Abstract Linear Dependence

O.Dogaru, M.Doroftei, I.Ţevy and V.Dogaru

Abstract

This paper is concerned in the topics of the foundations of mathematics. An important problem of an axiomatic theory is to establish the minimum number of axioms which are needed for that one obtains some standard properties.

In this paper we introduce and study two classes: the class of the D-spaces and the class of the DL-spaces. The class of the D-spaces contains the class of the linear vector spaces, the class of the afine spaces and the class of the projective spaces alike. The class of the DL-spaces, which is contained in that of the D-spaces, contains the linear vector spaces and the projective spaces. The main purpose of the paper is to prove that starting with three axioms, in the case of the DL-spaces, or four axioms, in the case of the DL-spaces we can obtain the basic properties of the linear vector spaces.

Key words: covering operator, dependence operator, D-space, DL-space.
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1 Dependence operators

Let M be a non empty set and $\mathcal{P}(M)$ the set of all parts of M.
1.1. Definition. A function $L : \mathcal{P}(M) \to \mathcal{P}(M)$ is a *covering operator* on M if:

(D_1) For any $A \in \mathcal{P}(M)$, $A \subset L(A)$;

(D_2) For any $A \in \mathcal{P}(M)$ and any $B \in \mathcal{P}(M)$ with $A \subset L(B)$, $L(A) \subset L(B)$.

Examples. 1. Let V be a real linear vector space. Then $L(A) = conv(A)$ (that is the convex covering of A) is a covering operator on V.

2. If M is an arbitrary set and $L(A) = M$ or $L(A) = A, \forall A \in \mathcal{P}(M)$, then in both cases L is a covering operator on M.

1.2. Proposition. *A covering operator L on M has the following properties: a) $A \subset B$ implies $L(A) \subset L(B)$, for $A, B \in \mathcal{P}(M)$;*

b) $L(L(A)) = L(A)$, for $A \in \mathcal{P}(M)$;

c) $L(\bigcup_{i\in I} A_i) = L(\bigcup_{i\in I} L(A_i))$;

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d) $L(\bigcap_{i\in I} A_i) \subset \bigcap_{i\in I} L(A_i)$;

e) If $a \in L(A \cup \{b_1, \dots, b_n\})$ *and* $b_i \in L(C)$, *where* $i = \overline{1,n}$ *then* $a \in L(A \cup C)$.

1.3. Definition. An application $L : \mathcal{P}(M) \to \mathcal{P}(M)$ is a *dependence operator* on M if it is a covering operator on M and it fulfills the extra axiom:

(D_3) If $a \notin L(A)$ and $a \in L(A \cup \{b\})$ then $b \in L(A \cup \{a\})$,

denoted as the *exchange axiom.*

In this case we will say that M is a *dependence space* or a *D-space.*

Examples 3. Let V be a linear vector space over the field K. Then $L : \mathcal{P}(V) \to \mathcal{P}(V)$ defined as $L(A) = \langle A \rangle = span(A)$ (the linear space generated by A) is a dependence operator. One can observe that in the case $K = R$, the convex covering operator is not a dependence operator since it doesn't fulfill the exchange axiom.

4. The covering operators on an arbitrary set M defined in the previous example, point b) are dependence operators too.

5. Let (X, τ) be a separated topological space. Then the Kuratowski's closure operator $A \to \bar{A}$ is a dependence operator.

1.4. Definition. For all $A \in \mathcal{P}(M)$ we will denote by $\mathcal{F}(A)$ the family of all finite parts of A. A dependence operator L on M is *finitely generated* if

$$L(A) = \bigcup_{F\in\mathcal{F}(A)} L(F).$$

Remark. On a linear vector space the linear covering operator is a finitely generated dependence operator, but the Kuratowski's operator on a topological space is not finitely generated since for any finite set $F = \bar{F}$.

1.5. Theorem. *Let* $L : \mathcal{F}(M) \to \mathcal{P}(M)$ *be an application fulfilling axioms* (D_1), (D_2), (D_3). *Then there exists an unique dependence operator*

$$\bar{L} : \mathcal{P}(M) \to \mathcal{P}(M),$$

which is finitely generated and extends L.

Proof. Unicity for $\bar{L}$ is obvious. To prove the existency, let us define

$$\bar{L}(A) = \bigcup_{F\in\mathcal{F}(A)} L(F), \quad \text{for } A \in \mathcal{P}(M),$$

