On Parallel Vector Fields of Subbundles of Vector Bundles with Linear and Non - Linear Connections

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Abstract

In this paper are considered the compatibility conditions for existing of a parallel vector field in vector subbundle of a vector bundle, and further if they are satisfied, the filed is found. The compatibility conditions and solutions are given with respect to the initial conditions. The results are generalized for non-linear connections.

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Key words and Phrases: Parallel vector fields, systems of partial differential equations, non-linear connections.

1 Introduction

This paper is a continuation of the papers [2-6], where [2],[3] and [4] due to the solving of systems of partial differential equations and [5],[6] due to their application in the differential geometry.

In [4] are considered linear and non-linear systems of partial differential equations. The compatibility conditions are found, and if they are satisfied, the solutions are found. All the solutions are given as functional series. The results in [4] are used in [5] for studying the non-linear connections. In [6] are considered parallel vector fields in vector bundles. Indeed the existence of such parallel fields depends on the choice of the initial conditions. The main features in [6] is that the compatibility conditions are found in dependence of the the initial conditions, and if they are satisfied, the vector field is found. It is generalized for non-linear connections and vector bundles with d-connection. In [2] and [3] are considered systems of partial differential equations with linear homogeneous algebraic constraints. It is given an algorithm for reducing to system of lower dimension without constraints and moreover the compatibility conditions for the reduced system are satisfied. This algorithm can also be applied in this paper.

This paper is of interest not only from viewpoint of the differential geometry, but also for differential equations or partial differential equations. Indeed the compatibility

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conditions and the solutions for a large class of partial differential equations are given. In this paper we convenient to use the summation convention.

2 On parallel vector fields of subbundles of vector bundles with linear connection

Suppose that the vector field z^i may not be prolonged in general case on the whole manifold M of dimension k, but can be prolonged on distribution η, i.e. on a subbundle of the tangent bundle $T(M)$. Let the rank of η be m ($m \leq k$), and let $Y_{(1)}, \cdots, Y_{(m)}$ be a local basis for η. Let ζ be the Lie subalgebra of $T(M)$ generated by $Y_{(1)}, \cdots, Y_{(m)}$, and let $Y_{(1)}, \cdots, Y_{(p)}$, ($m \leq p \leq k$) be basis of ζ. Then there exist functions $C^{\gamma}_{\alpha\beta}$ ($\alpha, \beta, \gamma \in \{1, \cdots, p\}$) such that

$$(2.1) \qquad [Y_{(\alpha)}, Y_{(\beta)}] = C^{\gamma}_{\alpha\beta} Y_{(\gamma)},$$

i.e. ζ is an integrable distribution. There exist also another basis $X_{(\alpha)} = p^{\delta}_{\alpha} Y_{(\delta)}$, $\alpha, \delta \in \{1, \cdots, p\}$, such that

$$(2.2) \qquad [X_{(\alpha)}, X_{(\beta)}] = 0,$$

i.e.

$$(2.3) \qquad p^{\delta}_{\alpha}(Y_{(\delta)} p^{\vartheta}_{\beta}) - p^{\delta}_{\beta}(Y_{(\delta)} p^{\vartheta}_{\alpha}) + p^{\delta}_{\alpha} p^{\gamma}_{\beta} C^{\vartheta}_{\delta\gamma} = 0.$$

Then there exist uniquely coordinate functions $u^i(x^1, \cdots, x^k)$, $i \in \{1, \cdots, p\}$ such that

$$(2.4) \qquad X_{(\alpha)} u^{\beta} = \delta^{\beta}_{\alpha}$$

and $u^{\alpha}(0, \cdots, 0) = 0$ for $\alpha, \beta \in \{1, \cdots, p\}$. We will prove that the system

$$(2.5) \qquad \nabla_{Y_{(\alpha)}} Y = 0, \qquad (1 \leq \alpha \leq m),$$

is equivalent to

$$(2.6) \qquad \nabla_{X_{(\alpha)}} Y = 0, \qquad (1 \leq \alpha \leq p).$$

Indeed, the system (2.6) is equivalent with $\nabla_{Y_{(\alpha)}} Y = 0$ for $1 \leq \alpha \leq p$, and it implies (2.5). Conversely, let the system (2.5) is satisfied. Since $R(Y_{(\alpha)}, Y_{(\beta)})Y = 0$, we obtain

$$\nabla_{[Y_{(\alpha)}, Y_{(\beta)}]} Y = R(Y_{(\alpha)}, Y_{(\beta)})Y + \nabla_{Y_{(\alpha)}}(\nabla_{Y_{(\beta)}} Y) - \nabla_{Y_{(\beta)}}(\nabla_{Y_{(\alpha)}} Y) = 0$$

for each $\alpha, \beta \in \{1, \cdots, m\}$, and finally we obtain that $\nabla_T Y = 0$ for each vector $T \in \zeta$ and the system (2.6) is satisfied.

