Variational and Conservative Principles in Rosen's Theory of Gravitation

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Abstract

A variational Action Principle is introduced, in the Rosen's Theory of Gravitation, in view of deriving field equations, motion equations and a canonical energy tensor. Using the constraint of metric invariance during the variational process along the trajectory, a certain relationship between the canonical tensor and the motion equations is established as a test for selfconsistency. Gravitostatics for spherical bodies is worked out, and the Equivalence Principle between gravitational mass and inertial mass does hold in a weak version (equality of masses, but not also of their space distributions).

Mathematics Subject Classification: 83CXX
Key words: General Relativity, Bimetric Theories, Field Equations, Gravitation

The field and motion equations, as well as the canonical energy tensor, may by derived, in the case of Rosen's theory of gravitation [1], from a certain Action Principle. Adopting the perfect fluid scheme for the Riemannian Matter tensor and specifying the field part of the Action as depending on Minkowskian quantities of definite variance, not exceeding the first order derivatives, the Action integral takes the form

(1) $$\mathcal{A} = \frac{\infty}{\rfloor} \int \mathcal{L}\sqrt{-\}}\left(\lceil^{\triangle}\S\right),$$

where,

(2) $$\begin{aligned} L &= L_m + L_f; \\ L_m &= \left[\left(c^2 + H\right)\rho - p\right]_R \cdot K,\ K = \frac{\sqrt{-\gamma}}{\sqrt{-g}}; \\ L_f &= \frac{c^4}{16\pi G} f\left(\chi^{\alpha\beta}, \gamma_{\alpha\beta|\lambda}; h^{\alpha\beta}\right). \end{aligned}$$

Here , a subscript R stands for specifying that the labelled quantity is defined in a Riemannian space-time, whose metric is

(3) $$\left(dS_R\right)^2 = \gamma_{\mu\nu} dx^\mu dx^\nu,\ \left(dS_R\right) > 0.$$

Balkan Journal of Geometry and Its Applications, Vol.4, No.1, 1999, pp. 31-43

Besides this, a flat space-time metric is adopted [2]

(4) $$(dS_M)^2 = g_{\mu\nu}dx^{\mu}dx^{\nu},\ (dS_M) > 0.$$

For ensuring coherence of the whole variational process, not only the coordinates x^{α} and the signatures (time-like) should be the same for the two metrics, but also the metric tensor $\gamma_{\mu\nu}$ of the Riemannian space-time must be considered as a tensor in M_4 (Minkowskian space time) [3], [4]. Arbitrary coordinates in M_4 are adopted (necessary for carrying out the variational calculations) and $x^0 = ct$ is taken as the time coordinate (with physical dimension of a length). So, the role of the two metrics is strongly dissymmetrized, $\gamma_{\mu\nu}$ are some quantities preserving only the meaning of gravitational potentials, and the metric $(dS_R)^2$ turns out to be a simple mathematical artifact necessary to formulate the specific coupling of gravitational field to its sources. This bimetric philosophy (which restricts the main role of Riemannian metric to the motion equations) entitles us to treat the quantities $\gamma_{\mu\nu}$ as true potentials, distinct from the metric functions $g_{\mu\nu}$. Now, performing the variations against $\gamma_{\mu\nu}$, in the action integral, we come to field equations; variation against $g_{\mu\nu}$ delivers a canonical energy tensor, while the variation against the coordinates of a fluid particle delivers motion equations.

For preventing confusion with respect to raising and lowering of indices, we denote distinctly the co- and contra- variant aspects of the metric tensors, always keeping in mind the Minkowskian character of the involved quantities. So, we write

(5) $$\begin{aligned} \gamma_{\mu\lambda}\chi^{\lambda\nu} &= \delta^{\nu}_{\mu},\ g_{\mu\lambda}h^{\lambda\nu} = \delta^{\nu}_{\mu} \\ \gamma^{\alpha\beta} &= \gamma_{\mu\nu}h^{\mu\alpha}h^{\nu\beta} \neq \chi^{\alpha\beta},\ \chi_{\alpha\beta} = \chi^{\mu\nu}g_{\mu\alpha}g_{\nu\beta} \neq \gamma_{\alpha\beta}\,. \end{aligned}$$

The quantity K (in (2)) is a Minkowskian scalar, while the quantities ρ (mass density), p (pressure) and H (Helmoltz potential) are scalar in both M_4 and R_4. H is defined as

(6) $$H = \int_0^{p(\rho)} \frac{dp}{\rho\,(p)},$$

this implying the existence of a reciprocal relationship between pressure and mass density

(7) $$p = p\,(\rho)\,,\ \rho = \rho\,(p)\,.$$

Finally, as a rule,

(8) $$\begin{aligned} \gamma &= Det\,\|\gamma_{\mu\nu}\|\,,\ g = Det\,\|g_{\mu\nu}\| \\ \gamma &< 0,\ g < 0 \end{aligned}$$

The Minkowskian covariant derivatives are denoted by a vertical bar followed by a certain (Greek) subscript, or (equivalently) by a derivative symbol (D) followed by the same subscript. For instance

(9) $$\gamma_{\alpha\beta|\lambda} \equiv D_{\lambda}\gamma_{\alpha\beta} \equiv \gamma_{\alpha\beta,\lambda} - G^{\sigma}_{\alpha\lambda}\gamma_{\sigma\beta} - G^{\sigma}_{\beta\lambda}\gamma_{\sigma\alpha},$$

where

(10) $$G^{\lambda}_{\mu\nu} = \frac{1}{2}h^{\lambda\sigma}\left(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}\right)$$

are the Christoffel symbols of the Minkowski space time, and $(,)$ stands for the usual partial derivative.