and show that it is a dependence operator. From axiom (D_1) for L we obtain $a \in L(\{a\})$, for $a \in A$, and since $L(\{a\}) \subset \bar{L}(A)$ we get $A \subset \bar{L}(A)$ and (D_1) for $\bar{L}$. Let now $G = \{g_1, \dots, g_n\} \subset A \subset \bar{L}(B)$ and if $F_1, \dots, F_n \in \mathcal{F}(B)$ such that $g_i \in L(F_i)$, $i = \overline{1,n}$, then $G \subset \bigcup_{i=1}^{n} L(F_i)$. But $F_i \subset \bigcup_{i=1}^{n} F_i$ implies $L(F_i) \subset L(\bigcup_{i=1}^{n} F_i)$ and $\bigcup_{i=1}^{n} L(F_i) \subset L(\bigcup_{i=1}^{n} F_i)$, hence $G \subset L(\bigcup_{i=1}^{n} F_i)$ and $L(G) \subset L(\bigcup_{i=1}^{n} F_i) \subset \bar{L}(B)$. The definition of $\bar{L}(A)$ leads to $\bar{L}(A) \subset \bar{L}(B)$ and (D_2). Let $a, b \in M$ and $A \in \mathcal{P}(M)$ with $a \notin \bar{L}(A), a \in \bar{L}(A \cup \{b\})$, then there exists a finite sef $F \subset A \cup \{b\}$ such that $a \in L(F)$. If we denote $G = A \cap F$, since $a \notin \bar{L}(A)$ then $a \notin L(G)$, but $F = G \cup \{b\}$, hence $a \in L(G \cup \{b\})$. From axiom (D_3) applied to L we get $b \in L(G \cup \{a\})$ and since $G \cup \{a\} \subset A \cup \{a\}$ we obtain $b \in \bar{L}(A \cup \{a\})$ and axiom (D_3).

1.6. Definition. A finite set F is said to be *dependent* if there exists $a \in F$ such that $a \in L(F \backslash \{a\})$. If not, F is *independent*. We consider that the empty set is independence.

It can be observed that if F and G are finite sets $F \subset G$, then F dependent implies G dependent. With this remark we can extend the concept of dependent set to an arbitrary set. Then, for $A \subset M$ an infinite set, A is *dependent* if it contains a dependent finite subset. If not, A will be called *independent*.

1.7. Definition. Let $a \in L(A)$. A set $S \subset A$ such that *a depends essentially on S*, that is $a \in L(S), a \notin L(S \backslash \{x\})$ for every $x \in S$, is said *support* of the element a.

1.8. Proposition. *If L is finitely generated, then every $a \in L(A)$ admits at least one finite support.*

Proof. From the hypothesis, there exists $F \subset A$, finite and $a \in L(F)$.Let $S \subset F$ be a minimal set with respect to the inclusion having the property that $a \in L(S)$ (eventually $S = \emptyset$!). Then S is a finite support for $a \in L(A)$.

Remark. If L is the closure operator of Kuratowski, on the topological space R and $A = \{\frac{1}{n} \,/\, n \in N^*\}$ then the element $0 \in L(A)$ doesn't have a support in A.

1.9. Proposition. *Suppose that the operator L is finitely generated. Then A is dependent if and only if there exists $a \in L(A)$ such that $a \in L(A \backslash \{a\})$.*

1.10. Proposition. *Let L be a dependence operator on M. The following statements are true:*

1) If $A \subset B$, then A dependent implies B dependent and B independent implies A independent;

2) $\{a\}$ is dependent if and only if $a \in L(\emptyset)$ (then if $L(\emptyset) = \emptyset$, $\{a\}$ is independent for every $a \in M$);

3) If $A \cap L(\emptyset) \neq \emptyset$, then A is dependent;

4) If $\{a\}$ is independent and $a \in L(\{b\})$ then $b \in L(\{a\})$;

5) A finite set A is dependent if and only if there exists $a \in A$ such that $L(A \backslash \{a\}) = L(A)$. If L is finitely generated, then the set A can be an arbitrary set;

6) For a finite set A if there exists $a \in L(A)$ and A is not the support of a, then A is independent. If L is finitely generated, then the set A can be an arbitrary set.

7) A is independent if and only if each finite sets $F_1, F_2 \subset A$ fulfilling $L(F_1) = L(F_2)$ are equal;

8) If A is independent and $F \subset A$ is finite and $L(F) = L(\emptyset)$, then $F = \emptyset$;

Proof. 7) Suppose that A is independent and let $F_1, F_2 \subset A$ be two finite, different sets with $L(F_1) = L(F_2)$. If $a \in F_1 \backslash F_2$, then by hypothesis $a \in L(F_2)$ and so $G = \{a\} \cup F_2 \subset A$ is dependent, which is a contradiction. Reciprocally,let's suppose that A is dependent. Then there exists a dependent finite set $F \subset A$. Hence there exists $a \in F$ such that $L(F \backslash \{a\}) = L(F)$, which is also a contradiction.

1.11. Proposition. *Let A be independent and $A \cup \{a\}$ dependent, where $a \in M$. Then $a \in L(A)$.*

1.12. Lemma. *Let F be a finite, non void set. Then, for every $a \in L(F) \backslash L(\emptyset)$ there exists $b \in F$ that fulfills $b \in L(F \backslash \{a\})$. If L is finitely generated, then the set F can be arbitrary.*

1.13. Proposition. *Let F be a finite set in the D-space M, such that*

$$F \cap L(\emptyset) = \emptyset.$$

The following statements are equivalent:

1) F is dependent;

2) There exists $F' \subset F, F' \neq F$, such that $L(F') = L(F)$;

3) There exist $F_1, F_2 \subset F$, $F_1 \cap F_2 = \emptyset$, $F = F_1 \cup F_2$ and $L(F_1) \cap L(F_2) \neq L(\emptyset)$.

Proof. "1) $\Longrightarrow$ 2)" By the hypothesis there exists $a \in F$, fulfilling $a \in L(F\backslash\{a\})$. Let now $F' = F\backslash\{a\}$ and we obtain $L(F') = L(F)$.