Indeed the system (2.6) can be obtained from (2.5) by generating new equations $\nabla_{X_{(\alpha)}} Y = 0$, $(m+1 \leq \alpha \leq p)$ using the Jacobi brackets and then changing the basis $\{Y_1, \cdots, Y_p\}$ by $\{X_1, \cdots, X_p\}$ such that (2.2) holds. Having this in mind, similarly to the theorem 2.1 [6], the following theorem can be proven.

Theorem 2.1. *Let the analytical system (2.6) with the initial conditions*

$$Y(0, \cdots, 0) = Z, \tag{2.7}$$

be given.

(i) The system (2.6) with (2.7) is integrable if and only if the following system of linear equations

$$\begin{aligned} &R(X_{(\alpha)}, X_{(\beta)})T = 0, \\ &(\nabla_{X_{(\gamma_1)}} R(X_{(\alpha)}, X_{(\beta)}))T = 0, \\ &(\nabla_{X_{(\gamma_2)}} \nabla_{X_{(\gamma_1)}} R(X_{(\alpha)}, X_{(\beta)}))T = 0, \\ &\cdots\cdots\cdots \\ &(\nabla_{X_{(\gamma_{n-1})}} \cdots \nabla_{X_{(\gamma_1)}} R(X_{(\alpha)}, X_{(\beta)}))T = 0, \end{aligned} \tag{2.8}$$

where $\alpha, \beta, \gamma_1, \cdots, \gamma_{n-1} \in \{1, \cdots, p\}$, *has an analytic solution* T, *such that* $T(0, \cdots, 0) = Z$.

(ii) Let the compatibility conditions (2.8) be satisfied and let T *be an arbitrary analytical solution of the system (2.8). Then there exist vector fields* $Q^{<v_1, \ldots, v_p>}$ *such that*

$$Q^{<0, \ldots, 0>} = T, \tag{2.9}$$

$$Q^{<v_1, \ldots, v_u + 1, \ldots, v_p>} = \nabla_{X_{(u)}} Q^{<v_1, \ldots, v_p>}, \tag{2.10}$$

and the solution of (2.6) with (2.7) is given by

$$Y = \sum_{v_1=0}^{\infty} \sum_{v_2=0}^{\infty} \cdots \sum_{v_p=0}^{\infty} \frac{(-u^1)^{v_1}}{v_1!} \frac{(-u^2)^{v_2}}{v_2!} \cdots \frac{(-u^p)^{v_p}}{v_p!} \cdot Q^{<v_1, \ldots, v_p>}. \tag{2.11}$$

We notice that $Q^{<v_1, \ldots, v_p>}$ is a vector field and $u^1, \cdots, u^p$ are scalars, invariant of the coordinate system, and hence the solution (2.11) has covariant form.

In the theorem 2.1 we found the solution of (2.6) on the distribution η, and the solution was unique. Now if there exist r parametric families of such distributions $(r \leq k - p)$, then we obtain r parametric solution Y in (2.11), except if the additional restrictions on the initial conditions are required. Specially, if $m = 1$, the compatibility condition vanish, and it is $p = 1$ and $r = k - 1$ in this case.

3 On parallel vector fields of subbundles of vector bundles with non-linear connections

In this section we will consider non-linear connection such that

$$\nabla_{X+Y} \neq \nabla_X + \nabla_Y \tag{3.1}$$

in general case.

Let $X = (X^1, \cdots, X^k)$ be a vector. Then we define an operator ∇_X by

$$\nabla_X y^j = X^i \frac{\partial}{\partial x^i} y^j + G^j_s(X) y^s. \tag{3.2}$$

The operator ∇_X acts over the tensor fields, and it holds

$$\nabla_X (T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2). \tag{3.3}$$

Now we will introduce the curvature tensor for the connection ∇_X.

Using (3.2), by direct calculation one obtains

$$(\nabla_Y \circ \nabla_X - \nabla_X \circ \nabla_Y - \nabla_{[Y,X]}) y^j = R^j_q(X, Y) y^q, \tag{3.4}$$

where

$$R^j_q(X,Y) = Y^r \frac{\partial}{\partial x^r} G^j_q(X) - X^r \frac{\partial}{\partial x^r} G^j_q(Y) + \tag{3.5}$$

$$+ G^j_s(Y) G^s_q(X) - G^j_s(X) G^s_q(Y) - G^j_q([Y,X]).$$

Note that

$$R^j_q(X,Y) \neq Y^r \frac{\partial}{\partial x^r} G^j_q(X) - X^r \frac{\partial}{\partial x^r} G^j_q(Y) +$$