For obtaining the field equations and the energy canonical tensor, we resort to the following identity

$$\frac{1}{\sqrt{-g}}\delta\left(\sqrt{-g}L\right) \equiv D_\lambda q^\lambda - \frac{1}{2}K\left(T_{\alpha\beta} + \frac{c^4}{8\pi G}E_{\alpha\beta}\right)\delta\chi^{\alpha\beta} - \frac{1}{2}\mathcal{T}_{\alpha\beta}\delta\langle^{\alpha\beta}, \tag{11}$$

where

$$\begin{aligned}
q^\lambda &= P^{\alpha\beta\|\lambda}\delta\gamma_{\alpha\beta} + 2\gamma_{\sigma\nu}P^{\mu\nu\|\eta}\Omega^{\lambda\sigma}_{\eta\mu\|\alpha\beta}\delta h^{\alpha\beta}, \\
P^{\alpha\beta\|\lambda} &\equiv \frac{\partial L_f}{\partial\gamma_{\alpha\beta|\lambda}}, \\
\Omega^{\lambda\sigma}_{\mu\nu\|\alpha\beta} &\equiv \frac{1}{4}\left(\delta^\lambda_\beta\delta^\sigma_\nu g_{\mu\alpha} + \delta^\lambda_\alpha\delta^\sigma_\nu g_{\mu\beta} + \delta^\lambda_\beta\delta^\sigma_\mu g_{\nu\alpha} + \right. \\
&\quad \left. + \delta^\lambda_\alpha\delta^\sigma_\mu g_{\nu\beta} - h^{\lambda\sigma}g_{\mu\alpha}g_{\nu\beta} - h^{\lambda\sigma}g_{\mu\beta}g_{\nu\alpha}\right), \\
T_{\alpha\beta} &= \left\{\left(c^2 + H\right)\rho U_\alpha U_\beta - p\gamma_{\alpha\beta}\right\}_R , \\
E_{\alpha\beta} &\equiv R_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}\left(\chi^{\mu\nu}R_{\mu\nu}\right), \\
R_{\alpha\beta} &= -\frac{1}{K}\left\{\left(\frac{\partial f}{\partial\chi^{\alpha\beta}} - \frac{1}{2}\gamma_{\alpha\beta}\chi^{\mu\nu}\frac{\partial f}{\partial\chi^{\mu\nu}}\right) + \right. \\
&\quad \left. + \left(\gamma_{\mu\alpha}\gamma_{\nu\beta} - \frac{1}{2}\gamma_{\alpha\beta}\gamma_{\mu\nu}\right)D_\lambda p^{\mu\nu\|\lambda}\right\}, \\
p^{\alpha\beta\|\lambda} &= \frac{16\pi G}{c^4}P^{\alpha\beta\|\lambda}, \\
\mathcal{T}_{\alpha\beta} &= D_\sigma Q^\sigma_{\alpha\beta} - 2\left(\frac{\partial L_f}{\partial h^{\alpha\beta}} - \frac{1}{2}g_{\alpha\beta}L_f\right), \\
Q^\sigma_{\alpha\beta} &= 4\gamma_{\lambda\zeta}P^{\mu\zeta\|\nu}\Omega^{\lambda\sigma}_{\mu\nu\|\alpha\beta}.
\end{aligned} \tag{12}$$

The (trivially) conservative tensor Ω is coming from variation against $g_{\alpha\beta}$ of the Minkowskian affine connections [5].

Putting the variation of the action integral against $\chi^{\alpha\beta}$ to vanish, one obtains the field equations

$$R_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}\left(\chi^{\mu\nu}R_{\mu\nu}\right) = -8\pi\frac{G}{c^4}T_{\alpha\beta}\,. \tag{13}$$

The denotation are intentional, so as to reach just the Einsteinian expression. However, $R_{\alpha\beta}$ is not, necessarily, the Ricci curvature tensor, excepting the case when a certain special function is chosen for L_f. Even so, the tensor $E_{\alpha\beta}$ (defined in (12)) has more in common with its Einsteinian homologous, because it satisfies the Einsteinian conservative condition

$$\nabla_\mu\left(\chi^{\mu\alpha}\chi^{\nu\beta}E_{\alpha\beta}\right) = 0; \tag{14}$$

justifying the statement that Rosen's Theory is the most likely theory to the Einstein's one. The result (14) may be proved in the following way. Performing the variation of the action integral against the coordinates of a fluid particle one obtains (just as in Einstein's theory) [6] the geodetic type motion equation

$$\delta\int\left(1+\frac{1}{c^2}H\right)\sqrt{\gamma_{\mu\nu}dx^\mu dx^\nu}=0. \tag{15}$$

At the same time, the respective motion equations may be obtained by putting the Riemannian covariant divergence of the tensor T to vanish, accounting equally for the particle current conservation in both R_4 and M_4. Accordingly, $\nabla_\mu\left(\chi^{\mu\alpha}\chi^{\nu\beta}T_{\alpha\beta}\right)=0$ emerges as a consequence of the motion equation (15) and, through the result (13) one reach the result (14). To a certain extent, this may justify the interpretation of the bimetric theories as a natural extension of the Einstein's Gravitation Theory. Einstein's theory itself may be considered as a special bimetric theory for which the following identity does hold

$$\nabla_\mu\left(\chi^{\mu\alpha}\chi^{\nu\beta}E_{\alpha\beta}\right)\equiv 0; \tag{16}$$