"2) $\Longrightarrow$ 3)" Let $F'' = F\backslash F'$, then $L(F') \cap L(F'') = L(F) \cap L(F'') = L(F'') \neq L(\emptyset)$.Otherwise, it would result $F'' \subset L(\emptyset)$, which is in contradiction with the main hypothesis $F \cap L(\emptyset) = \emptyset$.

"3) $\Longrightarrow$ 1)" Let $a \in L(F_1) \cap L(F_2)$, $a \notin L(\emptyset)$. Then $a \in L(F_1)\backslash L(\emptyset)$, and from the previous lemma we get $a_1 \in F_1$ with

$$a_1 \in L(F_1 \setminus \{a_1\} \bigcup \{a\}) \subset L(F \setminus \{a_1\}).$$

Hence F is dependent.

2 Basis in a D-space

2.1. Definition. Let M be a D-space and L its dependence operator. A set $B \subset M$ is called *basis* if it is independent and it is a *system of generators* for M, that is $L(B) = M$. If M admits a finite system of generators, then M is called a *finitely generated D-space.*

2.2. Theorem (the basis theorem). *Let M be a D-space and L its dependence operator. Let S be a system of generators on M, $S \subset M$ and let $X \subset S$ be an independent set. Then, there exists a basis B, such that $X \subset B \subset S$.*

Proof. Let $\mathcal{A}$ be the family of all independent subsets of S, containing X. Since $X \in \mathcal{A}$, $\mathcal{A}$ is non void. Let us show that $\mathcal{A}$ is inductively ordered with respect to the inclusion. Consider $\{B_i\}_{i\in I}$, a totally ordered subset of $\mathcal{A}$ and let $E = \bigcup\limits_{i \in I} B_i$. For E we have $X \subset E \subset S$ and we have to prove that it is independent. Suppose the contrary, that there exists a finite dependent subset F of E. Since $\{B_i\}_{i\in I}$ is totally ordered there exists $i_0 \in I$ such that $F \subset B_{i_0}$, which leads to a contradiction: B_{i_0} dependent. We obtain that $E \in \mathcal{A}$ and it is a majorant for $\{B_i\}_{i\in I}$, so $\mathcal{A}$ is inductively ordered. From Zorn's axiom we obtain that $\mathcal{A}$ contains at least a maximal element, denoted by B. Let us prove that B is a basis. We only have to prove that $L(B) = M$. Let $a \in S \setminus B$. Since B is maximal it results that $B \bigcup \{a\}$ is dependent and from 1.11 we obtain $a \in L(B)$. Hence $S \subset L(B)$ and therefore $L(S) \subset L(B)$, and finally, $L(B) = M$.

2.3. Corollary. *Any independent set in a D-space can be completed to a basis.*

2.4. Corollary. *From every system of generators we can extract a basis.*

Proof. Let S be a system of generators for M. Then $\emptyset \subset S$ and $\emptyset$ is independent, then we can apply the basis theorem for $X = \emptyset$.

2.5. Theorem (the exchange theorem. *Let $r, n \in \mathbf{N}^*$ and $A = \{a_1, \ldots, a_r\}$ an independent set fulfilling $A \subset L(B)$, where $B = \{b_1, \ldots, b_n\}$. Then:*

1) $r \leq n$;

2) $L(\{a_1, \ldots, a_r, b_{r+1}, \ldots, b_n\}) = L(B)$, for a suitable ordering.

Proof. We will use the induction method. Suppose $r = 1$. Condition 1) is fulfilled and let $G \subset B$ be a maximal set with respect to the inclusion such that $a_1 \notin L(G)$ (eventually $G = \emptyset$, since $A = \{a_1\}$ is independent). Obviously $G \neq B$ and let's suppose $b_1 \notin G$. Then $a_1 \in L(G \cup \{b_1\})$ (G is maximal) and from the exchange axiom we get $b_1 \in L(G \cup \{a_1\}) \subset L(\{a_1, b_2, \ldots, b_n\})$. We proved the inclusion $L(B) \subset L(\{a_1, b_2, \ldots, b_n\})$ and since the inverse inclusion is obvious we obtain condition 2).

Suppose now the conclusion true for $r - 1$ (that is $r - 1 \leq n$ and

$$L(\{a_1, \ldots, a_{r-1}, b_r, \ldots, b_n\}) = L(B)).$$

If $r - 1 = n$ then $a_r \in L(\{a_1, \ldots, a_{r-1}\})$ and it would result that A is not independent. Hence $r \leq n$. Consider now a maximal set $G \subset \{a_1, \ldots, a_{r-1}, b_r, \ldots, b_n\}$ fulfilling $\{a_1, \ldots, a_{r-1}\} \subset G$ and $a_r \notin L(G)$. Then $G \neq \{a_1, \ldots, a_{r-1}, b_r, \ldots, b_n\}$ and we can suppose that $b_r \notin G$. From G maximal we get $a_r \in L(G \cup \{b_r\})$, hence $b_r \in L(G \cup \{a_r\}) \subset L(\{a_1, \ldots, a_r, b_{r+1}, \ldots, b_n\})$. Hence $L(B) = L(\{a_1, \ldots, a_{r-1}, b_r, \ldots, b_n\}) \subset L(\{a_1, \ldots, a_r, b_{r+1}, \ldots, b_n\})$ and finally we obtain 2).