$$+ G^j_s(Y) G^s_q(X) - G^j_s(X) G^s_q(Y) + G^j_q([X,Y]),$$

$$R^j_q(X,Y) \neq -R^j_q(Y,X),$$

and $R^j_q(X,Y)$ is not linear with respect to X and Y. But using the properties (3.3) and (3.4), one can generalize the theorem 2.1 in this case. We will not do that, but we will obtain a further generalization considering a non-linear connection such that $\nabla(X+Y) \neq \nabla X + \nabla Y$ in general case also. Now instead of (3.2) we have

$$\nabla_X y^j = X^i \frac{\partial}{\partial x^i} y^j + F^j(x^1, \cdots, x^k, X^1, \cdots, X^k, y^1, \cdots, y^n). \tag{3.6}$$

Let $Y_{(\alpha)}$ $(1 \leq \alpha \leq m)$ be given m vector fields on M. In order to find vector field parallel along $Y_{(\alpha)}$ $(1 \leq \alpha \leq m)$ we have to solve the following system

$$Y^i_{(\alpha)} \frac{\partial}{\partial x^i} y^j + F^j_\alpha(x^1, \cdots, x^k, y^1, \cdots, y^n) = 0 \tag{3.7}$$

$$(1 \leq \alpha \leq m, 1 \leq j \leq n)$$

of partial differential equations.

We will find the compatibility conditions and the solution of (3.7). In order to do that, first we should transform the given system as follows. The fields $Y_{(\alpha)}$ generate an integrable distribution ζ of rank p $(m \leq p \leq k)$. If $p > m$, then for arbitrary α and β $(\alpha \neq \beta)$ we obtain new equations from (3.7) using the Jacobi brackets. Indeed it follows from (3.7) that

$$Y^r_{(\beta)} \frac{\partial}{\partial x^r} \Big(Y^i_{(\alpha)} \frac{\partial}{\partial x^i} y^j + F^j_\alpha \Big) - Y^r_{(\alpha)} \frac{\partial}{\partial x^r} \Big(Y^i_{(\beta)} \frac{\partial}{\partial x^i} y^j + F^j_\beta \Big) = 0, \text{ i.e.}$$

$$[Y_{(\beta)}, Y_{(\alpha)}]^i \frac{\partial}{\partial x^i} y^j + F^j = 0,$$

where

$$F^j = Y^r_{(\beta)} \frac{\partial}{\partial x^r} F^j_\alpha - Y^r_{(\alpha)} \frac{\partial}{\partial x^r} F^j_\beta$$

$$= Y^r_{(\beta)} \Big[\frac{\partial}{\partial x^r} F^j_\alpha (x^1, \cdots, x^k, y^1, \cdots, y^n) + \frac{\partial y^s \partial}{\partial x^r \partial y^s} F^j_\alpha (x^1, \cdots, x^k, y^1, \cdots, y^n) \Big]$$

$$-Y^r_{(\alpha)} \Big[\frac{\partial}{\partial x^r} F^j_\beta (x^1, \cdots, x^k, y^1, \cdots, y^n) + \frac{\partial y^s}{\partial x^r} \frac{\partial}{\partial y^s} F^j_\beta (x^1, \cdots, x^k, y^1, \cdots, y^n) \Big]$$

$$= Y^r_{(\beta)} \frac{\partial}{\partial x^r} F^j_\alpha (x^1, \cdots, x^k, y^1, \cdots, y^n) -$$

$$-F^s_\beta (x^1, \cdots, x^k, y^1, \cdots, y^n) \frac{\partial}{\partial y^s} F^j_\alpha (x^1, \cdots, x^k, y^1, \cdots, y^n) -$$

$$-Y^r_{(\alpha)} \frac{\partial}{\partial x^r} F^j_\beta (x^1, \cdots, x^k, y^1, \cdots, y^n) +$$

$$+F^s_\alpha (x^1, \cdots, x^k, y^1, \cdots, y^n) \frac{\partial}{\partial y^s} F^j_\beta (x^1, \cdots, x^k, y^1, \cdots, y^n).$$