Rosen's gravitation theory may be obtained out of the general bimetric theory so far presented by specifying the function f

$$\begin{aligned} f &= -\frac{1}{8}h^{\alpha\beta}W^{(\lambda\sigma)(\rho\tau)}\gamma_{\lambda\sigma|\alpha}\gamma_{\rho\tau|\beta}, \\ W^{(\lambda\sigma)(\rho\tau)} &\equiv \left(\chi^{\lambda\rho}\chi^{\sigma\tau}+\chi^{\sigma\rho}\chi^{\lambda\tau}-\chi^{\lambda\sigma}\chi^{\rho\tau}\right). \end{aligned} \tag{17}$$

The expression of the tensor $R_{\mu\nu}$ acquires the form

$$R_{\mu\nu}=\frac{1}{2K}\left(\Box\gamma_{\mu\nu}-\chi^{\lambda\sigma}\gamma_{\mu\lambda|\alpha}\gamma_{\nu\sigma}^{\ \ |\alpha}\right),\ \Box\equiv h^{\alpha\beta}D_\alpha D_\beta, \tag{18}$$

and the tensor $\mathcal{T}_{\alpha\beta}$ turns out to be

$$\begin{aligned} \mathcal{T}_{\alpha\beta} &= \frac{c^4}{32\pi G}\Big\{\chi^{\mu\lambda}\chi^{\nu\sigma}\left(\gamma_{\mu\nu|\alpha}\gamma_{\lambda\sigma|\beta}-\frac{1}{2}g_{\alpha\beta}\gamma_{\mu\nu|\eta}\gamma_{\lambda\sigma}^{\ \ |\eta}\right)- \\ &-2\left[(\ln K)_{,\alpha}(\ln K)_{,\beta}-\frac{1}{2}g_{\alpha\beta}h^{\mu\nu}(\ln K)_{,\mu}(\ln K)_{,\nu}\right]-P_{\alpha\beta}\Big\}, \end{aligned} \tag{19}$$

$$\begin{aligned} P_{\alpha\beta} \equiv\ & \left(\gamma_{\mu\beta|\alpha}\chi^{\mu\nu}\right)_{|\nu}+\left(\gamma_{\mu\alpha|\beta}\chi^{\mu\nu}\right)_{|\nu}+\left(\gamma_{\mu\beta}^{\ \ |\nu}\chi_\alpha^{\ \mu}\right)_{|\nu}+ \\ &+\left(\gamma_{\mu\alpha}^{\ \ |\nu}\chi_\beta^{\ \mu}\right)_{|\nu}-\left(\gamma_{\mu|\alpha}^{\ \nu}\chi_\beta^{\ \mu}\right)_{|\nu}-\left(\gamma_{\mu|\beta}^{\ \nu}\chi_\alpha^{\ \mu}\right)_{|\nu}-2g_{\alpha\beta}\Box\ln K. \end{aligned} \tag{20}$$

The contravariant derivative $\left(...^{|\alpha}\right)$ is lift via $h^{\alpha\beta}$.

There is a close analogy between the tensors $\mathcal{T}_{\alpha\beta}$ and $T_{\alpha\beta}+\dfrac{c^4}{8\pi G}E_{\alpha\beta}$, both being canonical energy tensors-the first one defined in M_4 and the second one in R_4. Their conservative character is conditioned-through the intermediary of motion equations-by the semi-local connection (i. e. along the trajectory of a fluid particle) of the metrical functions variation to the variation of the fluid particle coordinates [7]

$$\delta h^{\alpha\beta}=D_\lambda\left(h^{\lambda\alpha}\delta x^\beta+h^{\lambda\beta}\delta x^\alpha\right), \tag{21}$$

$$\delta\chi^{\alpha\beta}=\nabla_\lambda\left(\chi^{\lambda\alpha}\delta x^\beta+\chi^{\lambda\beta}\delta x^\alpha\right). \tag{22}$$

Here, ∇_λ stands for the covariant derivative in R_4. The formulas (21), (22) are a specific feature of the bimetric doctrine, because they mean conservation, under variational changes, of both curved and flat metrics. Out of (22) and (11) we obtain the identity

$$
\begin{aligned}
(23)\ \delta_\gamma\left(\sqrt{-g}L\right) &\equiv \frac{\partial}{\partial x^\lambda}\left(\sqrt{-g}P^{\alpha\beta\|\lambda}\delta\gamma_{\alpha\beta}\right)-\frac{1}{2}\sqrt{-\gamma}\left(T_{\alpha\beta}+\frac{c^4}{8\pi G}E_{\alpha\beta}\right)\delta\chi^{\alpha\beta}\\
&\equiv \frac{\partial}{\partial x^\lambda}\left(\sqrt{-g}P^{\alpha\beta\|\lambda}\delta\gamma_{\alpha\beta}\right)-\frac{\partial}{\partial x^\lambda}\left\{\sqrt{-\gamma}\left(\delta x^\sigma\right)\left[\left(\frac{T^{\lambda}_{\cdot\sigma}+T^{\cdot\lambda}_{\sigma}}{2}\right)+\right.\right.\\
&+\left.\left.\frac{c^4}{8\pi G}\left(\frac{E^{\lambda}_{\cdot\sigma}+E^{\cdot\lambda}_{\sigma}}{2}\right)\right]\right\}+\sqrt{-\gamma}\left(\delta x^\sigma\right)\nabla_\lambda\left[\left(\frac{T^{\lambda}_{\cdot\sigma}+T^{\cdot\lambda}_{\sigma}}{2}\right)+\right.\\
&+\left.\frac{c^4}{8\pi G}\left(\frac{E^{\lambda}_{\cdot\sigma}+E^{\cdot\lambda}_{\sigma}}{2}\right)\right],
\end{aligned}
$$

where $T_{\alpha\beta}$ is a symmetrical tensor in both R_4 and M_4 and the index raising is made with the help of $\chi^{\mu\nu}$. By asking the condition

$$(24)\qquad \delta_\gamma\mathcal{A}=\prime,$$

we come to a (Riemannian) conservative tensor

$$(25)\qquad \tau_{\alpha\beta}\equiv T_{\alpha\beta}+\frac{c^4}{8\pi G}E_{\alpha\beta},\ \nabla_\mu\left(\chi^{\mu\alpha}\chi^{\nu\beta}\tau_{\alpha\beta}\right)=0.$$

