Beil Metrics Associated to a Finsler Space

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1 Introduction

Let $F^n = (M, F)$ be a Finsler space with M a smooth i.e. C^∞ manifold and $F : TM \to R$, $(x, y) \to F(x, y)$. Assume that F^n is endowed with a Finsler 1–form $\beta_i(x, y)$ and set $\beta = \beta_i(x, y)y^i$. Here $i, j, k, ...$ will run from 1 to $n = \dim M$ and the Einstein convention on summation is implied. Then ${}^*F = L(F, \beta)$ in some conditions on L is so that ${}^*F^n = (M, {}^*F)$ is a new Finsler space. It is said that ${}^*F^n$ is obtained from F^n by a β–change [7],[10].

Typical for ${}^*F^n$ are the Randers and Kropina spaces which are obtained from a Riemannian space by particular β–changes.

Let $g_{ij}(x, y)$ be the Finsler metric tensor of F^n. If one wishes the construction of a new Finsler metric ${}^*g_{ij}$ which depends on $g_{ij}(x, y)$, then because of the linear structure of the set of Finsler tensor fields of a given type, the most general choice is

$$ {}^*g_{ij}(x, y) = \rho(x, y)g_{ij}(x, y) + \sigma(x, y)B_{ij}(x, y), \tag{1.1} $$

for ρ and σ two Finsler scalars and $B_{ij}(x, y)$ a symmetric Finsler tensor field of type $(0, 2)$. We may say that ${}^*g_{ij}$ is obtained from g_{ij} by a B–change.

It is clear that ${}^*g_{ij}$ is no longer a Finsler metric except if some strong conditions on ρ, σ and B_{ij} are imposed. Metrics similar to (1.1) appear in [2] and [5] from physical considerations. See also [11].

In order to relax such conditions we do not ask ${}^*g_{ij}$ be a Finsler metric but a generalized Lagrange metric in Miron' sense, shortly a GL–metric. For the theory of the GL–metrics we refer to [9], ch.X.

As such $({}^*g_{ij})$ has to satisfy

a) $\det({}^*g_{ij}) \neq 0$ and

b) the quadratic from ${}^*g_{ij}(x, y)\xi^i\xi^j$, $(\xi^i) \in \mathrm{R}^n$, to be of constant signature.

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Even this minimal requirements are not easy to be fullfiled except for some particular σ, ρ and B_{ij}.

By our best knowledge the following two particular forms of the GL–metric (1.1) were studied:

$$^*g_{ij}(x,y) = e^{2\alpha(x,y)} g_{ij}(x,y). \tag{1.2}$$

This class of GL–metrics contains the Miron–Tavakol metrics used by them in General Relativity and the Antonelli metrics which were introduced by P.L. Antonelli for some studies in Biology and Ecology. For details see [9], ch.XI, and reference therein;

$$^*g_{ij}(x,y) = g_{ij}(x,y) + \sigma(x,y) y_i y_j, \; y_i = g_{ij}(x,y) y^j. \tag{1.3}$$

Particular forms of the GL–metric (1.3) were used by R. Miron in Relativistic Geometrical Optics. See also [9], ch.XII.

Some particular forms of the GL–metric

$$^*g_{ij}(x,y) = g_{ij}(x,y) + \sigma(x,y) B_i(x,y) B_j(x,y), \tag{1.4}$$

with $B_i(x,y) = g_{ij}(x,y) B^j(x,y)$, for $B^j(x,y)$ a given Finsler vector field, were introduced by R.G. Beil in order to develop his interesting unified field theory ([4]). These were called Beil metrics. As such we refer to $^*g_{ij}$ in (1.4) as to the Beil metric, too. The following comment of R.G.Beil is illuminating on (1.4). "Since in my unified theory the quantity k which correspond to your σ is related to the gravitational constant, this means that a possible physical interpretation of your theory with a y-dependent σ is that gravitation itself is velocity dependent. This possibility is mentioned , for example, in Section 40.8 of the famous book "Gravitation" by Misner, Thorne and Wheeler". See [13].

The particular form of (1.4) obtained for $\sigma = 1$ and $B_i = \dfrac{\partial f}{\partial x^i}, f : M \to I\!R$ was considered by C. Udrişte in [14]. He proved that if f is proper i.e. $f^{-1}(K)$ is a compact set whenever K is compact, then the Finsler manifold $(M, ^* g_{ij}(x,y))$ is complete. A Riemannian version of (1.1), that is, was used by T. Aubin in order to prove that any compact Riemannian manifold of dimension greater then 2 admits a metric whose scalar curvature is a negative constant. See [3] and for other connected results.