Continuing this procedure, we obtain the following system

$$Y^i_{(\alpha)} \frac{\partial}{\partial x^i} y^j + F^j_\alpha (x^1, \cdots, x^k, y^1, \cdots, y^n) = 0 \tag{3.8}$$

$$(1 \leq \alpha \leq p, 1 \leq j \leq n)$$

which is equivalent to (3.7), and where $Y_{(\alpha)}$ $(1 \leq \alpha \leq p)$ is basis of ζ. Since ζ is an integrable distribution, there exist p vectors $X_{(\alpha)}$ $(1 \leq \alpha \leq p)$, $X_{(\alpha)} = C^\beta_\alpha Y_{(\beta)}$ such that $[X_{(\alpha)}, X_{(\beta)}] = 0$, and moreover there exist p functions $u^i(x^1, \cdots, x^k)$ $(1 \leq i \leq p)$ such that $X_{(\alpha)} u^i = \delta^i_\alpha$. Indeed $u^1, \cdots, u^k$ are coordinates on the integrable distribution ζ, and $X_{(\alpha)} = \partial / \partial u^\alpha$. Now form (3.8) we obtain the following system

$$X^i_{(\alpha)} \frac{\partial}{\partial x^i} y^j + C^\beta_\alpha F^j_\beta (x^1, \cdots, x^k, y^1, \cdots, y^n) = 0, \text{ i.e.}$$

$$X^i_{(\alpha)} \frac{\partial}{\partial x^i} y^j + \Phi^j_\alpha (x^1, \cdots, x^k, y^1, \cdots, y^n) = 0 \tag{3.9}$$

$$(1 \leq \alpha \leq p, 1 \leq j \leq n)$$

which is equivalent to (3.7). In order to consider the system (3.9), assume that Φ^j_β are analytical functions, and let

$$\Phi^j_\beta = \sum_{t\geq 0}\frac{1}{t!}\Gamma^j_{q_1\ldots q_t\beta}(x^1,\cdots,x^k)y^{q_1}\cdots y^{q_t}. \tag{3.10}$$

Considering a covariant derivative analogous to (3.19) from [6] and and also the curvature tensor, similarly to the theorem 3.2 [6] we have the following

Theorem 3.1. *Let the analytical system (3.9) with the initial conditions*

$$Y(0,\cdots,0)=Z \tag{3.11}$$

be given.

(i) The system (3.9) with (3.11) is integrable if and only if the following system of non-linear equations

$$\sum_{s\geq 0}\frac{1}{s!}R^i_{r_1\ldots r_s\alpha\beta}t^{r_1}\cdots t^{r_s}=0$$

$$\sum_{s\geq 0}\frac{1}{s!}R^i_{r_1\ldots r_s\alpha\beta;\gamma_1}t^{r_1}\cdots t^{r_s}=0$$

$$\sum_{s\geq 0}\frac{1}{s!}R^i_{r_1\ldots r_s\alpha\beta;\gamma_1;\gamma_2}t^{r_1}\cdots t^{r_s}=0 \tag{3.12}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{s\geq 0}\frac{1}{s!}R^i_{r_1\ldots r_s\alpha\beta;\gamma_1;\ldots;\gamma_{n-1}}t^{r_1}\cdots t^{r_s}=0$$

has an analytical solution t^j, such that

$$t^j\quad(0,\cdots,0)=z^j\quad(1\leq j\leq n),$$

where

$$\alpha,\beta,\gamma_1,\cdots,\gamma_{n-1}\in\{1,\cdots,p\}.$$

(ii) Let the compatibility conditions for the system (3.9) be satisfied, and let t^j be an arbitrary analytical solution of the system (3.12) with the initial conditions (3.11). Then there exist functions

$$Q^{r_1\ldots r_s<v_1,\ldots,v_p>}\quad(s\geq 0,r_1,\cdots,r_s\in\{1,\cdots,n\},v_1,\cdots,v_p\in N_0)$$

such that

$$Q^{<0,\cdots,0>}=1,\quad Q^{r_1\ldots r_s<0,\ldots,0>}=t^{r_1}\cdot t^{r_2}\cdots t^{r_s}, \tag{3.13}$$

$$Q^{<v_1,\ldots,v_u+1,\ldots,v_p>}=0,$$

$$Q^{r_1\ldots r_s<v_1,\ldots,v_u+1,\ldots,v_p>}=X^i_{(u)}\frac{\partial}{\partial x^i}Q^{r_1\ldots r_s<v_1,\ldots,v_p>}+ \tag{3.14}$$

$$+\sum_{t\geq 0}\sum_{i=1}^{s}\frac{1}{t!}\Gamma^{r_i}_{j_1\ldots j_t u}Q^{j_1\ldots j_t r_1\ldots\hat{r}_i\ldots r_s<v_1,\ldots,v_p>}$$

and the solution of (3.9) with (3.11) is given by

$$y^r=\sum_{v_1=0}^{\infty}\sum_{v_2=0}^{\infty}\cdots\sum_{v_p=0}^{\infty}\frac{(-u^1)^{v_1}}{v_1!}\frac{(-u^2)^{v_2}}{v_2!}\cdots\frac{(-u^p)^{v_p}}{v_p!}\times \tag{3.15}$$

$$\times Q^{r<v_1,\ldots,v_p>}\qquad (1\leq r\leq n).$$

The theorem 3.1 can be generalized for parallel tensor fields.

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