On the other hand, taking in view that (in R_4)

$$
\begin{aligned}
(26)\qquad \delta_x L_R &= \frac{\partial L_R}{\partial\rho}\delta_x\rho,\ \delta_x\rho=\rho U_\sigma U^\nu\nabla_\nu\left(\delta x^\sigma\right)-\nabla_\sigma\left(\rho\delta x^\sigma\right);\\
\nabla_\nu\left(\rho U^\nu\right) &= 0,\ U^\alpha=\frac{dx^\alpha}{dS},\ dS=\left(\gamma_{\mu\nu}dx^\mu dx^\nu\right)^{\frac{1}{2}},\ \sqrt{-\gamma}L_R=\sqrt{-g}L,
\end{aligned}
$$

we have equally

$$
\begin{aligned}
(27)\qquad \delta_x\left(\sqrt{-g}L\right) &= \frac{\partial}{\partial x^\nu}\left\{\sqrt{-\gamma}\rho\frac{\partial L_R}{\partial\rho}\delta x^\sigma\left(U^\nu U_\sigma-\delta^\nu_\sigma\right)\right\}-\\
&- \sqrt{-\gamma}\left(\delta x^\sigma\right)\rho\frac{\partial L_R}{\partial\rho}\gamma_{\sigma\lambda}\left\{\frac{d^2x^\lambda}{dS^2}+\Gamma^\lambda_{\alpha\beta}U^\alpha U^\beta+\right.\\
&+ \left.\left(U^\lambda U^\nu-\chi^{\lambda\nu}\right)\nabla_\nu\ln\left(\frac{1}{c^2}\frac{\partial L_R}{\partial\rho}\right)\right\}.
\end{aligned}
$$

Now, taking in view that (onto the trajectory and with geodetic constraint (22))

$$\delta_\gamma\mathcal{A}+\delta_{\S}\mathcal{A}=\prime,$$

we come to the relevant result

$$(28)\qquad \nabla_\lambda\left(T^{\lambda\mu}+\frac{c^4}{8\pi G}E^{\lambda\mu}\right)=\rho\frac{\partial L_R}{\partial\rho}\omega^\mu,$$

where

$$\omega^{\mu} \equiv \frac{d^2x^{\mu}}{dS^2} + \Gamma^{\mu}_{\alpha\beta}U^{\alpha}U^{\beta} + \left(U^{\mu}U^{\nu} - \chi^{\mu\nu}\right)\frac{\partial}{\partial x^{\nu}}\ln\left(\frac{1}{c^2}\frac{\partial L_R}{\partial\rho}\right). \tag{29}$$

But, equation (28) may be written (equivalently) as

$$\nabla_{\lambda}\left(T^{\lambda\mu} + \frac{c^4}{8\pi G}E^{\lambda\mu}\right) = -\frac{1}{m_0}\rho\chi^{\mu\lambda}Q_{\lambda}, \tag{30}$$

where

$$\begin{aligned} Q_{\lambda} &\equiv \frac{d}{dS}\left(\frac{\partial\mathcal{L}}{\partial U^{\lambda}}\right)_x - \left(\frac{\partial\mathcal{L}}{\partial x^{\lambda}}\right)_U, \\ \mathcal{L} &\equiv -m_0\frac{\partial L_R}{\partial\rho}\sqrt{\gamma_{\alpha\beta}U^{\alpha}U^{\beta}}, \\ Q_{\lambda} &\equiv -m_0\frac{\partial L_R}{\partial\rho}\gamma_{\mu\lambda}\omega^{\mu}, \end{aligned} \tag{31}$$

and it is transparent that $Q_{\lambda} = 0$ are the Euler-Lagrange equations of motion coming from the variational problem

$$\delta\int\mathcal{L}dS \equiv \delta\left(-m_0\int\frac{\partial L_R}{\partial\rho}\sqrt{\gamma_{\alpha\beta}dx^{\alpha}dx^{\beta}}\right) = 0. \tag{32}$$

Asking the condition (25) in (28) we come to the geodetic motion equation (32). As the equation (32) may be recovered asking only $\nabla_{\lambda}T^{\lambda\mu} = 0$, we reach the result (14), Q.E.D.

One of the specific features of the General Relativity Theory, namely, the derivation of the motion equations out of the field equations is to be found equally in Rosen's Bimetric Theory. Indeed, by putting the tensor $\tau_{\alpha\beta}$ to vanish all over the 4-dimensional space-time, we get the field equations (13). At the same time, this condition entails the vanishing of the covariant derivative of the respective tensor as well (25), which, through the intermediary of (28) or (30), delivers the motion equations ($\omega^{\mu} = 0$ or, equivalently, $Q_{\lambda} = 0$).