The geometry of the GL–metrics (1.4) was not investigated in a systematic way. It is our purpose to fill this gap. After some preliminaries in Section 2, we show in Section 3 that $(^*g_{ij})$ from (1.4) is a GL–metric and we point out cases when it reduces to a Lagrange or to a Finsler metric. In Section 4 we discuss possibilities for introducing metrical connections for the GL–space $(M, ^* g_{ij})$. In Section 5 we digress on parallel and resp. concurrent Finsler vector fields showing that the usual definitions for these notions are also justified from the viewpoint of the almost Hermitian model of a GL–space. For such a model see [9], ch.X. Section 6 is devoted to the analysis of the GL–metric (1.4) for B^i a concurent Finsler vector field. For σ a constant we rediscover a modification of a Finsler function studied by M. Matsumoto and K. Eguchi in [8]. The case when σ is a solution of the so–called Tavakol–Van der Berg equation is investigated, too. In Section 7 we treat a Beil metric associated to a Finsler space with (α, β)–metric. It is a future task to find properties of the GL–metric (1.4) when F^n is a particular Finsler space or its dimension is low (2 or 3).

2 Preliminaries

Let M be a smooth i.e. C^∞ manifold, paracompact and of dimension n, TM its tangent manifold and $\tau : TM \to M$ its tangent bundle. If $x = (x^i)$, $i, j, k, ... = 1, ..., n$ are local coordinates on M, then the induced coordinates on TM will be $(x, y) = (x^i : x^i \circ \tau, y^i)$ with (y^i) provided by $u_x = y^i \left.\frac{\partial}{\partial x^i}\right|_x$, $u \in T_xM$, $x \in M$. The change of coordinates $(x, y) \to (\tilde{x}, \tilde{y})$ on TM are as follows.

$$(2.1)\qquad \begin{aligned} \tilde{x}^i &= \tilde{x}^i(x^1, ..., x^n),\ \text{rank}\left(\frac{\partial \tilde{x}^i}{\partial x^k}\right) = n \\ \tilde{y}^i &= \frac{\partial \tilde{x}^i}{\partial x^k}(x) y^k. \end{aligned}$$

The geometrical objects on TM whose local components change by (2.1) as on M i.e. ignoring their dependence on y, will be called Finsler objects as in [7] or d–objects as in [9].

We set $\partial_i := \frac{\partial}{\partial x^i}$, $\dot{\partial}_i := \frac{\partial}{\partial y^i}$ and notice that the vertical subspace of T_uTM i.e. $V_uTM = \text{Ker}\,(D\tau)_u$, $u \in TM$, where $D\tau$ means the differential of τ, is spanned by $(\dot{\partial}_i)$. The d–objects can be expressed using $(\dot{\partial}_i)$.

A function $F : TM \to \mathbb{R}$ which is positive, smooth on $TM \setminus 0$ and only continuous in the rest, positively homogeneous of degree 1 with respect to y i.e. $F(x, \lambda y) = \lambda F(x, y)$, $\lambda > 0$ and with the quadratic form $g_{ij}(x, y)\xi^i\xi^j$, $(\xi^i) \in \mathbb{R}^n$ nondegenerate and of constant signature, where

$$(2.2)\qquad g_{ij}(x, y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2,$$

is called a fundamental Finsler function. The pair $F^n = (M, F)$ is called a Finsler space.

The function $g_{ij}(x, y)$ are the components of a Finsler tensor field called the Finsler metric of F^n.

A supplement H_uTM of V_uTM i.e. the decomposition in a direct sum $T_uTM = H_uTM \oplus V_uTM$ holds, will be called the horizontal space and the distribution $u \to H_uTM$ will be called a horizontal distribution. A basis of it of the form $\delta_i = \partial_i - N_i^k(x, y)\dot{\partial}_k$, provides the functions $(N_i^k(x, y))$ called the local coefficients. These functions have a special rule of change by (2.1) and in turn they completely determine the horizontal distribution called also a nonlinear connection. Then $(\delta_i, \dot{\partial}_i)$ is a basis adapted to the previous decomposition of T_uTM. The Finsler objects may be also expressed by using (δ_i). We notice that (δ_i) are Finsler vector fields. For more details we refer to [7],[9].

