Dynamical Systems on Vector Bundles

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Abstract

This paper contains our current research on techniques for the construction of fractal objects using random transformations. Special emphasis is put on the study of invariant probability measures on vector bundle.

Mathematics Subject Classification: 60A10; 65C20.
Key words: random dynamical system, invariance.

1 Introduction

We consider W and X be two measurable spaces, $\mathcal{W}$ and $\mathcal{X}$ be the Borelian spaces of W and X respectively. In this paper we shall refer to W as the *state space* and to X as the *parameters space*. If f is an $(\mathcal{W} \otimes \mathcal{X}, \mathcal{W})$-measurable application and q is a probability measure on X then the quadruple (W, X, f, q) is called *random dynamical system* ([12], [14]). To a random dynamical system one associates a Markov chain ξ, with an arbitrary set of states that is the space W. The asymptotic behaviour of random dynamical system is closely related to the asymptotic behaviour of the Markov chain ξ, hence to the study of ξ-invariant measures. In this note, we shall define another Markov chain on $\mathbf{R}^k$ which will be a repartee of ξ.

2 Random dynamical systems on vector bundles

Let W be a measurable space and $E = W \times \mathbf{R}^k$ be the cartesian product between W and the k-dimensional euclidean space.

Definition 1 Suppose $f : W \to W$ is a measurable mapping on W. The pair $F = (f, A_f)$ is called *vector bundle mapping* (with respect to f) if

$$F : E \to E, \quad F(w,s) = (f(w), A_f(w)s); \quad w \in W, \quad s \in \mathbf{R}^k,$$

where $A_f(w)$ is an $k \times k$ real matrix whose elements are measurable functions of W.

Balkan Journal of Geometry and Its Applications, Vol.3, No.2, 1998, pp. 47-56

We shall denote by $\mathcal{T}$ the space of vector bundle mappings on E. We remark that $\mathcal{T}$ is endowed with a measurable structure given by the measurable product structure on E,

$$(F, u) \to Fu, \quad F \in \mathcal{T}, \quad u \in E.$$

In the following, we consider the discrete set $\mathcal{F} = \{f_1, f_2, \ldots, f_r\}$ of mappings $W \to W$ and the set $\mathcal{A} = \{A_1, A_2, \ldots, A_r\}$ of corresponding $k \times k$ matrices. Let q be a probability measure on $\mathcal{F}$

$$q(B) = \sum_{i \in B} \rho_i, \quad \rho_i \geq 0, \quad \sum_{i=1}^{r} \rho_i = 1, \quad B \in \mathcal{X},$$

where $X = \{1, 2, \ldots, r\}$, and $\mathcal{X} = \mathcal{P}(X)$ is the set of all parts of X.

Definition 2 Let ν be a probability measure on $\mathcal{T}$. A $\mathcal{T}$-random variable F whose distribution is ν is called *random bundle mapping*.

By definition, any random bundle mapping $F = (f, A_f)$ is generated by a random transformation $f : W \times X \to W$. Let us introduce the well-known notations

$$\begin{aligned} f^{(n)} &= f_{x_n} \circ \cdots \circ f_{x_1}, \quad f_{x_i} \in \mathcal{F}, \quad x_i \in X, \quad i \in \{1, 2, \ldots, n\}, \\ F^{(n)} &= F_{x_n} \circ \cdots \circ F_{x_1}, \quad F_{x_i} \in \mathcal{T}, \quad x_i \in X, \quad i \in \{1, 2, \ldots, n\}, \\ A^{(n)} &= A_{x_n} \cdots A_{x_1}. \end{aligned}$$

If $F_x(w, s) = (f_x(w), A_x s)$, where $w \in W$, $s \in \mathbf{R}^k$, then we obtain $F^{(2)}(w, s) = (f_{x_2} \circ f_{x_1} w, A_{x_2} A_{x_1} s) = (f^{(2)}(w), A^{(2)} s)$ and, for all $n \geq 1$, we have

$$\begin{aligned} F^{(n)}(w, s) &= (f_{x_n} \circ \cdots \circ f_{x_1} w, A_{x_n} \cdots A_{x_1} s) \\ &= (f^{(n)}(w), A^{(n)} s). \end{aligned}$$

