Rational Representations of Some Algebraic Varieties of M-Orthogonal Groups

Mircea Geanău

Abstract

The purpose of this paper is the study of the algebraic variety of the M-orthogonal group, insisting on the rational representations of M-orthogonal matrices. We shall follow the study of a certain subgroup of the M-orthogonal group and we will give a parametric rational representation of this variety.

Mathematics Subject Classification: 20G20
Key words: algebraic variety, rational representation, M-orthogonal group.

Let $GL(n, \mathbf{R})$ be the group of real non-singular matrices of n-th order and let M be a fixed matrix of n-th order that doesn't necessarily belong to $GL(n, \mathbf{R})$. We shall consider the following nonempty set denoted by

$$G_M = \{A \in GL(n, R) | AM\tilde{A} = M\},$$

where $\tilde{A}$ is the transpose of A. It is almost obvious that the set G_M having as a law composition the multiplication of the matrices is a group.

By definition, G_M is the M-orthogonal group. In the study of the group G_M, two cases, M non-singular and M singular assert themselves.

a) Let suppose that M is non-singular and symmetric.

Because it exists $1 + 2 + ... + n = \dfrac{n(n+1)}{2}$ independent relations, it results that the variety depends of $n^2 - \dfrac{n(n+1)}{2} = \dfrac{n(n-1)}{2}$ parameters.

Proposition 1. *If M is non-singular and symmetric matrix, the relation $A = (M + X)^{-1}(M - X)$ with $\tilde{X} = -X$ and $M^2X = XM^2$, defines a rational parametric representation of the M-orthogonal group G_M in the neighbourhood determined by the condition $|M + X| \neq 0$.*

Proof. We must show that the equality

$$(M + X)^{-1}(M - X)M(M + X)(M - X)^{-1} \tag{1}$$

Balkan Journal of Geometry and Its Applications, Vol.3, No.2, 1998, pp. 69-74

is true.

Because $M^2X = XM^2$, the equality

$$(M - X)M(M + X) = (M + X)M(M - X)$$

is fulfilled which implies the valability of the relation (1).

Note: By keeping fulfilled the conditions from Proposition 1, it shows as above that the relation $A = (M + X)(M - X)^{-1}$, with $|M - X| \neq 0$, gives a parametric rational representation of the M-orthogonal group in the neighbourhood determined by $|M - X| \neq 0$.

From the equality $A = (M + X)^{-1}(M - X)$, multiplied at left by $(M + X)$ it results that

$$X = M(E - A)(E + A)^{-1} \text{ with } |A + E| \neq 0$$

and E is the identity matrix.

Example 1. Taking

$$M = E = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

it is obtained a parametric representation of the orthogonal group,

$$A = (E + X)^{-1}(E - X) \text{ with } |E + A| \neq 0, \quad A \in O(n, R).$$

2. In the particular case

$$M = \begin{pmatrix} -1 & & & & & 0 \\ & -1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ 0 & & & & & 1 \end{pmatrix},$$

where -1 appears p times, with $M^2 = E$, it is obtained the parametric representation of the pseudo-orthogonal group.

b) Let's take the case when the matrix M is non-singular and antisymmetric.

Because $M = AM\tilde{A}$ is antisymmetric it results that the dimension $n = 2p$ and it exists $\dfrac{n(n-1)}{2}$ independent relations and in that case the algebraic variety has $n^2 - \dfrac{n(n-1)}{2} = \dfrac{n(n+1)}{2}$ parameters. Considering X a symmetric matrix and making similar reasonings with those from above its is proved just like in the case of Proposition 1 the following result

Proposition 2. *If the matrix M is non-singular and antisymmetric, the equality $A = (M + X)^{-1}(M - X)$, with $\tilde{X} = X$ and $M^2X = XM^2$, defines a rational parametric representation of the M-orthogonal group G_M in the neighbourhood limited by $|M + X| \neq 0$.*

Example. Let

$$M = \begin{pmatrix} 0 & E \\ \dots & \dots \\ -E & 0 \end{pmatrix}$$

be an antisymmetric matrix of $2n$-th order where E is the identity matrix of n-th order. Because are fulfilled the conditions of Proposition 2 we get a parametric representation of the symplectic group $S_p(2n, \mathbf{R})$.

c) Let's suppose that M is a singular and symmetric matrix. There are

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

independent and distinct relations so the variety has

$$n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$$

parameters.