3 The Beil metric

Let $F^n = (M, F)$ be a Finsler space and $g_{ij}(x, y)$ its Finsler metric. Assume that F^n is endowed with a Finsler vector field $B = B^i(x, y)\dot{\partial}_i$ and let $B_i(x, y)dx^i$ the Finsler 1-form with $B_i = g_{ik}B^k$. The lowering and rising of indices will be done with (g_{ij}) and

(g^{jk}), where $g^{jk}g_{ki} = \delta^j_i$, respectively. Let $\sigma : TM \to \mathbb{R}$, $(x,y) \to \sigma(x,y)$ a Finsler scalar. We set

$$ {}^*g_{ij}(x,y) = g_{ij}(x,y) + \sigma(x,y)B_i(x,y)B_j(x,y). \tag{3.1} $$

The functions $({}^*g_{ij})$ from (3.1) define for $\sigma > 0$ a positive definite GL–metric called *the Beil metric.*

It is clear that $({}^*g_{ij})$ are the components of a symmetric d–tensor field. We look for the inverse of the matrix $({}^*g_{ij})$ in the form ${}^*g^{jk} = {}^*g^{jk} - {}^*\sigma B^j B^k$ with ${}^*\sigma$ to be determined. From ${}^*g_{ij}{}^*g^{jk} = \delta^k_i$ it follows that ${}^*\sigma = \dfrac{\sigma}{1+\sigma B^2}$, with $B^2 = B_iB^i = g_{ij}B^iB^j$ (the length of B with respect to g_{ij}). Thus we have

$$ {}^*g^{jk} = g^{jk} - \frac{\sigma}{1+\sigma B^2}B^jB^k. \tag{3.2} $$

Consequently, we have $\det(g_{ij}) \neq 0$.

The quadratic from $\Phi(\xi) = {}^*g_{ij}\xi^i\xi^j = g_{ij}\xi^i\xi^j + \sigma(B_k\xi^k)^2$ is clear positive definite in our hypothesis. **q.e.d.**

We notice that (3.2) holds in the weaker condition $\sigma \neq -\dfrac{1}{B^2}$ and if $g_{ij}\xi^i\xi^j$ is only of constant signature, the signature of $\Phi(\xi)$ will be constant for some σ and (B^k) at least locally.

Remark 3.1. The GL–metric (3.1) appears in papers by R.G. Beil ([4]) for F^n a pseudo–Riemannian space or a Minkowski space. It was called Beil's metric.

We notice that for $B^i = y^i$ in (3.1) one obtains a general version of the Synge metric which was used by R. Miron for a geometrical theory of Relativistic Optics (cf. [9], ch.XI).

In the following we shall assume $B^i \neq y^i$ and use the ideas and techniques from [9], ch.XI.

One says that ${}^*g_{ij}$ is reducible to a Lagrange metric, shortly an L–metric if there exists a Lagrangian $L : TM \to \mathbb{R}$ such that ${}^*g_{ij} = \dfrac{1}{2}\dot\partial_i\dot\partial_j L$. A necessary and sufficient condition for ${}^*g_{ij}$ be reducible to an L–metric is the symmetry in all indices of the Cartan tensor field ${}^*C_{ijk} = \dfrac{1}{2}\dot\partial_k{}^*g_{ij}$ i.e.

$$ \dot\partial_k{}^*g_{ij} = \dot\partial_i{}^*g_{kj}. \tag{3.3} $$

Using (3.1) this condition becomes

$$ \begin{gathered} \dot\sigma_k B_iB_j - \dot\sigma_i B_kB_j + \sigma(\dot\partial_k B_i \cdot B_j - \dot\partial_i B_k \cdot B_j) + \\ +\sigma(B_i \cdot \dot\partial_k B_j - B_k \cdot \dot\partial_i B_j) = 0, \ \dot\sigma_k := \dot\partial_k\sigma. \end{gathered} \tag{3.4} $$

Multiplying it by B^j one gets

$$ \begin{gathered} B^2(\dot\sigma_k B_i - \dot\sigma_i B_k) + \sigma B^2(\dot\partial_k B_i - \dot\partial_i B_k) + \\ +\sigma(B_i \cdot \dot\partial_k B_j \cdot B^j - B_k\dot\partial_i B_j \cdot B^j) = 0. \end{gathered} \tag{3.5} $$

If (3.4) is an identity, then (3.5) should be an identity for any σ and B_i. But for $B_i = B_i(x)$ and $\sigma = F^2$, (3.5) reduces to $y_k B_i - y_i B_k = 0$ which is not an identity for any B_i. Thus in general ${}^*g_{ij}(x,y)$ is not reducible to an L–metric.

We have a case when ${}^*g_{ij}(x,y)$ is an L–metric as follows.

Proposition 3.1. *Assume $B_i = B_i(x)$. If $\sigma(x,y) = f(B_i(x)y^i)$ for a smooth function $f : \mathbb{R} \to \mathbb{R}$, then ${}^*g_{ij}$ is an L–metric.*

Indeed, it is easy to check that in these hypothesis (3.4) identically holds. Notice that we do not know which is L such that ${}^*g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L$.