We now consider $\Omega = \mathcal{T} \times \mathcal{T} \times \cdots$, $\mathcal{K}$ be the σ-algebra of Ω and $p = \nu^{\mathbf{N}}$ with ν a probability measure on $\mathcal{T}$. Then $(\Omega, \mathcal{K}, p)$ is a probability space. An element of Ω is a sequence $\omega = (F_1, F_2, \ldots)$, $F_i \in \mathcal{T}$ while the probability measure p is given by

$$p\{\omega : F_l(\omega) \in \psi_0, \ldots, F_{l+n}(\omega) \in \psi_n\} = \prod_{i=0}^{n} \nu(\psi_{l+i}).$$

If $\sigma : \Omega \to \Omega$ is a shift on Ω and

$$F_n(\sigma\omega) = F_{n+1}(\omega), \quad f_n(\sigma\omega) = f_{n+1}(\omega)$$

are shift mappings, then we can define the skew products T and τ as follows:

$$\begin{aligned} T &: E \times \Omega \to E \times \Omega, \quad T(u, \omega) = (F_1(\omega)u, \sigma\omega), \\ \tau &: W \times \Omega \to W \times \Omega, \quad \tau(w, \omega) = (f_1(\omega)w, \sigma\omega). \end{aligned}$$

If $\pi : W \times \mathbf{R}^k \to W$ is the natural projection from E on W then from $F = (f, A_f)$ we have $f = \pi F \pi^{-1}$. Therefore

$$q(B) = \nu\{F \in \mathcal{T} : \pi F \pi^{-1} \in B\}$$

As in [9] we consider the Markov chain ξ given by

$$w_n = f^{(n)}(w_0) = \pi F^{(n)} \pi^{-1} w_0$$

and the transition probability

$$P(w, A) = \sum_{x \in X} \rho_x I_A(f_x(w)). \tag{1}$$

If $A_{f_1(\omega)}(w) = A(w, \omega)$ then $A \circ \tau(w, \omega) = A(\tau(w, \omega))$ and

$$A^{(n)}(w, \omega) = A \circ \tau^{n-1}(w, \omega) \cdots A \circ \tau(w, \omega) \cdot A(w, \omega).$$

Remark 1 . On the space $\mathcal{T}$ of random vector bundles we built the Markov chain ξ on W with transition probabilities (1). Also, we can build a Markov chain on $\mathbf{R}^k$ given by the product of random matrices

$$s_n = A_n(w, \omega) \cdots A_1(w, \omega) s_0,$$

where $s_0 \in \mathbf{R}^k$ is fixed. This may be written as

$$s_n = A_{(n)} s_{n-1}, \quad n \geq 1,$$

where $A_{(n)} = A_{f_n(w)} \in \mathcal{A}$ and transition probabilities

$$Q(s, G) = \iint I_G(A_x(w)s) d\rho(x) d\eta(w),$$

where ρ is a probability measure on X and η is ξ-invariant probability measure. In the discrete case, we have

$$Q(s, G) = \sum_{x \in X} \rho_x \int I_G(A_x(w)s) d\eta(w),$$

with $G \subset \mathbf{R}^k$ and $s \in \mathbf{R}^k$. We shall denote by ξ^* this Markov chain on $\mathbf{R}^k$. This one will be a repartee, on $\mathbf{R}^k$, of ξ

Definition 3 . Let μ and α be probability measures on W and $R_1^k = \{s \in \mathbf{R}^k / \|s\| = 1\}$. We say that $\mu \times \alpha$ *is F_x-invariant* if

$$\mathcal{E}_x(\mu \times \alpha)(F_x^{-1}(E_1 \times E_2)) = (\mu \times \alpha)(E_1 \times E_2),$$

where we denoted by $\mathcal{E}_x$ the expectation over the probability measure ρ on X, $F_x(w, s) = (f_x(w), A_x(w)s)$, $E_1 \in \mathcal{W}$, $E_2 \in \mathcal{K}$, $\mathcal{K}$ being the Borelian space of R_1^k.