In the case when $n = 2$, due to the fact that M^2 is also symmetric and singular, if it exists an antisymmetric matrix X with $M^2 X = X M^2$, it results that $M^2 = 0$, this one implies $M = 0$, which shows that it doesn't exist a parametric representation of the form $A = (M + X)^{-1}(M - X)$.

If $n \geq 3$, using a similar computation with the one used at Proposition 1, it is shown that the equality $A = (M + X)^{-1}(M - X)$, with $\tilde{X} = -X$ and $M^2 X = X M^2$, defines a rational parametric representation of the M- orthogonal group G_M in the neighbourhood given by the $|M + X| \neq 0$.

In the hypothesis that M is singular and antisymmetric, $n \geq 3$, analogous to the Proposition 2, the relation $A = (M+X)^{-1}(M-X)$, with $\tilde{X} = X$ and $M^2 X = X M^2$, defines a rational parametric representation of the group G_M in the neighbourhood defined by $|M + X| \neq 0$.

Keeping the above results and notations we shall treat a more general case, that of the subset of G_M, noted $H_{M,N}$ defined such that,

$$H_{M,N} = \{A \in GL(n, R), \quad AM\tilde{A} = M, \quad AN = NA\},$$

where M and N are two fixed matrices of n-th order.

It is obvious that H is nonempty and is a subgroup of G_M. We shall try to find rational parametric representations of these varieties. Because M and N are two fixed matrices, without specific forms, we shall assume that the variety obtained has p parameters.

Analogous to the M-orthogonal group, we shall treat more situations concerning the fact that M is non-singular or singular and $MN = NM$ or $MN = -NM$.

Proposition 3. *Let $H_{M,N}$ be the subgroup of the M-orthogonal group defined by*

$$\begin{cases} AM\tilde{A} = M \\ AN = NA. \end{cases}$$

If M is non-singular and symmetric and N is involutive $(N = N^{-1})$ and verifies the equality $MN = NM$, then the algebraic variety V_p is rational, the formula $A = (M + X)^{-1}(M - X)$, with $\tilde{X} = -X$ and $XN = NX$, $XM^2 = M^2X$, $|M + X| \neq 0$, defines *a rational parametric representation of this variety in the neighbourhood determined by the condition* $|M+X| \neq 0$.

Proof. From Proposition 1, we know that it is verified the equality $AM\tilde{A} = M$. We shall show that there exists the relation $AN = NA$ if are fulfilled the conditions from Proposition 3.

The relation $AN = NA$ is equivalent to $(M+X)^{-1}(M-X)N = N(M+X)^{-1}(M-X)$. Because $MN = NM$ and $XN = NX$, it results $(M-X)N = N(M-X)$, $(M+X)N = N(M+X)$. Then

$$\begin{aligned} &(M+X)^{-1}(M-X)N = (M+X)^{-1}N(M-X) = \\ &= (M+X)^{-1}N^{-1}(M-X) = [N(M+X)^{-1}](M-X) = \\ &= [(M+X)N]^{-1}(M-X) = N^{-1}(M+X)^{-1}(M-X) = \\ &= N(M+X)^{-1}(M-X) \end{aligned}$$

which shows that $AN = NA$. In the hypothesis $MN = -NM$, we have the following results

Proposition 4. *Let* $H_{M,N}$ *be the subgroup of the* M*-orthogonal group* G_M*, defined by the relations* $AM\tilde{A} = M$*,* $AN = NA$*.*

If M *is symmetric non-singular matrix and* $N = N^{-1}$ *verifies the equality* $MN = -NM$*, then the equality* $A = (M+X)^{-1}(M-X)$*, with* $\tilde{X} = -X$ *and*

$$\begin{cases} XM^2 = M^2X \\ XN = -NX \end{cases}$$

defines a rational parametric representation of the group $H_{M,N}$ *in the neighbourhood determined by* $|M+X| \neq 0$*.*

Proof. Analogous to the Proposition 3, we must show that is fulfilled the equality $AN = NA$. This one shows that the equality $(M+X)^{-1}(M-X)N = N(M+X)^{-1}(M-X)$ is true.

