_r \omega^a = \theta_r^a , \quad f_r \in \Lambda^0 M , \quad a = 1, \dots, 2m$$

and

$$(2.2) \quad \begin{aligned} d\eta^{2m+1} &= u \wedge \eta^{2m+2} \\ d\eta^{2m+2} &= -u \wedge \eta^{2m+1} \end{aligned} ,$$

where u is some closed 1-form. In the same way, (0.4) ensures that $d\gamma^r = 0$ holds. (2.2) can be written as

$$(2.3) \quad u = \theta_{2m+1}^{2m+2} .$$

Connections satisfying (2.1) are called *principal connections* [12].

One may split the soldering form dp in a unique manner as

$$(2.4) \quad dp = dp^\top \otimes dp^\perp ,$$

where $dp^\top := \omega^a \otimes e_a$ and $dp^\perp := \eta^r \otimes \xi_r$ are called the *horizontal* and the *vertical* component of dp , respectively. From (2.3) and (2.1) one finds

$$(2.5) \quad \begin{aligned} \nabla \xi_{2m+1} &= f_{2m+1} dp^\top + u \otimes \xi_{2m+2} \\ \nabla \xi_{2m+2} &= f_{2m+2} dp^\top - u \otimes \xi_{2m+1} \end{aligned} .$$

Hence we have

$$\begin{aligned} \nabla_{\xi_{2m+2}} \xi_{2m+1} &= u(\xi_{2m+2}) \xi_{2m+2} \\ \nabla_{\xi_{2m+1}} \xi_{2m+2} &= -u(\xi_{2m+1}) \xi_{2m+1}, \end{aligned}$$

and referring to [1] one may say that the structure vector fields ξ_r define a *concircular pairing*. Then (2.5) and the well-known formula

$$\operatorname{div} Z = \operatorname{tr}(\nabla Z) = \sum_{a=1}^{2m} \omega^a (\nabla_{e_a} Z) + \sum_{r=2m+1}^{2m+2} \eta^r (\nabla_{\xi_r} Z), \quad Z \in \mathcal{X}M$$

yield

$$\begin{aligned} \operatorname{div} \xi_{2m+1} &= 2m f_{2m+1} + u(\xi_{2m+2}) \\ \operatorname{div} \xi_{2m+2} &= 2m f_{2m+2} + u(\xi_{2m+1}) \end{aligned} .$$

If u is a *basic form*, i.e. if $u(\xi_r) = 0$, then (2.2) entails

$$i_{\xi_r} d\eta^r = 0.$$

Therefore, according to a well known definition, we may say that ξ_r move to Reeb vector fields (in the large).

In the general case, i.e. $u(\xi_r) \neq 0$, we shall say that the manifold $M(\Phi, \Omega, \eta^r, \xi_r, g)$ is endowed with a *2-framed Godbillon - Vey structure*, (abbreviated 2FG-V structure). Referring to [11] we call the distribution $D^\perp := \{\xi_r; r = 2m+1, 2m+2\}$ the *vertical distribution*, and its orthogonal complement $D^\top := \{e_a, a = 1, \dots, 2m\}$ the *horizontal distribution* on M . Similarly

$$\varphi^\perp := \eta^{2m+1} \wedge \eta^{2m+2}$$

and

$$(2.6) \quad \varphi^\top := \omega^1 \wedge \dots \wedge \omega^{2m}$$

are called the *vertical* and the *horizontal form*, respectively. With these definitions, (2.2) gives immediately

$$d\varphi^\perp = 0 .$$

Therefore it follows from *Frobenius'* theorem that the horizontal distribution D^\top is involutive. Setting

$$(2.7) \quad \eta := \sum_{r=2m+1}^{2m+2} f_r \eta^r ,$$

(2.6) and (2.1) yield

$$(2.8) \quad d\varphi^\top = 2m \eta \wedge \varphi^\top .$$

This shows that φ^\top is an exterior recurrent form [5] and consequently D^\perp is also involutive. Hence any 2FG-V manifold is the local Riemannian product

$$M = M^\top \times M^\perp ,$$

where M^\top is a $2m$ -dimensional manifold tangent to D^\top and M^\perp is a surface tangent to D^\perp .