In the following, we shall generalize Lemma 3.1 of Morita given in [10]. In this Lemma, Morita states that a measure μ on $\mathcal{W}$ is ξ-invariant if and only if $\mu \times P$ is T-invariant, where P is a product measure on $X^{\mathbf{N}}$ and T.

Theorem 1 . *Let μ, ρ be probability measures on W and X respectively. Then μ is ξ-invariant if and only if $\mu \times \alpha$ is F_x-invariant, where α is a probability measure on R_1^k, $(\alpha(R_1^k) = 1)$.*

Proof. Let μ be a ξ-invariant probability measure, $E_1 \in \mathcal{W}$, $E_2 \in \mathcal{K}$. Then we have

$$\begin{aligned}
\mathcal{E}_x(\mu \times \alpha)(F_x^{-1}(E_1 \times E_2)) &= \int_X \left[\iint I_{E_1 \times E_2}(F_x(w,s))d\mu(w)d\alpha(s)\right] d\rho(x) \\
&= \int_X \left[\iint I_{E_1}(wx) I_{E_2}(A_x(w)s)d\mu(w)d\alpha(s)\right] d\rho(x) \\
&= \int_X \left[\iint I_{E_1}(wx) I_{A_x^{-1}E_2}(s)d\mu(w)d\alpha(s)\right] d\rho(x) \\
&= \int_X \left[\iint I_{E_1}(wx)\alpha\left(A_x^{-1}E_2\right) d\mu(w)d\alpha(s)\right] d\rho(x) \\
&= \alpha(E_2)\iint \left[\int_X I_{E_1}(wx)d\rho(x)\right] d\mu(w)d\alpha(s) \\
&= \alpha(E_2)\iint P(w,E_1)d\mu(w)d\alpha(s) \\
&= \alpha(E_2)\int_W P(w,E_1)d\mu(w)\int_{R_1^k} d\alpha(s) \\
&= \alpha(E_2)\mu(E_1)\alpha\left(R_1^k\right) \\
&= (\mu \times \alpha)(E_1 \times E_2)
\end{aligned}$$

that is $\mu \times \alpha$ is F_x-invariant.

To prove the converse, we suppose $\mu \times \alpha$ is F_x-invariant. Then

$$\begin{aligned}
\mu(E_1) &= (\mu \times \alpha)(E_1 \times R_1^k) = (\mu \times \alpha)\mathcal{E}_x F_x^{-1}(E_1 \times R_1^k) \\
&= \iint \left[\int_X I_{E_1}(wx) I_{R_1^k}(A_x(w)s)d\mu(w)d\alpha(s)\right] d\rho(x) \\
&= \iint \left[\int_X I_{E_1}(wx)d\rho(x)\right] d\mu(w)d\alpha(s) = \iint P(w,E_1)d\mu(w)d\alpha(s) \\
&= \int_{R_1^k} d\alpha(s)\int_W P(w,E_1)d\mu(w) = \int P(w,E_1)d\mu(w)
\end{aligned}$$

hence μ is ξ-invariant.

An interesting particular case is that when the matrices from $\mathcal{A}$ are triangular

$$A_f(w) = \begin{pmatrix} a_f^{(1)}(w) & * & * & \dots & * \\ 0 & a_f^{(2)}(w) & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_f^{(k)}(w) \end{pmatrix}$$

with zeros under diagonal and $a_f^{(1)} \dots a_f^{(k)} \neq 0$. In this case the diagonal elements are eigenvalues and the dominant eigenvalue is the spectral radius. For $k = 2$ we have

$$A_f(w) = \begin{pmatrix} a_f(w) & b_f(w) \\ 0 & c_f(w) \end{pmatrix}$$

We consider μ as ξ-invariant measure and we put

$$a = \iint \log|a_f(w)| d\mu(w) d\nu(f) = \sum_{x\in X} \int \rho_x \log|a_{f_x}(w)| d\mu(w)$$
$$b = \iint \log|b_f(w)| d\mu(w) d\nu(f) = \sum_{x\in X} \int \rho_x \log|a_{f_x}(w)| d\mu(w).$$

Then they are characteristic exponents.