2.6. Corollary. *Let M be a finitely generated D-space. Then M admits at least one finite basis. More, every basis is finite and any two bases have the same number of elements.*

2.7. Lemma. *Let L be a dependence operator finitely generated on M and $B \subset M$. If B is independent, then any $a \in L(B)$ admits an unique finite support in B .Reciprocally, if $B \cap L(\emptyset) = \emptyset$ and any $a \in L(B)$ admits an unique finite support in B, then B is independent.*

Proof. Suppose B is independent and let $a \in L(B)$ such that there exist $F_1, F_2 \subset B$ two different, finite sets, a depending essentially on F_1 and F_2 (Proposition 1.8 assures the existence of a finite support). Choosing $b \in F_1 \backslash F_2$, since a depends essentially on F_1, we obtain $a \notin L(F_1 \backslash \{b\})$ and $a \in L(F_1)$. From the substitution axiom it results $b \in L(F_1 \backslash \{b\} \cup \{a\})$. But $a \in L(F_2)$, therefore $b \in L((F_1 \cup F_2) \backslash \{b\})$. Hence $F_1 \cup F_2$ is dependent, which implies B dependent. Contradiction!

Reciprocally, suppose B is dependent, but $B \cap L(\emptyset) = \emptyset$ and any $a \in L(B)$ admits an unique finite support in B. Consequently, there exists $F \subset B$ finite and dependent and we can choose $a \in F$ with $a \in L(F \backslash \{a\})$. Let $G \subset F \backslash \{a\}$ such that a depends essentially on G. But a depends essentially on $\{a\}$ too, otherwise we would get $a \in L(\emptyset)$, which contradicts the hypothesis $B \cap L(\emptyset) = \emptyset$. Hence a admits no unique finite support. Contradiction!

2.8. Theorem. *Let M be a D-space and L its dependence operator. Suppose L is finitely generated. Then any two bases on M have the same cardinal.*

Proof. Let B and B' be two bases in M. If one of them is finite, we apply Corollary 2.6. Let's suppose that B and B' are infinite. For any $e \in B$ we denote $s(e) \subset B'$ the support of e in B'. We'll show that $B' = \bigcup_{e \in B} s(e)$. Suppose that there exists $f_0 \in B'$ such that $f_0 \notin \bigcup_{e \in B} s(e)$ and let $t(f_0) \subset B$ be the support of f_0 in B. If $t(f_0) = \{e_1, \ldots, e_p\}$, then $f_0 \in L(\{e_1, \ldots, e_p\})$. Since $e_i \in L(s(e_i)), \forall i = \overline{1, p}$, it results

$f_0 \in L(\bigcup_{i=1}^{p} s(e_i))$. But $f_0 \notin s(e_i), \forall i = \overline{1,p}$, therefore the set $\{f_0\} \cup (\bigcup_{i=1}^{p} s(e_i)) \subset B'$ is dependent, which is a contradiction.

From $B' = \bigcup_{e \in B} s(e)$ it results

$$card\ B' \leq \aleph_0 = \sum_{e \in B} card\ s(e) \leq \aleph_0 card\ B = card B,$$

because $card\ s(e) \leq \aleph_0$. In the same way we can prove that $card\ B \leq card\ B'$.

2.9. Definition. Let M be a D-space and L its dependence operator. A set $S \subset M$ is called a *subspace* if $L(S) = S$.

2.10. Remark. If S is a subspace, then is obvious that for any $A \in \mathcal{P}(S)$ it results $L(A) \in \mathcal{P}(S)$. Hence the restriction of L to $\mathcal{P}(S)$ will define a dependence operator on S and we obtain that any subspace of a D-space is also a D-space. Obviously, $L(A)$ is a subspace, for any $A \subset M$.

2.11. Definition. Let $\{S_i\}_{i \in I}$ be a family of subspaces in the D-space M. Then the subspace $L(\bigcup_{i \in I} S_i)$ will be called *the sum of the family* $\{S_i\}_{i \in I}$ and it will be denoted as $\sum_{i \in I} S_i$. If S_1 and S_2 are two subspaces, the sum of the family $\{S_1, S_2\}$ will be denoted as $S_1 + S_2$.

2.12.Theorem. *Let M be a D-space. The the family of all subspaces of M (denoted by $\mathcal{L}$) is a complete lattice with respect to the inclusion.*

Proof. Consider $\{S_i\}_{i \in I}$ a family of subspaces and it is obvious that

$$\bigcap_{i \in I} S_i = \inf\{S_i\}_{i \in I}.$$

Since $L(\bigcap_{i \in I} S_i) \subset \bigcap_{i \in I} L(S_i) = \bigcap_{i \in I} S_i$ and $\bigcap_{i \in I} S_i \subset L(\bigcap_{i \in I} S_i)$, it results that $\bigcap_{i \in I} S_i$ is a subspace. Finally let's prove that $\sum_{i \in I} S_i = \sup\{S_i\}_{i \in I}$. From the definition of the subspace $\sum_{i \in I} S_i$ we get $S_i \subset \sum_{i \in I} S_i$, for any $i \in I$. Let now S be a subspace such that $S_i \subset S$, for any $i \in I$. It results $\bigcup_{i \in I} S_i \subset S$, hence $L(\bigcup_{i \in I} S_i) \subset L(S) = S$.