Since η is the recurrence form of φ^\top (see (2.8)), it is closed. (Generally, we shall call an exterior recurrent form *strictly recurrent*, if its recurrence form is closed.) This fact together with (2.7) and (2.2) give

$$(2.9) \quad \begin{aligned} df_{2m+1} &= f_{2m+2} u \\ df_{2m+2} &= -f_{2m+1} u . \end{aligned}$$

Therefore the Poisson bracket $\{ \quad \}_P$ of the function f_r , i.e.

$$\{f_{2m+1}, f_{2m+2}\}_P := \Omega(\nabla f_{2m+1}, \nabla f_{2m+2})$$

vanishes. Defining

$$\xi := \sum_{r=2m+1}^{2m+2} f_r \xi_r ; \quad \eta := \sum_{r=2m+1}^{2m+2} f_r \eta^r = \xi^b$$

one easily deduces from (2.9) that

$$(2.10) \quad \|\xi\|^2 = (f_{2m+1})^2 + (f_{2m+2})^2 =: 2f = \text{const.}$$

and further from (2.9), (2.4), and (2.5):

$$(2.11) \quad \nabla \xi = 2f dp^\top .$$

On the other hand using (2.3), (2.1), (1.9), $du = 0$ (see (2.2)) and the fact that $\theta_{2m+2}^a = -\theta_a^{2m+2}$ holds because of $g(e_{2m+2}, e_a) = 0$, one finds

$$\Theta_{2m+1}^{2m+2} = 0 .$$

It is easily seen that Θ_{2m+1}^{2m+2} is the curvature form of M^\perp . Therefore this surface is *flat*. Further, because of (0.1), the horizontal connection forms satisfy the Kähler relations

$$(2.12) \quad \theta_j^i = \theta_{j^*}^{i^*} ; \quad \theta_j^{i^*} = \theta_i^{j^*} ; \quad i = 1, \dots, m ; \quad i^* = i + m .$$

Recalling the standard expression for the structure 2-form Ω

$$(2.13) \quad \Omega = \sum_{i=1}^m \omega^i \wedge \omega^{i^*} ; \quad i^* = i + m ,$$

we find with the help of (2.1) and (2.7), after some calculation,

$$(2.14) \quad d\Omega = 2\eta \wedge \Omega .$$

This shows the important fact that the 2FG-V manifold under discussion is endowed with a *locally conformal symplectic structure* $CSp(m+1, \mathbb{R})$, with $\eta = \xi^b$ as covector of Lee. Since $i_\xi \Omega = 0$ and $f = \text{const.}$ (see (2.10)), one gets from (2.13):

$$(2.15) \quad \mathcal{L}_\xi \Omega = 2f \Omega ,$$

which shows that ξ defines an *infinitesimal homothety* of Ω .

On the other hand, $\Omega|_{M^\top}$ is of rank $2m$. Therefore it is the symplectic form of the Kähler submanifold M^\top of M . Next let H be the mean curvature vector field

associated with the immersion $x : M^\top \rightarrow M$. If γ_{BC}^A denote the coefficients of the connection θ , the vector field H is given by

$$H = \frac{1}{2m} \sum_{a=1}^{2m} \gamma_{aa}^r \xi^r .$$

(We denote the induced elements by the same letters.) Now using (2.1) and (2.10), an easy calculation gives

$$H = -\xi \quad \Rightarrow \quad \|H\|^2 = 2f = \text{const.}$$

Hence one deduces the following important fact: M^\top is a Kähler submanifold of M of *constant mean curvature*. Moreover, since dp^\top is the soldering form of M^\top , it follows from (2.4) that the second quadratic forms associated with the immersion $x : M^\top \rightarrow M$ are

$$l_r = - \langle dp^\top, \nabla \xi_r \rangle = -f_r g^\top .$$

This means that the immersion $x : M^\top \rightarrow M$ is *umbilical*.
Summing up we state

Theorem 1. *Let $M(\Phi, \Omega, \xi_r, \eta^r, g)$ be a $(2m+2)$ -dimensional Riemannian manifold endowed with a 2 FG-V structure defined by (0.1) - (0.3). Such a manifold admits a locally conformal symplectic structure with ξ^b as covector of Lee, i.e.*

$$d\Omega = 2\xi^b \wedge \Omega .$$

Furthermore M is the local Riemannian product

$$M = M^\perp \times M^\top ,$$

where

1. M^\perp is a flat surface tangent to the structure vector fields ξ_r .
2. M^\top is a $2m$ -dimensional Kählerian submanifold, and the immersion $x : M^\top \rightarrow M$ has the following properties:
 - (a) M^\top is of constant mean curvature.
 - (b) The immersion $x : M^\top \rightarrow M$ is umbilical.