It is said that ${}^*g_{ij}(x,y)$ is weakly regular if its absolute energy

$$\mathcal{E}(x,y) := {}^*g_{ij}(x,y)y^iy^j = F^2(x,y) + \sigma(x,y)(B_iy^i)^2 \tag{3.6}$$

is a regular Lagrangian i.e. the matrix with the entries

$$a_{kh}(x,y) = \frac{1}{2}\dot{\partial}_k\dot{\partial}_h\mathcal{E}, \tag{3.7}$$

is of rank n.

A direct calculation yields

$$a_{kh} = g_{kh} + \frac{1}{2}\dot{\sigma}_{kh}\beta^2 + \beta(\dot{\sigma}_k\dot{\beta}_h + \dot{\sigma}_h\dot{\beta}_k) + \sigma\dot{\beta}_k\dot{\beta}_h + \sigma\beta\dot{\beta}_{kh}, \tag{3.8}$$

$$\beta := B_i(x,y)y^i, \dot{\beta}_k := \dot{\partial}_k\beta, \dot{\beta}_{kh} := \dot{\partial}_k\dot{\partial}_h\beta, \dot{\sigma}_{kh} := \dot{\partial}_k\dot{\partial}_h\sigma,\ \ \dot{\sigma}_k := \dot{\partial}_k\sigma \tag{3.8$'$}$$

It is hopeless to decide if a_{kh} is invertible or not. However we have some interesting particular cases.

Proposition 3.2

a) *If B is orthogonal to the Liouville vector field $\mathbb{C} = y^i\dot{\partial}_i$, then ${}^*g_{ij}$ is weakly regular and $a_{kh}(x,y) = g_{kh}(x,y)$.*

b) *If $B_i = B_i(x)$ and $\sigma(x,y) = f(\beta)$ for some smooth function $f : R \to R$, then ${}^*g_{ij}$ is weakly regular if and only if $1 + \varphi(\beta)B^2 \neq 0$, where $2\varphi(\beta) = \beta^2 f'' + 4\beta f' + 2f$, $f' = \dfrac{df}{d\beta}$, $f'' = \dfrac{d^2f}{d\beta^2}$ and we have*

$$a_{kh}(x,y) = g_{kh}(x,y) + \varphi(x,y)B_k(x)B_h(x). \tag{(3.9}$$

Proof. a) The condition B orthogonal to $\mathbb{C}$ is equivalent to $\beta = 0$. Thus $\mathcal{E}(x,y) = F^2(x,y)$ and so $a_{kh} = g_{kh}$.

b) By a direct calculation one finds ((3.9). Hence (a_{kh}) has the same form as ${}^*g_{kh}$ with σ replaced by φ. The conclusion follows.

We keep the hypothesis $B_i = B_i(x)$ and $\sigma = f(\beta)$, $\beta \neq 0$. From ((3.9) we see that we have again $a_{kh} = g_{kh}$ when $\varphi = 0$. The differential equation $\beta^2 f'' + 4\beta f' + 2f = 0$

takes the form $(\beta^2 f' + 2\beta f)' = 0$ and so its general solution is $f(\beta) = \dfrac{a}{\beta} + \dfrac{b}{\beta^2}$, $a, b \in \mathbb{R}$. The metric ${}^*g_{ij}$ becomes

$$((3.10) \qquad {}^*g_{ij} = g_{ij} + \left(\frac{a}{B_i(x)y^i} + \frac{b}{(B_s(x)y^s)^2}\right) B_i(x)B_j(x).$$

Notice that although ${}^*g_{ij}$ is an L–metric, we do not yet know the Lagrangian L.

The absolute energy of ${}^*g_{ij}$ is now $\mathcal{E} = F^2 + a(F_i(x)y^i) + b$ and the Lagrange space $L^n = (M, \mathcal{E})$ is called an almost Finslerian–Lagrange space (see Section 6, ch.IX of [9]).

We may put ((3.9) into the form

$$((3.9)' \qquad a_{kh}(x,y) = {}^*g_{kh} + \left(\frac{1}{2}\beta^2 f'' + 2\beta f'\right) B_k B_h.$$

Thus we see that $a_{kh} = {}^*g_{kh}$ if and only if f is a solution of the differential equation

$$\frac{1}{2} f''\beta^2 + 2\beta f' = 0 \text{ i.e. } f(\beta) = c - \frac{d}{\beta^3},\ c, d \in \mathbb{R}.$$

We know that ${}^*g_{kh}$ is an L–metric (in previous hypothesis). The condition $a_{kh} = {}^*g_{kh}$ gives L in the form $L(x,y) = \mathcal{E}(x,y) + A_i(x)y^i + \psi(x)$, where A_i is a covector and ψ a scalar. Inserting here $\mathcal{E}$ we get

$$((3.10)' \quad L(x,y) = F^2(x,y) + c(B_i(x,y)y^i)^2 - \frac{d}{B_i(x)y^i} + A_i(x)y^i + \psi(x),\ c, d \in \mathbb{R}.$$

Therefore we found a case when ${}^*g_{ij}$ is an L–metric with L of explicit form ((3.10)$'$.