From those conditions it results that the relation $N(M-X) = -(M-X)$ is true. The following equalities are true:

$$\begin{aligned} &(M+X)^{-1}(M-X)N = -(M+X)^{-1}N(M-X) = \\ &-(M+X)^{-1}N^{-1}(M-X) = -[N(M+X)^{-1}](M-X) = \\ &= [(M+X)N]^{-1}(M-X) = N^{-1}(M+X)^{-1}(M-X) = \\ &= N(M+X)^{-1}(M-X). \end{aligned}$$

We'll continue to study the case when M is non-singular and antisymmetric.

Proposition 5. *If* M *is a non-singular antisymmetric matrix and* N *is an involutive matrix with* $MN = -NM$*, then the formula*

$$A = (M+X)^{-1}(M-X) \text{ with } |M+X| \neq 0 \text{ and } XM^2 = M^2X,$$

$XN = -NX$*,* $\tilde{X} = X$*, defines a rational parametric representation of the subgroup* $H_{M,N}$*.*

Proof. We know that the equality $AM\tilde{A} = M$ is verified (Proposition 2).

It is remained to be shown that the equality

$$(M+X)^{-1}(M-X)N = N(M+X)^{-1}(M-X) \tag{2}$$

is true. We shall show that the right member of the equality (2) is the same as that from the left. We have then

$$\begin{aligned}
&N(M+X)^{-1}(M-X) = N^{-1}(M+X)^{-1}(M-X) = \\
&= [(M+X)N]^{-1}(M-X) = (-NM-NX)^{-1}(M-X) = \\
&= [-N(M+X)]^{-1}(M-X) = -(M+X)^{-1}N(M-X) = \\
&= -(M+X)^{-1}(-MN-NX) = (M+X)^{-1}(MN+NX) = \\
&= (M+X)^{-1}(MN-XN) = (M+X)^{-1}(M-X)N
\end{aligned}$$

which shows that the equality (2) it's true.

Example. Let

$$M_0 = \begin{pmatrix} I_2 & 0 & \dots & 0 \\ 0 & I_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & I_2 \end{pmatrix} \text{ and } N_0 = \begin{pmatrix} J_2 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & J_2 \end{pmatrix}$$

be cellular matrices of $2n$-th order with

$$I_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It verifies that $M_0N_0 = -N_0M_0$, $N_0^2 = E_{2n}$, where E_{2n} is the identity matrix of $2n$-th order. In this particular case the subgroup H_{M_0,N_0} depends of n^2 parameters. If X is a non-singular symmetric matrix of $2n$-th order which verifies the relation $XN_0 = -N_0X$, then the variety of the group H_{M_0,N_0} formed by the matrices A which $AM\tilde{A} = M_0$, $AN_0 = N_0A$, admits a rational parametric representation of the form $A = (M_0+X)^{-1}(M_0-X)$ in the neighbourhood determined by $|M_0+X| \neq 0$.

Remark. Let $H_{M,N}$ be the subgroup of G_M and let $B \in G_M$ be. In these condition the set $S = \{B^{-1}AB | A \in H_{M,N}\}$ is a subgroup of G_M, isomorphic which $H_{M,N}$.

Obviously the set S is nonempty. Let $B^{-1}A_1B \in S$ and $B^{-1}A_2B \in S$, then

$$(B^{-1}A_1B)(B^{-1}A_2B)^{-1} = B^{-1}A_1BB^{-1}A_2^{-1}B = B^{-1}(A_1A_2)^{-1}B \in S$$

because $H_{M,N}$ is a subgroup of G_M. The function $f : H_{M,N} \to S$ defined by $f(A) = B^{-1}AB$ is an isomorphism of $H_{M,N}$ on S.

References

[1] J.Dieudonné, *Sur les groupes classiques*, Hermann, Paris, 1967.

[2] E.Grecu, *Sur le groupe symplectique pseudo - ortogonal*, Bulletin Mathématiques de la Societé des Sciences Mathématiques de la Roumanie, Tome 24 (72), nr.3, 1980.

[3] O.T. O'Meara, *Symplectic groups*, Amer. Math. Soc. Providence, RI 1978.

[4] J.M.Smith, *on the existence of certain matrices*, Portugal Math., 30 (1971).

[5] C.Teleman, *Geometrie diferenţială şi globală*, Ed. Tehnică, Bucureşti, 1974.

[6] G.Vrănceanu, *Leçons de géométrie differentielle*, III-e vol, Ed. Academiei, 1964.

University Politehnica of Bucharest
Department of Mathematics I
Splaiul Independenţei 313
77206 Bucharest, Romania