An alternative way of calculating the total variation of action integral, along a fluid particle trajectory, against the particle coordinates variations, is to resort to the geodetic constraint (21) instead of (22). To notice that the condition (21) means the keeping of the Minkowskian metrics invariance during the variational process, while the condition (22) implies the same status for the Riemannian metrics. The putting on the same footing the two metrics may be considered as a basic feature of the bimetrism, regarded from the standpoint of the variational process. Thus, we can write

$$\begin{aligned} \delta_g\left(\sqrt{-g}L\right) + \delta_x\left(\sqrt{-g}L\right) &= \frac{\partial}{\partial x^{\lambda}}\Big\{\sqrt{-g}\Big[2\gamma_{\sigma\mu}P^{\mu\nu\|\eta}\cdot\Omega^{\lambda\sigma}_{\eta\nu\|\alpha\beta}\delta h^{\alpha\beta} - \\ &- \frac{1}{2}\left(\mathcal{T}^{\lambda}_{\cdot\alpha} + \mathcal{T}^{\cdot\lambda}_{\alpha}\right)\delta x^{\alpha} + \\ &+ \left(\rho\frac{\partial L}{\partial\rho}\right)_R\left(U^{\lambda}U_{\alpha} - \delta^{\lambda}_{\alpha}\right)\delta x^{\alpha}\Big]\Big\} + \end{aligned} \tag{33}$$

$$+ \quad \sqrt{-g}\,(\delta x^{\alpha}) \left\{ D_{\lambda} \left(\frac{\mathcal{T}^{\lambda}_{\cdot\alpha} + \mathcal{T}^{\cdot\lambda}_{\alpha}}{2} \right) - \right.$$

$$\left. - \quad K \left(\rho \frac{\partial L}{\partial \rho} \right)_{R} \gamma_{\alpha\lambda} \omega^{\lambda} \right\};$$

(34)
$$\delta_{g} \mathcal{A} + \delta_{\S} \mathcal{A} = /;$$

(35)
$$D_{\lambda} \mathcal{T}^{\lambda\mu} - \mathcal{K} \cdot \left(\rho \frac{\partial \mathcal{L}}{\partial \rho} \right)_{\mathcal{R}} \langle^{\alpha\mu} \gamma_{\alpha\lambda} \omega^{\lambda} = /;$$

whence (accounting for the motion equations $\omega^{\lambda} = 0$), we conclude about the Minkowskian conservativity of tensor $\mathcal{T}^{\alpha\beta}$.

An unsatisfactory aspect of the tensor $\mathcal{T}^{\alpha\beta}$ is the presence in its structure of the second order derivatives of the potentials. These appear through the intermediary of the tensor $P_{\alpha\beta}$ and cannot be put all in D'Alembert form, in order to be avoided by resorting to the field equations. Indeed, we can write

(36)
$$P_{\alpha\beta} = \left(\chi^{\mu}_{\beta} \Box \gamma_{\mu\alpha} + \chi^{\mu}_{\alpha} \Box \gamma_{\mu\beta} - 2 g_{\alpha\beta} \Box \ln K \right) + \left(\gamma_{\mu\beta}^{\;\;|\nu} \chi^{\;\mu}_{\alpha\;|\nu} + \gamma_{\mu\alpha}^{\;\;|\nu} \chi^{\;\mu}_{\beta\;|\nu} \right) + \Pi_{\alpha\beta},$$

(37)
$$\Pi_{\alpha\beta} = \left(\gamma_{\mu\beta|\alpha} \chi^{\mu\nu} + \gamma_{\mu\alpha|\beta} \chi^{\mu\nu} - \gamma_{\mu\;|\alpha}^{\;\nu} \chi^{\mu}_{\beta} - \gamma_{\mu\;|\beta}^{\;\nu} \chi^{\mu}_{\alpha} \right)_{|\nu}.$$

If the tensor $\Pi_{\alpha\beta}$ (which gathers all the non-D'Alembertian second order terms of $\mathcal{T}^{\alpha\beta}$) would have an identically vanishing covariant divergence, any difficulty will no longer survive. Unfortunately, this is not the case, because we have instead

(38)
$$\xi^{\alpha} \equiv \Pi^{\alpha\beta}_{\;\;\;|\beta} = \left(\gamma_{\mu}^{\;\alpha|\nu} \chi^{\mu\beta} - \gamma_{\mu}^{\;\beta|\nu} \chi^{\mu\alpha} \right)_{|\nu\beta} \neq 0\,,$$

(39)
$$\xi^{\alpha}_{\;|\alpha} \equiv 0.$$

Further on , we shall prove that, however, the tensor $\mathcal{T}^{\alpha\beta}$ does agree with the equivalence principle in a weak version (i. e. equality between inertial and gravitational masses, but not between their spatial densities). In this respect, we need first to write down the field equations for a static spherically symmetric mass distribution. We denote

(40)
$$R_{\mu\nu} = \frac{1}{K} N_{\mu\nu},\; N_{\mu\nu} = \frac{1}{2} \left(\Box \gamma_{\mu\nu} - \varepsilon_{\mu\nu} \right),\; \varepsilon_{\mu\nu} = \chi^{\lambda\sigma} \gamma_{\mu\lambda|\alpha} \gamma_{\nu\sigma}^{\;\;|\alpha},$$

and write down the field equations in the form

(41)
$$N_{\alpha\beta} = -\frac{8\pi G}{c^4} K \left(T_{\alpha\beta} - \frac{1}{2} \gamma_{\alpha\beta} T \right)_{R}.$$

Thereafter we adopt polar coordinates and a diagonal form for metrical tensor $\gamma_{\alpha\beta}$