3 Skew symmetric conformal vector fields

In this section we assume that the 2FG-V manifold under consideration carries a *horizontal skew symmetric conformal* (abr. SSC) *vector field* C . The generative of C is assumed to be the Reeb vector field ξ . This means [9]

$$(3.1) \quad \nabla C = \lambda dp + C \wedge \xi .$$

Here \wedge denotes the wedge product of vectors: $C \wedge \xi := \xi^b \otimes C - C^b \otimes \xi$. One may set

$$C = C^a e_a \in D^\top ; \quad a, b \in \{1, \dots, 2m\} .$$

Then it follows from (2.1), (3.1), and (1.7):

$$(3.2) \quad dC^a + C^b \theta_b^a = \lambda \omega^a + C^a \eta .$$

Clearly, from

$$(3.3) \quad C^b = \sum_{a=1}^{2m} C^a \omega^a$$

one obtains

$$(3.4) \quad dC^b = 2 \eta \wedge C^b .$$

This agrees with Rosca's lemma [9]. As a simple consequence of (3.2), one derives

$$(3.5) \quad d\|C\|^2 = 2 \lambda C^b - 2 \|C\|^2 \eta .$$

Denote now by Σ the exterior differential system which defines the vector field C . Then because of $d\eta = 0$, (3.4) and (3.5), the characteristic numbers of Σ are $r = 3$, $s_0 = 1$, $s_1 = 2$. Since $r = s_0 + s_1$ holds, it follows that Σ is in involution (in the sense of E. Cartan [3]). Therefore Cartan's test states that C exists and depends on two arbitrary functions of one argument. On the other hand, recall that the symplectic isomorphism (see also [8]) is expressed as

$$(3.6) \quad Z \rightarrow -i_Z \Omega = {}^b Z =: \Omega^b(Z) , \quad \Omega(Z, Z') =: \langle Z', Z \rangle .$$

So one may write

$$i_C \Omega = -{}^b C = \sum_{i=1}^m (C^i \omega^{i*} - C^{i*} \omega^i) =: \beta ,$$

where we have set $\beta := -{}^b C$. From (2.12), (2.14), and (3.2), one derives:

$$d\beta = 2 \lambda \Omega + 2 \eta \wedge \beta .$$

Again an exterior derivation yields $\lambda = const$ (remember $d\eta = 0$.) On the other hand, from

$$\mathcal{L}_Z g = \frac{2 \operatorname{div} Z}{\operatorname{dim} M} g = \rho g ; \quad Z \in \mathcal{X}(M)$$

(cf. (1.11)) and from (3.1), one quickly finds

$$(3.7) \quad \rho = 2 \lambda .$$

This means that C defines an *infinitesimal homothety* of M , because using (2.13) and (2.15), one obtains at once

$$\mathcal{L}_C \Omega = \rho \Omega$$

and

$$\mathcal{L}_\xi \Omega = 2 f \Omega$$

(remember $f = const$.) Furthermore, let L be the operator of type (1,1) given by

$$L u := u \wedge \Omega ; \quad u \in \Lambda^1 M$$

and define (cf. [6])

$$L^q u := u_q := u \wedge \Omega^q \in \Lambda^{2q+1} M .$$

Coming back to the case under discussion, (3.4) yields

$$\mathcal{L}_C C^b = \rho C^b .$$

This shows that C^b is a self-conformal form. A standard calculation gives

$$\mathcal{L}_C (C^b)_q = (q+1)(C^b)_q .$$

Therefore C defines an infinitesimal homothety of all these $(2q+1)$ -forms. With Yano's formulas (1.12) and (1.13), one finds

$$\mathcal{L}_C K = -\rho K$$

and

$$\mathcal{L}_C R(Z, Z') = 0 ; \quad Z, Z' \in \mathcal{X}(M) ,$$

where K and R denote the scalar curvature of M and the Ricci tensor field, respectively. Now, for any vector field Z , one has

$$(\nabla\Phi)Z = \nabla(\Phi Z) - \Phi \nabla Z .$$

Therefore (0.1) and (3.1) yield

$$\begin{aligned} (\nabla\Phi)C &= \left(\frac{\rho}{2} - \lambda - \eta(C)\right) \Phi dp - (\Phi C)^b \otimes \xi \\ &= \nabla(\Phi C) - \lambda \Phi dp - \eta(\Phi C) . \end{aligned}$$

Hence

$$\begin{aligned} \nabla(\Phi C) &= \left(\frac{\rho}{2} - \eta(C)\right) \Phi dp + \eta(\Phi C) - (\Phi C)^b \otimes \xi \\ (3.8) \quad &= \left(\frac{\rho}{2} - \eta(C)\right) \Phi dp + \Phi C \wedge \xi \end{aligned}$$

(\wedge : wedge product of vector fields). From the inner product $\langle Z, \Phi dp \rangle = \Phi Z$, and from (3.8), one derives

$$\langle \nabla_Z \Phi C, Z' \rangle + \langle \nabla_{Z'} \Phi C, Z \rangle = 0 ; \quad Z, Z' \in \mathcal{X}(M) .$$

Furthermore, since C is a horizontal vector field, it is easily seen that

$${}^b\Phi C = C^b$$

holds. So together with (2.13), this leads to

$$\mathcal{L}_{\Phi C} \Omega = 0 .$$

Therefore ΦC defines an infinitesimal automorphism of Ω .