2.13. Proposition. *Let M be a finitely generated D-space. Then any subspace of M is also a finitely generated D-space.*

Proof. Let $S \subset M$ be a subspace and consider $A \subset S$ a system of generators for S (we can consider the case $A = S$). Applying Corollary 2.4 to the D-space S we get that there exists $F \subset A$, such that F is a basis for S. Since F is independent, we can use Corollary 2.3 for M to obtain that $F \subset B$, where B is a finite basis in M. Hence F is also finite.

2.14. Corollary. *Let M be a D-space and L its dependence operator. Then, if M is finitely generated, L is also finitely generated.*

3 DL-spaces

Let M be a non empty set and $L : \mathcal{P}(M) \to \mathcal{P}(M)$ an arbitrary map. Related with L, besides the axioms $(D_1), (D_2)$ and (D_3) (Definitions 1.1 and 1.3) we will consider some extra axioms:

$(D_3)'$: (*the exchange weak* axiom)
Let $F \subset M$, F being a set with at most one element.
If $a \notin L(F)$ and $a \in L(F \cup \{b\})$, then $b \in L(F \cup \{a\})$.

(D_4) :
Let $F \subset M$. If $a \in L(F \cup \{b\})$ and $a \notin L(\{b\})$, then $L(\{a, b\}) \cap L(F) \neq L(\emptyset)$.
For $F \subset M$ and $a \in M$ we denote the set

$$aL(F) = \bigcup_{x \in L(F)} L(\{a, x\})$$

and obtain $aL(F) \subset L(\{a\} \cup F)$.

(D_5) :
For any $F \subset M$ and any $a \in M$ we get $aL(F) = L(\{a\} \cup F)$.

As a remark, axiom (D_5) is equivalent with $aL(F)$ being a subspace, if L is a dependence operator.

In the sequel we will prove some implications between these axioms, emphasizing an important class of dependence operators.

3.1. Proposition. *From axioms* $(D_1), (D_2), (D_3)'$ *and* (D_4) *we get* (D_3) *and* (D_5).

Proof. To prove (D_5) we consider $F \subset M$ and $a \in M$. We have to show that $L(\{a\} \cup F) \subset aL(F)$ If $a \in L(F)$ the inclusion is obvious. If we choose now $a \notin L(F)$. Let $x \in L(\{a\} \bigcup F)$. In the case $x \in L(\{a\})$ it results $x \in L(\{a, b\})$ for any $b \in L(F)$ and this proves the inclusion. In the case $x \notin L(\{a\})$ from (D_4) we get $L(\{x, a\}) \cap L(F) \neq L(\emptyset)$ and we can consider $b \in L(\{x, a\}) \cap L(F) \backslash L(\emptyset)$. Choosing now $b \in L(\{a\})$ since $b \notin L(\emptyset)$ we obtain from $(D_3)'$ that $a \in L(\{b\}) \subset L(F)$, which is absurd. If we choose $b \notin L(\{a\})$ and $b \in L(\{x, a\})$ we can apply again $(D_3)'$, obtaining $x \in L(\{a, b\}) \subset aL(F)$. So (D_5) is true.

To prove (D_3) let's take $a \in L(A \cup \{b\})$ and $a \notin L(A)$. From (D_5) we get that $a \in bL(A)$. Then there exists $x \in L(A)$ with $a \in L(\{b, x\})$ and since $a \notin L(\{x\})$ we get from $(D_3)'$ that $b \in L(\{a, x\}) \subset L(A \cup \{a\})$.

3.2. Proposition. *From axioms* $(D_1), (D_2), (D_3)'$ *and* (D_5) *we get* (D_3) *and* (D_4).

Proof. Let $a \in L(F \cup \{b\})$ with $a \notin L(\{b\})$. From (D_5) it results $a \in L(\{x, b\})$ $x \in L(F) \backslash L(\emptyset)$. From axiom $(D_3)'$ we get $x \in L(\{a, b\})$, then (D_4) is fulfilled.

To prove (D_3), let $a \in L(F \cup \{b\})$ with $a \notin L(F)$.If $a \in L(\{b\})$ we get $b \in L(\{a\})$, since we can use $(D_3)'$ and $a \notin L(\emptyset)$. Hence in this case (D_3) is fulfilled. On the other hand, if $a \notin L(\{b\})$ we can apply (D_5) to get $a \in L(\{b, x\})$ with $x \in L(F)$. From axiom $(D_3)'$ it results $b \in L(\{x, a\}) \subset L(\{a\} \cup F)$.

3.3. Definition. Let M be a set. A dependence operator $L : \mathcal{P}(M) \to \mathcal{P}(M)$ (it fulfills axioms $(D_1), (D_2)$ and (D_3)) is called *linear dependence operator* if it fulfills axiom (D_4) too. In this case M is called a *DL-space*. From Proposition 1 it results that in a *DL-space* axiom (D_5) is also true.