(42)
$$\begin{aligned} x^0 &= ct,\;\; x^1 = r,\;\; x^2 = \theta,\;\; x^3 = \varphi; \\ \gamma_{00} &= e^{+\nu},\;\; \gamma_{11} = -e^{+\lambda},\;\; \gamma_{22} = -r^2 e^{+\sigma},\;\; \gamma_{33} = -r^2 \sin^2\theta e^{+\sigma}, \end{aligned}$$

and, after a long and tedious calculation, we obtain [8]

$$\begin{aligned}\Box e^{\nu} &= -\triangle_r e^{\nu},\ -\Box e^{\lambda} = \triangle_r e^{\lambda} - \frac{4}{r^2}\left(e^{\lambda} - e^{\sigma}\right), \\ -\Box\left(r^2 e^{\sigma}\right) &= r^2 \triangle_r e^{\sigma} + 2\left(e^{\lambda} - e^{\sigma}\right), \\ -\Box\left(r^2 \sin^2\theta e^{\sigma}\right) &= \sin^2\theta\left[r^2 \triangle_r e^{\sigma} + 2\left(e^{\lambda} - e^{\sigma}\right)\right];\end{aligned} \tag{43}$$

$$\begin{aligned}\varepsilon_{00} &= -\nu'^2 e^{\nu},\ \varepsilon_{11} = \lambda'^2 + \frac{2}{r^2} e^{-\sigma}\left(e^{\sigma} - e^{\lambda}\right)^2, \\ \varepsilon_{22} &= r^2 \sigma'^2 e^{\sigma} + e^{-\lambda}\left(e^{\sigma} - e^{\lambda}\right)^2, \\ \varepsilon_{33} &= \sin^2\theta\left[r^2 \sigma'^2 e^{\sigma} + e^{-\lambda}\left(e^{\sigma} - e^{\lambda}\right)^2\right];\end{aligned} \tag{44}$$

$$\begin{aligned}N_{00} &= -\frac{1}{2} e^{\nu} \triangle_r \nu,\ N_{11} = \frac{1}{2} e^{\lambda}\left[\triangle_r \lambda + \frac{4}{r^2}\sinh(\sigma - \lambda)\right], \\ N_{22} &= \frac{1}{2} r^2 e^{\sigma}\left[\triangle_r \sigma - \frac{2}{r^2}\sinh(\sigma - \lambda)\right],\ N_{33} = N_{22}\sin^2\theta .\end{aligned} \tag{45}$$

But, as a result of the specific form of the tensor $T_{\alpha\beta}$ in the perfect fluid scheme, we get the subsidiary condition among the components of the $N_{\alpha\beta}$ tensor

$$N_{22} = r^2 e^{\sigma - \lambda} N_{11} . \tag{46}$$

The previous condition may be written, in explicit form as

$$\triangle_r(\sigma - \lambda) - \frac{6}{r^2}\sinh(\sigma - \lambda) = 0 . \tag{47}$$

So, we come to the solution

$$\sigma = \lambda . \tag{48}$$

Now, by denoting

$$\nu = -2\frac{\psi_1}{c^2},\quad \lambda = +2\frac{\psi_2}{c^2}, \tag{49}$$

we reach the so called *exponential metric*

$$(dS)_R^2 = e^{-2\frac{\psi_1}{c^2}}(cdt)^2 - e^{+2\frac{\psi_2}{c^2}}\left\{(dr)^2 + r^2\left[(d\theta)^2 + \sin^2\theta\,(d\varphi)^2\right]\right\}, \tag{50}$$

where

$$\begin{aligned}\triangle_r \psi_1 &= -4\pi G e^{\frac{3\psi_2 - \psi_1}{c^2}}\left[\rho\left(1 + \frac{1}{c^2}H\right) + 2\frac{p}{c^2}\right]_R . \\ \triangle_r \psi_2 &= -4\pi G e^{\frac{3\psi_2 - \psi_1}{c^2}}\left[\rho\left(1 + \frac{1}{c^2}H\right) - 2\frac{p}{c^2}\right]_R .\end{aligned} \tag{51}$$

Taking the source body to be a sphere of radius R, we define two masses, namely

$$M_1 = \int_0^R 4\pi r^2 \rho_1 dr\ ,\ M_2 = \int_0^R 4\pi r^2 \rho_2 dr, \tag{52}$$

where the densities $(\rho_1,\ \rho_2)$ are those from (51), i. e. $\triangle_r\psi_1(r) = -4\pi G\rho_1,\ \triangle_r\psi_2(r) = -4\pi G\rho_2$. To notice that, even outside the mass distribution, the metric $(dS_R)^2$ contains two potential functions, (unlike the simple exponential metric of Yilmaz and Rastall, which contains a single such function) [9], [10]

$$(dS_R)^2 = e^{-2\frac{\mu_1}{r}}(cdt)^2 - e^{+2\frac{\mu_2}{r}}(d\vec{r})^2\,, \tag{53}$$

$$\mu_1 = \frac{GM_1}{c^2},\ \mu_2 = \frac{GM_2}{c^2},\ M_1 \neq M_2\,. \tag{54}$$

The presence of two masses M_1 and M_2 in the static metric may be considered as a second unsatisfactory aspect of the Rosen's bimetric theory of gravitation.