The product matrix is

$$A_f^{(n)} = \begin{pmatrix} a_1 \dots a_n & \sum\limits_{i=1}^{n} a_n \dots a_{i+1} b_i c_{i-1} \dots c_1 \\ 0 & c_1 \dots c_n \end{pmatrix}$$

where $a_i = a_{f_{x_1}} \dots a_{f_{x_i}}$, $b_i = b_{f_{x_1}} \dots b_{f_{x_i}}$, $c_i = c_{f_{x_1}} \dots c_{f_{x_i}}$.

If $a < b$ then for μ-almost any initial point w_0 and any non-null vector from the Ox-axis the increasing rate of the eigenvector of product matrix is of the order e^{na} and any non-null vector from the real plane is linearly independent and has the increasing rate e^{nb}, [9].

From the product matrix $A_f^{(n)}$ we deduce that the second vector has the form

$$v = \left(\sum_{i=1}^{n} a_n \dots a_{i+1} b_i c_{i-1} \dots c_1, \quad a_1 \dots a_n - c_1 \dots c_n \right)$$

Kaijser [7], defines the sequence of random variables on the unit sphere $\mathbf{R}^k$ as follows

$$s_n(s) = \frac{A^{(n)} \cdot s}{\|A^{(n)} s\|}, \quad n \geq 1$$

and proves that this sequence is a Markov chain whose distribution at the moment n, μ_n, converges to an invariant measure of process. Moreover, there exist $\lim\limits_{n\to\infty} \frac{1}{n} \log\|A^{(n)} s\|$ and $\lim\limits_{n\to\infty} \frac{1}{n} \log\|A^{(n)}\|$ a.s.

3 Illustrations

The snowflake in Fig. 1 is obtained as invariant of just three Möbius transformations. We remark that any Möbius transformation may be written as

$$f(z) = \frac{\alpha z + \beta}{z + \gamma}, \quad \alpha, \beta, \gamma \in \mathbf{C}.$$

It generates a random number q. If $q \in (0, 0.8)$ then $\alpha = -3 + 5i$, $\beta = 0.8 - i$, and $\gamma = -4 + 5i$. If not, for $q \in [0.8, 0.86)$ choose $\alpha = 2 + 0.4i$, $\beta = -0.7 - i$, $\gamma = 1 - 2i$ else $\alpha = -4 + 4i$, $\beta = 0.7 - i$, $\gamma = 5 + 2i$. Compute and plot a random orbit of z_0 under f. In Fig. 1 the result is displayed after 150,000 iterations (for details, see [4]).

Now we shall transform the points of the snowflake using some affine transformations.

To state our computer algorithm, in the following we deal with the set of affine mappings

$$\mathcal{F} = \{f / f(w) = Tw + b, \quad T \text{ is } k \times k \text{ matrix}, \quad b \in \mathbf{R}^k\}.$$

By all means, the pair $\{\mathbf{R}^k, f(w)\}$ represents a dynamical system.

Let us consider the set $\mathcal{T} = \{T_0, T_1, \ldots, T_r\}$ of $(r+1)$ non-singular $k \times k$ matrices which correspond to some affine transformations of $\mathcal{F}$. If we use the method studied in [4] we obtain the representation of this sequence

$$\begin{aligned} & w_0 \quad \text{given}, \\ & w_{k+1} = T^{(k)} w_0 + T^{(k)} \left[(T_k \cdots T_0)^{-1} b_k + \cdots + (T_1 T_0)^{-1} b_1 + T_0^{-1} b_0 \right], \end{aligned}$$

where $T^{(k)} = T_k T_{k-1} \cdots T_0$ and $b_i \in R^k$, $i = 0, \ldots n$.