Examples. 1. Let V be a linear space. Then V is a DL-space, since the operator $L(A) = spanA$ is a linear dependence operator.

2. Let Π be a projective plane. Then Π is a DL-space since $L(M) = [M]$ (the linear manifold generated by M) is a linear dependence operator.

3. Let α be an affine plane. Then A is a D-space defined by the dependence operator $L(M) =< M >$ (the linear manifold generated by M), but α is not a DL-space since the above operator does not fulfill axiom (D_5). Indeed, if we consider a line d and a point A which does not belong d, then the set $A < d >= \bigcup_{A\in d} AB$ does not coincide with the set $< \{A\} \bigcup d >= \alpha$.

3.4. Proposition. *Let M be a DL-space and L its linear dependence operator. Let F and G be two sets from M such that at least one is finite. Then*

$$L(F \cup G) = \bigcup_{\substack{a\in L(F)\\ b\in L(G)}} L(\{a, b\}).$$

Proof. Let's suppose that F is finite and let n be the number of elements from F To prove the inclusion "$\subset$" we shall use the induction after n. Statement for $n = 1$ results from axiom (D_5). Next we suppose that the statement is true for any set F with $n-1$ elements and we consider a set F having n elements. Using the affirmation true for $n = 1$ it results:

$$L(F \cup G) = L(\{a\} \cup [F\backslash\{a\} \cup G]) = aL(F\backslash\{a\} \cup G)$$

Let $x \in L(F \cup G)$ We exclude the trivial cases $x \in L(F)$ or $x \in L(G)$, consequently $x \in L(\{a, b\})$ where $b \in L(F\backslash\{a\} \cup G)$. From the induction hypothesis we get $b \in L(\{a', c\})$ where $a' \in L(F\backslash\{a\})$ and $c \in L(G)$. Then $x \in L(\{a, a', c\}) = cL(\{a, a'\})$ and we obtain

$$L(F \cup G) \subset \bigcup_{\substack{a\in L(F)\\ b\in L(G)}} L(\{a, b\}).$$

3.5. Corollary. *If the operator L is finitely generated then the result from the previous theorem remains true for two arbitrary sets F and G.*

Proof. Let $x \in L(F \cup G)$. Since L is finitely generated there exists a finite set V such that $x \in L(V)$. Taking $F' = V \cap F$ and $G' = G \cap V$ we get $F' \neq \emptyset$ and $x \in L(F' \cup G')$ (we exclude the trivial cases $x \in L(F)$ or $x \in L(G)$). Since $'F$ is finite we can apply the previous theorem, obtaining $x \in L(\{a', b'\})$ where $a' \in L(F') \subset L(F)$ and $b' \in L(G') \subset L(G)$.

3.6. Lemma. *Let M be a DL-space and L its linear dependence operator. Let F and G be two sets from M such that at least one of them is finite. If $a \in L(F \cup G)$ and $a \notin L(F)$, then $L(F \cup \{a\}) \cap L(G) \neq L(\emptyset)$.*

Proof. Let $a \in L(F \cup G)$ and $a \notin L(F)$. From theorem 3.4 we get that $a \in L(\{x, y\})$ where $x \in L(F)$ and $y \in L(G)$.From $a \notin L(F)$ it results $a \notin L(\{x\})$. Hence, applying axiom (D_4) we get $L(\{a, x\}) \cap L(\{y\}) \neq L(\emptyset)$. In this way the conclusion in the statement is proved.

3.7. Corollary. *If the operator L is finitely generated, then the previous lemma remains also true in the case of two arbitrary sets F and G.*

3.8. Proposition. *Let M be a DL-space and L its linear dependence operator.Let F_1 and F_2 be two finite sets such that $F_1 \cap F_2 = \emptyset$. Then $F_1 \cup F_2$ is independent if and only if*

$$L(F_1) \cap L(F_2) = L(\emptyset).$$

Proof. To prove the necessity we suppose $L(F_1) \cap L(F_2) \neq L(\emptyset)$.Then, by Proposition 1.13, $F_1 \cup F_2$ is dependent. For the sufficiency, we suppose that $F_1 \cup F_2$ is dependent. Then there exists $a \in F_1 \cup F_2$ with $a \in L(F_1 \cup F_2 \setminus \{a\})$. If we choose $a \in F_1$, from F_1 independent we get $a \notin L(F_1 \setminus \{a\})$. Now, applying Lemma 3.6 to $a \in L(F_1 \setminus \{a\}) \cup F_2$ it results $L(F_1) \cap L(F_2) \neq L(\emptyset)$, which is a contradiction.

3.9 Corollary. *If the operator L is finitely generated, then the previous proposition remains also true in the case of two arbitrary sets F_1 and F_2.*

3.10. Definition. Let M be a D-space and S be a finitely generated subspace in M. The number of elements in a basis in S is called *the rank* of S and is denoted by $r(S)$. By the Theorem 2.8, the number $r(S)$ does not depend on the basis we consider. Obviously, $r(L(\emptyset)) = 0$.