Coming back to the canonical energy tensor, and calculating the $\mathcal{T}_{\prime\prime}$ component, we obtain, in the static case of a spherical source body

$$\mathcal{T}_{\prime\prime} = \frac{\rfloor^{\triangle}}{\ni\in\pi\mathcal{G}}\left[\mathcal{A}+\mathcal{B}-\mathcal{P}_{\prime\prime}\right], \tag{55}$$

$$\begin{aligned} A &= -\frac{1}{2}\chi^{\lambda\sigma}\chi^{\mu\nu}\gamma_{\lambda\mu|\alpha}\gamma_{\sigma\nu}^{\ \ |\alpha} = \frac{2}{c^4}\left(3\psi_2'^2+\psi_1'^2\right)\,, \\ B &= (\ln K)_{|\alpha}(\ln K)^{|\alpha} = -\frac{1}{c^4}\left(3\psi_2'-\psi_1'\right)^2\,, \\ P_{00} &= \frac{2}{c^2}\triangle_r\left(3\psi_2+\psi_1\right) = -\frac{8\pi G}{c^2}\left(3\rho_2+\rho_1\right)\,. \end{aligned} \tag{56}$$

We notice the elimination of the non-D'Alembertian terms of $\mathcal{T}_{\prime\prime}$ in the static case. Thus

$$\mathcal{T}_{\prime\prime} = \left(\frac{\ni}{\triangle}\rho_{\in}+\frac{\infty}{\triangle}\rho_{\infty}\right)\rfloor^{\in} + \frac{\infty}{\ni\in\pi\mathcal{G}}\left(\psi_\infty'^{\in}+\psi_\infty'\psi_\in'-\ni\psi_\in'^{\in}\right)\,. \tag{57}$$

Using the Eddington's definition of the gravitational mass together with the correspondence principle between a relativistic theory of gravitation and the Newtonian theory, we come to the formula

$$M_g = \lim_{r\to\infty}\left(\frac{1}{G}r\psi_1\right) = \int_0^R 4\pi r^2\rho_1(r)\,dr\,. \tag{58}$$

At the same time, the inertial mass is defined as

$$M_i = \frac{1}{c^2}\int_0^\infty \mathcal{T}_{\prime\prime}\triangle\pi\nabla^{\in}\lceil\nabla\,. \tag{59}$$

The weak version of the equivalence principle between the two kinds of mass requires the vanishing of the following integral

$$I \equiv \int_0^\infty\left(\frac{1}{c^2}\mathcal{T}_{\prime\prime}-\rho_\infty\right)4\pi r^2dr\,. \tag{60}$$

Accounting for (57), the respective integral becomes

$$I = \int_0^\infty\left\{\frac{3}{4}\left(\rho_2-\rho_1\right)+\frac{1}{32\pi Gc^2}\left(\psi_1'^2+6\psi_1'\psi_2'-3\psi_2'^2\right)\right\}4\pi r^2dr\,. \tag{61}$$

But from (49) and (50), we obtain for the difference $\rho_2 - \rho_1$ the expression

$$\rho_2 - \rho_1 = -4\frac{p_R}{c^2} e^{\frac{3\psi_2 - \psi_1}{c^2}} \tag{62}$$

so that the integral I may be thought to the form

$$I = -\frac{1}{c^2}\int_0^\infty \left\{ 3p_R e^{\frac{3\psi_2-\psi_1}{c^2}} - \frac{1}{32\pi G}\left(\psi_1'^2 + 6\psi_1'\psi_2' - 3\psi_2'^2\right)\right\} 4\pi r^2 dr \,. \tag{63}$$

In the Newtonian approximation, we can write

$$p_R = p_N + O\left(\frac{1}{c^2}\right),\ \psi_1 = \psi_N + O\left(\frac{1}{c^2}\right),\ \psi_2 = \psi_N + O\left(\frac{1}{c^2}\right), \tag{64}$$

and the integral (63) becomes

$$I = -\frac{1}{c^2}\int_0^\infty \left(3p_N - \frac{\psi_N'^2}{8\pi G}\right) 4\pi r^2 dr + O\left(\frac{1}{c^4}\right) . \tag{65}$$

The equilibrium pressure p and the gravitational potential ψ are expressed, in the nonrelativistic approximation, by the formulas

$$p_N = -\int_r^R \rho\psi' dr,\ \triangle_r \psi_N = -4\pi G \rho_N , \tag{66}$$

$$\rho_N = \begin{cases} \rho_N(r) & r \in (0,R) \\ 0 & r \in (R,\infty) \end{cases} \quad p_N = \begin{cases} p_N(r) & r \in (0,R) \\ 0 & r \in (R,\infty) \end{cases} \tag{67}$$

so that the integral we want to estimate may be transformed in a suitable way

$$I = -\frac{1}{c^2}\int_0^R \left(3p_N - \frac{1}{2}\rho_N \psi_N\right) 4\pi r^2 dr + O\left(\frac{1}{c^4}\right) . \tag{68}$$

Now, inserting p_N from (66) in (68) and working out the necessary calculations we come to the result we look for

$$\int_0^R \left(3p_N - \frac{1}{2}\rho_N\psi_N\right) 4\pi r^2 dr \equiv 0,\ \to I = O\left(\frac{1}{c^4}\right), \tag{69}$$

id est

$$M_i - M_g = O\left(\frac{1}{c^4}\right) . \tag{70}$$

This is the quantitative expression of the statement made by Nathan Rosen (without proof), that his theory does comply with the equivalence principle.

Conclusions

The variational formulation of the Rosen's Theory of Gravitation in the perfect fluid scheme and the derivation of the energy tensor are entirely original. The proof of the equivalence principle between the inertial mass and the gravitational mass is also given here for the first time. In the original Rosen's work what is proved, in connection with this principle, is the equivalence between the inertial force and the gravitational force. The similarity between the Einstein's Theory and the Rosen's one is given in our paper, pointing out the bimetric feature of the two theories. Finally, a clear difference between the Rosen's exponential metric and the similar metrics from the literature is put into evidence.