For practical reasons, let us now consider $W = \mathbf{R}^2$, $X = \mathcal{T}$, $\mathcal{W}$ the Borelians of W, and $\mathcal{X}$ the set of all parts of X. If $T_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \mathcal{T}, \quad n \geq 1$ then we build the sequence $(w_n)_{n \geq 0}$ as follows:

$$w_0 \text{ given}, \quad w_1 = f_1(z_0), \quad w_2 = (f_2 \circ f_1)(w_0), \ldots.$$

Let us suppose, without loss of generality, that the matrices T_k are positive (the general case is similar). For practical reasons we have to study the asymptotic behaviour of the following product of random matrices

$$T^{(k)} = T_k \cdots T_1 T_0.$$

For this purpose, we consider a positive matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its decomposition as a product of a diagonal matrix and a stochastic one as follows:

$$D = \begin{pmatrix} a+b & 0 \\ 0 & c+d \end{pmatrix}, \quad P = \begin{pmatrix} \frac{a}{a+b} & \frac{b}{a+b} \\ \frac{c}{c+d} & \frac{d}{c+d} \end{pmatrix}.$$

Therefore we have $T = D \cdot P$. Using this decomposition for $T^{(k)}$ we have

$$\begin{aligned} T^{(k)} &= D_k P_k T_{k-1} \cdots T_0 \\ &= D_k \bar{T}_{k-1} T_{k-2} \cdots T_0 \\ &= D_k D_{k-1} P_{k-2} T_{k-3} \cdots T_0 \\ &\vdots \\ &= D_0 \cdots D_k P_k. \end{aligned}$$

Here $D_0, \ldots, D_k$ are diagonal matrices, P_k is a stochastic one and $\bar{T}_{k-1} = P_k T_{k-1}$. In general, if we denote by $\bar{T}$ the product between a general matrix and a stochastic one, that is, $\bar{T} = TP$, thus from the componentwise interpretation we get

$$\min_{i,j} t_{ij} < \min_{i,j} \bar{t}_{ij} < \max_{i,j} \bar{t}_{ij} < \max_{i,j} t_{ij}. \tag{2}$$

If we denote

$$T^{(k)} = \begin{pmatrix} a^{(k)} & b^{(k)} \\ c^{(k)} & d^{(k)} \end{pmatrix}, \quad D_0 \cdots D_k = \begin{pmatrix} h^{(k)} & 0 \\ 0 & g^{(k)} \end{pmatrix},$$

$$P_k = \begin{pmatrix} p_{1k} & 1 - p_{1k} \\ 1 - p_{2k} & p_{2k} \end{pmatrix},$$

then we have

$$T^{(k)} = \begin{pmatrix} h^{(k)} p_{1k} & h^{(k)} (1 - p_{1k}) \\ g^{(k)} (1 - p_{2k}) & g^{(k)} p_{2k} \end{pmatrix}.$$

Based on the inequalities (2) and taking into account the positivity of T_k, it follows that $\delta > 0$ and $k_0 > 1$ exist such that $p_{1k} > \delta$, $p_{2k} > \delta$ for all $k > k_0$. Then we obtain

Proposition 1 *In the conditions described above, there exists the limit*

$$\lim_{k \to \infty} \frac{1}{k} \log a^{(k)}.$$

By Proposition 1 it follows that if k is large enough then $T^{(k)}$ approaches to a matrix with proportional rows $T^{(k)} = \begin{pmatrix} e^{kh} & e^{kg} \\ e^{kh} & e^{kg} \end{pmatrix}$, where h and g are positive random variables.

The theoretical approach given above, suggest the following procedure for finding self-similar sets:

Algorithm 1 *Let X be a general compact metric space and $w_i : X \to X$ be contraction mappings with*

$$d(w_i(x), w_i(y)) \leq r d(x, y), \quad \textit{for all } x, y \in X,$$

for $i = 1, 2, \ldots, N$, where $0 \leq r < 1$. Let $\{p_1, p_2, \ldots, p_N\}$ be probabilities with $p_i > 0$ and $\sum p_i = 1$. Choose $x_0 \in X$ and pick recursively

$$x_n \in \{w_1(x_{n-1}), w_2(x_{n-1}), \ldots, w_N(x_{n-1}), \}, \quad \textit{for } n = 1, 2, \ldots, M,$$

where M is a large integer and $p_i = P(x_n = w_i(x_{n-1}))$.