3.11. Theorem. *Let M be a DL-space. Let S_1 and S_2 be two finitely generated subspaces in M. Then*

$$r(S_1 + S_2) = r(S_1) + r(S_2) - r(S_1 \cap S_2).$$

Proof. Let F be a basis for $S_1 \cap S_2$ (it may be $F = \emptyset$). F can be completed to a basis F_1 in S_1 and to a basis F_2 in S_2. Obviously $F_1 \cup F_2$ is a system of generators for $S_1 + S_2$. If we consider $G = F_2 \setminus F$, then $F \cup F_2 = F_1 \cup G$. We have to prove that $L(F_1) \cap L(G) = L(\emptyset)$. Taking $a \in L(F_1) \cap L(G)$ we obtain $a \in S_1 \cap S_2 = L(F)$ and $L(F) \cap L(G) = L(\emptyset)$, applying Proposition 3.8, since $F_2 = G \cup F$, F and G are independent and $F_1 \cap G = \emptyset$. Due to the same proposition $F_1 \cup F_2 = F_1 \cup G$ is independent, hence a basis for $S_1 + S_2$. Now the conclusion is obvious.

3.12. Theorem. *Let M be a D-space and L its dependence operator. If we suppose that L is finitely generated and for any two finitely generated subspaces S_1 and S_2 in M relation $r(S_1 + S_2) = r(S_1) + r(S_2) - r(S_1 \cap S_2)$ holds, then M is a DL-space.*

Proof. We have to prove axiom (D_4). For this purpose we choose $a \in L(F \cup \{b\})$ such that $a \notin L(\{b\})$ Obviously $L(\{b\}) \neq L(\emptyset)$, hence $L(\{a, b\})$ is independent. Since L is finitely generated, it results that there exists a finite set G, $G \subset F \cup \{b\}$, such that $a \in L(G)$. Let now $F' = F \cap G$, hence $a \in L(F' \cup \{b\})$ and $L(F' \cup \{a, b\}) = L(F' \cup \{b\})$. Therefore

$$r(L(F' \cup \{a, b\})) = r(L(F')) + r(L(\{a, b\})) - r(L(F') \cap L(\{a, b\})).$$

It results

$$r(L(F') \cap L(\{a, b\})) = 2 - (r(L(F' \cup \{b\})) - r(L(F'))) \geq 1,$$

since $r(L(F' \cup \{b\})) - r(L(F')) \leq 1$. We obtain then $L(F') \cap L(\{a, b\}) \neq L(\emptyset)$, which leads to $L(F) \cap L(\{a, b\}) \neq L(\emptyset)$.

3.13. Remark. The Theorems 3.11 and 3.12 show that for D-spaces with finitely generated dependence operators, the Grassmann's formula becomes an axioms for DL-spaces.

3.14. Remark. Theorem 1.5 remains true for the case of a DL-space.

4 Subspaces in a finitely generated DL-space

Let M be a finitely generated DL-space and L its linear dependence operator. We suppose that $r(M) = n$, where $n \geq 2$.

4.1. Proposition. *If S' and S are two subspaces in M such that $S' \subset S$ and $r(S') = r(S)$, then $S' = S$.*
4.2. Definition. A subspace $H \subset M$ is called a *hyperplane* in M if $r(H) = n - 1$.
4.3. Proposition. *A subspace $H \subset M$ is a hyperplane if and only if for any subspace S fulfilling $H \subset S, H \neq S$ we get $S = M$. We can also obtain that in any subspace S there exists a hyperplane $H \subset S$.*
4.4. Proposition. *Let H be a hyperplane and S a subspace having $r(S) = p$. Then either $S \subset H$ or $r(H \cap S) = p - 1$.*
Proof. If we suppose $S \not\subset H$ then $S' = S + H$ is also a subspace which contains H but which is different from H. Hence $S' = M$ and $n = r(S+H) = r(S) + r(H) - r(S \cap H)$. Consequently $r(H \cap S) = p - 1$.
4.5. Proposition. *Let S and S' be two subspaces such that $S' \subset S$, $r(H \cap S) = p - 1$ and $r(S) = p$. Then there exists a hyperplane H which fulfills $S' = S \cap H$.*
Proof. Let F be a basis in S' which can be completed to a basis $F \cup \{a\}$ in S and also to a basis $F \cup \{a\} \cup G$ in M. Let $H = L(F \cup G)$ be the hyperplane which does not contain S since it does not contain a. Due to the previous proposition we get $r(H \cap S) = p - 1$.On the other side $S' \subset H \subset S$ and since they have the same rank we get $S' = S \cap H$.
4.6. Theorem. *Let S be a subspace having $r(S) = p, p = \overline{1, n-1}$. Then S is the intersection of no less than $n - p$ hyperplanes.*
Proof. We will show that S can not be the intersection of less than $n - p$ hyperplanes. For this we suppose that $S = H_1 \cap \ldots \cap H_k$, where H_i are hyperplanes. From Proposition 4.4 it results:

$$r(H_1 \cap H_2) \geq n - 2, r(H_1 \cap H_2 \cap H_3) \geq n - 3, \ldots, r(H_1 \cap \ldots \cap H_k) \geq n - k,$$

hence $p \geq n - k$, so $k \geq n - p$. Let now $\{a_1, \ldots, a_p\}$ be a basis in S which can be completed to a basis $F = \{a_1, \ldots, a_p, a_{p+1}, \ldots, a_n\}$ in M. For any $i = \overline{1, n-p}$ we consider the hyperplane $H_i = L(F \backslash \{a_{p+1}\})$. Then

$$S = H_1 \cap \ldots \cap H_{n-p} \subset H_1 \cap \ldots \cap H_{n-p-1} \subset \ldots \subset H_1 \cap H_2 \subset H_1.$$

All these inclusions , except the first one, are strict inclusions, since $a_{p+i+1} \in H_1 \cap \ldots \cap H_i$, but $a_{p+i+1} \notin H_1 \cap \ldots \cap H_{i+1}$. Hence $r(H_1 \cap \ldots \cap H_{n-p}) = p = r(S)$ and consequently $S = H_1 \cap \ldots \cap H_{n-p}$.