It should be noticed that (2.10), (3.1), and (3.8) entail

$$[\xi, \Phi C] = 0 ; \quad [C, \Phi C] = 0 ; \quad [C, \xi] = -\frac{\rho}{2} \xi .$$

So ξ and C commute with ΦC , and ξ admits an infinitesimal homothety of generators C [4].

Let now $\mathcal{C} : (M, g) \rightarrow (\tilde{M}, \tilde{g})$ be a *conformal diffeomorphism* (abr. CD) of argument t , i.e.

$$C : g \mapsto \tilde{g} := e^{2t} g .$$

One has (see also [10])

$$\tilde{\nabla} C = \nabla C + (\nabla t)^b \otimes C - C^b \otimes \nabla t + g(C, \nabla t) dp ,$$

and the scalar curvature \tilde{K} of \tilde{M} is given by

$$\tilde{K} = e^{-2t} (K + 2(2m + 1) \operatorname{div} \nabla t + (2m + 1) 2m \|\nabla t\|^2) .$$

Since $K = \operatorname{const.}$, the manifold \tilde{M} is homothetic to M , if it satisfies $\|\nabla t\|^2 = \operatorname{const.}$ and $\operatorname{div} \nabla t = \operatorname{const.}$ Furthermore

$$d\|C\|^2 = \rho C^b + 2\|C\|^2 \eta ,$$

and the gradient (which will also be denoted by ∇) of the function $\|C\|^2$ is expressed by

$$(3.9) \quad \nabla\|C\|^2 = \rho C + 2\|C\|^2 \xi .$$

Thus from

$$\operatorname{div} C = (m + 1) \rho = \operatorname{const.} ; \quad \operatorname{div} \xi = 4m f = \operatorname{const.}$$

(see (2.5), (2.9), and (2.10)) one quickly derives

$$(3.10) \quad \Delta\|C\|^2 = -\operatorname{div} \nabla\|C\|^2 = -\kappa f\|C\|^2 - (m + 1) \rho^2 ; \quad \kappa \in \mathbb{R} .$$

Therefore as an extension of a well-known definition (see e.g. [13]), we may say that $\|C\|^2$ is an *almost eigenfunction* of Δ with $-\kappa f$ as eigenvalue. We notice that if C is a Killing vector field, i.e. if $\rho = 0$ (see (3.1) and (3.7)), then $\|C\|^2$ becomes an eigenfunction of Δ . Since the eigenvalue is negative definite, the corresponding manifold cannot be compact.

We recall that a function $\nu : \mathbb{R} \rightarrow \mathbb{R}$ is isoparametric, iff both, $\|\nabla \nu\|^2$ and $\operatorname{div}(\operatorname{grad} \nu)$ are functions of ν [13]. Then from (3.9) and (3.10), it is quickly seen that $\|C\|^2$ is an *isoparametric function*.

Finally, setting

$$\nabla^2\|C\|^2 := \nabla \operatorname{grad} \|C\|^2$$

in (3.1), one deduces after a short calculation

$$[C, \nabla\|C\|^2] = 0 .$$

This shows that C is a module commuting vector field. Thus we have proven
Theorem 2. *Let C be a horizontal skew symmetric conformal vector field on the 2FG-V manifold defined by conditions (0.1) - (0.3). Such a C always exists; it is determined by an exterior differential system in involution. C infinitesimal homothety on M , i.e.*

$$\mathcal{L}_C K = -\rho K ; \quad K: \text{ scalar curvature of } M; \rho = \operatorname{const.}$$

Moreover:

1.

$$\mathcal{L}_C R(Z, Z') = 0, \quad Z, Z' \in \mathcal{X}M,$$

where R denotes the Ricci tensor field, and

$$\mathcal{L}_C(C^b)_q = (q+1)(C^b)_q.$$

Here $L^q : C^b \rightarrow (C^b)_q := C^b \wedge \Omega^q$ is the $(1,1)$ - Weyl operator.

2. ΦC defines an infinitesimal automorphism of Ω , i.e.

$$\mathcal{L}_{\Phi C} \Omega = 0,$$

and ξ and C commute with ΦC . In addition, ξ admits an infinitesimal homothety of generators C , i.e.

$$[\xi, \Phi C] = 0; \quad [C, \Phi C] = 0; \quad [C, \xi] = -\frac{\rho}{2} \xi.$$

3. $\|C\|^2$ is an almost eigenfunction of Δ , as well as an isoparametric function, and C is a module commuting vector field.

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