In this respect, the two shadows in Fig. 2 are obtained as invariant set of the following three affine transformations $w_i : \mathbf{R}^2 \to \mathbf{R}^2$ $(i = 1, 2, 3)$

$$\begin{aligned} w_1(u, v) &= \left((1.1x^2 + 0.1y)u + 3.5yv + 0.2; (0.3x + 0.7y)v + 1.2y^2 u + 0.3\right) \\ w_2(u, v) &= (0.12xv + (1.78y + 0.5)u + 1; (0.8x + 1.2y)u + 0.5yv + 0.5) \\ w_3(u, v) &= \left((x + 0.2y^2)u + yv; yv + (0.1x + 0.9xy)u\right). \end{aligned}$$

We generated $v_0 = (x_0, y_0) \in \mathbf{R}^2$ as random point, and we defined a set of points $\{v_n \in \mathbf{R}^2 / n = 0, 1, \ldots, 10^5\}$ recursively according to

$$v_{n+1} = \begin{cases} w_1 v_n & \text{with} \quad p_1 = 0.86 \\ w_2 v_n & \text{with} \quad p_2 = 0.13 \\ w_3 v_n & \text{with} \quad p_3 = 0.01 \end{cases}$$

If those points from the set which lie in the square $\{(x,y)/-1 \leq x,y \leq 1\}$ are plotted the result will be similar to Fig. 2.

The resulting picture in Fig. 2 appears to be the same no matter which initial point v_0 is chosen. Also, it is obvious that these sets share the self-similarity property.

The theoretical presentation in this paper is not exhaustive, so the reader is encouraged to consult another research works ([1], [2], [3], [5], [8], [13], [16], [17]).

References

[1] K.B. Athreya and P.E. Ney: *Branching Processes*, Springer-Verlag, Heidelberg and New York, 1972.

[2] M. Barnsley and S. Demko: Iterated functions system and the global construction of fractals, *Proc. R. Soc., London*, A 399 (1985), 243-275.

[3] M. Barnsley and J. Elton: A new class of Markov processes for image encoding, *Adv. Appl. Prob.*, 20 (1988), 14-32.

[4] S. Corbu and M. Postolache: Fractals generated by Möbius transformations and applications, in *New Frontiers in Algebra, Groups and Geometries*, (Gr. Tsagas (Ed.)), Hadronic Press, 1996, 407.

[5] P. Gora: Random comparing mappings, small stochastic perturbations and attractors, *Z. wahr. verw. Gabiete.*, 69 (1985), 137-160.

[6] M. Iosifescu and S. Grigorescu: *Dependence with Complete Connections and Its Applications*, Cambridge Univ. Press, 1991.

[7] T. Kaijser: A limit theorem for Markov chains in compact metric spaces with applications to products of random matrices, *Duke Math. J.*, 45 (1978), 311-349.

[8] N. Kaplan: Some result about multidimensional branching processes with random environments, *Ann. Prob.*, 2 (1974), 411-455.

[9] Y. Kifer: *Ergodic Theory of Random Transformations*, Birkhäuser, 1986.

[10] T. Morita: Asymptotic behaviour of one-dimensional random dynamical systems, *J. Math. Soc. Japan*, 37 (1985), 651-663.

[11] T. Morita: *Hiroshima Math. J.*, 18 (1988), 15.

[12] T. Ohno: Asymptopic behaviour of dynamical systems with random parameters, *Publ. RIMS Kyoto Univ.*, 19 (1983), 83-98.

[13] B. Oksendal: *Stochastic Differential Equations*, Springer-Verlag, Heidelberg and New York, 1995.

[14] V.I. Oseledec: A multiplicative ergodic theorem, *Trudîi Moscov Mat. Obsc.*, 19 (1968), 179-210.

[15] M. Postolache and T. Kawaguchi: Random affine transformations and fractal structures, *Tensor, N.S.*, 58 (1997), to appear.

[16] M. Postolache: *Numerical Methods (Second Revised Edition)*, Sirius Publishers, Bucharest, 1994.

[17] W. Smith and W. Wilkinson: On branching processes in random environment, *Ann. Math. Statistics*, 40 (1969), 814-827.

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