Acknowledgement. The authors would like to thank to the reviewer and editors of BJGA for several helpful comments on our preliminary version of this paper. One of the authors (I. D.) is obliged to Professor Ioan Gottlieb for his constant moral assistance and useful advice. Thanks are also conveyed to Phys. Dr. Marius Piso, Chief of the Gravitation Research Laboratory for his encouragement during the elaboration of this work and to Miss Sylvia Onofrei for her interest in this subject in the initial phase of the work.

Appendix

General static metric of spherical symmetry

Solving eq (47) for $\sigma \neq \lambda$ one obtains

$$\begin{aligned} u(\rho) &= 0.987\,606\,7 \times 10^{0}\rho^{2} + 2.675\,775\,0 \times 10^{-2}\rho^{6} \\ &+ 1.204\,552\,6 \times 10^{-3}\rho^{10} + 6.421\,986\,8 \times 10^{-5}\rho^{14} \\ &+ 3.718\,234\,4 \times 10^{-6}\rho^{18} + 2.260\,255\,4 \times 10^{-7}\rho^{22} \\ &+ 1.418\,832\,9 \times 10^{-8}\rho^{26} + 9.111\,626\,1 \times 10^{-10}\rho^{30} \end{aligned}$$

$$0 < \rho \leq 1\,,$$

$$\begin{aligned} u(\rho) &= 1.000\,000\,0 \times 10^{0}\rho^{-3} + 1.515\,151\,5 \times 10^{-2}\rho^{-9} \\ &+ 4.679\,144\,4 \times 10^{-4}\rho^{-15} + 1.707\,923\,6 \times 10^{-5}\rho^{-21} \\ &+ 6.725\,845\,2 \times 10^{-7}\rho^{-27} + 2.766\,368\,7 \times 10^{-8}\rho^{-33} \\ &+ 1.170\,356\,0 \times 10^{-9}\rho^{-39} + 5.050\,235\,9 \times 10^{-11}\rho^{-45} \end{aligned}$$

$$1 \leq \rho < \infty,$$

where $u = \sigma - \lambda$ and $\rho = r/r_0$, r_0 standing for an arbitrary constant with physical dimension of a length. The metric acquires the form

$$\begin{aligned} (dS)^2 &= e^{-\frac{2}{c^2}\psi_1}(cdt)^2 - e^{+\frac{2}{c^2}\psi_2} \cdot \\ &\cdot \left\{ e^{-\frac{2}{3}u} \cdot \frac{(\mathbf{r}\cdot d\mathbf{r})^2}{r^2} + e^{+\frac{1}{3}u}\frac{(\mathbf{r}\times d\mathbf{r})^2}{r^2} \right\}. \end{aligned}$$

For a constant mass density inside the spherical source, we have

$$\psi_1 = \frac{GM}{r},\ \psi_2 = \frac{GM}{r}\left(1 - \frac{4}{5}\frac{GM}{c^2 R}\right),\ r > R\,.$$

The equation

$$\frac{d^2u}{d\rho^2} + \frac{2}{\rho}\frac{du}{d\rho} - \frac{6}{\rho^2}\sinh u = 0$$

is fulfilled within a relative error of 10^{-6} and $u(\rho)$ is continuous in the point $\rho = 1$. No prescription is made in the framework of Rosen's bimetric theory for determining r_0. Fock's gauge does not apply because $\psi_1 \neq \psi_2$. However, if r_0 has the magnitude order of GM/c^2, then the gravitational tests are not changed by the presence of the function $u(\rho)$.

The following empirical rules concerning $u(\rho)$ may be established

$$
\begin{aligned}
u(\rho) &= \sum_{n=1}^{n=\infty} a_n \rho^{4n-2} \\
a_1 &= 0.987\,606\,7 \\
\frac{a_{n+1}}{a_n} &= \left(7.339\,333\,7 - 6.720\,578\,5\frac{1}{n} + 2.090\,597\,6\frac{1}{n^2}\right) \times 10^{-2}
\end{aligned}
$$

$$0 \leq \rho \leq 1,$$

$$
\begin{aligned}
u(\rho) &= \sum_{n=1}^{n=\infty} b_n \rho^{-6n+3} \\
b_1 &= 1 \\
\frac{b_{n+1}}{b_n} &= \left(4.829\,979\,3 - 3.652\,148\,1\frac{1}{n} + 0.337\,320\,3\frac{1}{n^2}\right) \times 10^{-2}
\end{aligned}
$$

$$1 \leq \rho < \infty .$$

For checking the efficiency of these approximations, we calculated

$$
\begin{aligned}
u_{1-0} &= u_{1+0} = 1.015\,637\,2 \\
\triangle u_{1-0} &= 7.196\,849\,4;\ \triangle u_{1+0} = 7.196\,847\,7 \\
\left(\frac{6}{\rho^2}\sinh u\right)_{1\mp 0} &= 7.196\,851\,9 .
\end{aligned}
$$

For determining the coefficients of the empirical formulas a_{n+1}/a_n, b_{n+1}/b_n, the first few terms of the series expansions of $u(\rho)$ were directly calculated

$$
\begin{aligned}
a_2 &= \frac{1}{66}a_1^3,\ a_3 = \frac{7}{14\,960}a_1^5,\ a_4 = \frac{181}{10\,597\,664}a_1^7 \\
b_2 &= \frac{1}{36}b_1^3,\ b_3 = \frac{1}{780}b_1^5,\ b_4 = \frac{281}{4\,009\,824}b_1^7 .
\end{aligned}
$$

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