If in a DL-space M we renounce to axiom (D_4) the results given above can be generalized, proofs being similar:
4.7. Theorem. *Let M be a finitely generated D-space having $r(M) = n \geq 2$.*

1) If S_1 and S_2 are two subspaces, then

$$r(S_1 + S_2) \leq r(S_1) + r(S_2) - r(S_1 \cap S_2).$$

2) If S is a subspace having $r(S) = p$ and if H is a hyperplane, then either $S \subset H$ or $r(H \cap S) \leq p - 1$.

3) If S is a subspace having $r(S) = p < n$, then S is the intersection of $n - p$ hyperplanes in M.

Let now M be a DL-space and let L be its linear dependence operator. We suppose that $r(M) = n > 1$ and we consider M^* the family of all the hyperplanes of M. Let $L^* : \mathcal{F}(M^*) \to \mathcal{P}(M^*)$ defined by

$$L^*(\emptyset) = \emptyset,\ L^*(\{H_1,\dots,H_p\}) = \{H \in M^* | \bigcap_{i=1}^{p} H_i \subset H\}.$$

4.8. Theorem. *If the above hypotheses are fulfilled the extension of the operator L^* to $\mathcal{P}(M^*)$ is a linear dependence operator, hence M^* becomes a DL-space.*
Proof. Due to Remark 3.14 it is sufficient to prove that operator L^* verifies axioms $(D_1),(D_2),(D_3)$ and (D_4) only in the case of finite sets. Axiom (D_1) is obvious, so we shall prove (D_2): let

$$\{H'_1,\dots,H'_q\} \subset L^*(\{H_1,\dots,H_p\}).$$

It results

$$\bigcap_{i=1}^{p} H_i \subset \bigcap_{j=1}^{q} H'_j,$$

hence

$$L^*(\{H'_1,\dots,H'_q\}) \subset L^*(\{H_1,\dots,H_p\}).$$

To prove (D_4) we consider

$$H \in L^*(X \cup \{H'\}), \quad H \notin L^*(\{H'\}) = \{H'\},$$

where $X = \{H_1,\dots,H_p\}$. We exclude the trivial case $H \in L^*(X)$. If $A = \bigcap_{i=1}^{p} H_i$ then $A \not\subset H$, $A \cap H' \subset H$ and $H \neq H'$. Obviously $q = r(A) \geq 1$. In the same time $(A \cap H') \cap H = A \cap H'$, hence $r(A \cap H' \cap H) \geq q-1$. Let now $S = L((H \cap H') \cup A)$, then $r(S) = n-2+q-r(A \cap H' \cap H) \leq n-1$. Consider a hyperplane H'' such that $S \subset H''$. Consequently $H'' \in L^*(\{H,H'\}) \cap L^*(X)$.

To prove axiom (D_3), as in Proposition 3.1, it is sufficient to prove that $H \in L^*(\{H_1,H_2\})$ with $H \neq H_1$, implies $H_2 \in L^*(\{H_1,H\})$. Let us suppose that $H_1 \cap H_2 \subset H$ and $H \neq H_1$, hence $r(H \cap H_1) = n-2$, and let's suppose that $H_2 \notin L^*(\{H_1,H\})$. Hence $H \cap H_1 \not\subset H_2$ and $r(H \cap H_1 \cap H_2) = n-3$. But $H \cap H_1 \cap H_2 = H_1 \cap H_2$ and it has rank equal to $n-2$. This leads to a contradiction.

The DL-space M^* constructed above is called *the dual* of M.
4.9. Proposition. *If all the above hypotheses are fulfilled we get $r(M^*) = r(M)$.*
Proof. Let $F = \{a_1,\dots,a_n\}$ be a basis in M and let $F^* = \{H_1,\dots,H_n\}$, where for any $i = \overline{1,n}$ we consider the hyperplane $H_i = L(F \backslash \{a_i\})$. Then F^* is a basis for M^* since from the fact that the inclusions

$$H_1 \cap \dots \cap H_n \subset H_1 \cap \dots \cap H_{n-1} \subset \dots \subset H_1$$

are strict inclusions it results $\bigcap_{i=1}^{n} H_i = L(\emptyset)$ and consequently $L^*(F^*) = M^*$. On the other side it is obvious that $H_i \notin L^*(F^* \backslash \{H_i\})$ since $a_i \notin H_i$, therefore F^* is independent.

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University Politehnica of Bucharest
Department of Mathematics I
Splaiul Independenţei 313
77206 Bucharest, Romania