Remark 3.2 In the hypothesis of a) in Proposition 3.2, ${}^*g_{ij}$ is not necessarily an L–metric. If $\sigma(x,y)$ and $B_i(x,y)$ are positively homogeneous of degree 0, then ${}^*g_{ij}(x,y)$ is so and $(M, {}^*g_{ij})$ is a generalized Finsler space in Izumi' sense (see [6]).

Remark 3.3. The condition B orthogonal to $\mathbb{C}$ is equivalent with the condition B is tangent to the indicatrix bundle $I(M) \subset TM$.

Caution. The conditions $\beta = 0$ and $B_i = B_i(x)$ are incompatible since they lead to $B = 0$.

Remark 3.4. If in ((3.10) we take $d = 0$, $A_i = 0$, $\psi = 0$, $c > 0$, then ${}^*F^2 := L(x,y)$ is positively homogeneous of degree 2 and so ${}^*F^n = (M, {}^*F)$ becomes a Finsler space. Notice that ${}^*F$ is getting from F by a β–change and in this case ${}^*g_{ij}$ reduces to a Finsler metric.

Remark 3.5. An interesting Beil metric can be associated to a Finsler space F^n with an (α, β)–metric. Here $\alpha^2 = a_{ij}(x)y^i y^j$ and $\beta = b_i(x)y^i$, where a_{ij} is a Riemannian metric an b_i a covector field on M. One may consider

$$(3.12) \qquad {}^*g_{ij}(x,y) = a_{ij}(x) + \sigma(x,y)b_i(x)b_j(x),$$

where σ is a Finsler scalar such that $1 + \sigma b^2 \neq 0$ for $b^2 = a^{ij}b_i b_j$. This GL–metric is not reducible to an L–metric or a Finsler metric. The previous discussion applies, too.

4 Metrical connections for $GL = (M, {}^*g_{ij}(x,y))$

In Finsler geometry as well as in their generalizations, the nonlinear connections play an important role. For instance these connections allow us to work with *d*– or Finsler objects and so to keep and check easily the geometrical meaning of calculation in local coordinates.

A nonlinear connection always exists if M is paracompact. But the nonlinear connections derived from or associated in a way to a GL–metric are much more useful. There are no possibilities to find nonlinear connections for any GL–metric. But there are some classes of GL–metrics for which such possibilities exist. One is that of weakly regular GL–metrics and as it is well known there exist nonlinear connections canonically derived from a Lagrangian, a Finslerian or a Riemannian metric. See [9] for details.

We recall here the Cartan nonlinear connection for F^n. Set

$$\gamma^i_{jk}(x,y) = \frac{1}{2}g^{ih}(\partial_j g_{hk} + \partial_k g_{hj} - \partial_h g_{jk}),\ \gamma^i_{00} := \gamma^i_{jk}y^j y^k. \tag{4.1}$$

Then $\overset{\circ}{N}{}^i_j = \frac{1}{2}\dot\partial_j\gamma^i_{00}$ are the local coefficients of the Cartan nonlinear connection.

For any Finsler connection $F\Gamma(N)$ we denote by $_{|k}$ and $\big|_k$ its h– and v–covariant derivatives. Then $F\Gamma(N)$ is called h–metrical if $g_{ij|k} = 0$ and v–metrical if $g_{ij}\big|_k = 0$.

We consider

$$\begin{aligned} F^i_{jk} &= \frac{1}{2}g^{ih}(\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk}),\\ C^i_{jk} &= \frac{1}{2}g^{ih}(\dot\partial_j g_{hk} + \dot\partial_k g_{jh} - \dot\partial_h g_{jk}), \end{aligned} \tag{4.2}$$

where $\delta_j = \partial_j - \overset{\circ}{N}{}^k_j\dot\partial_k$. For F^n we have four remarkable Finsler connections based on $(\overset{\circ}{N}{}^i_j)$.

We mention here only the Cartan connection $C\Gamma(\overset{\circ}{N}) = (\overset{\circ}{N}{}^i_j, F^i_{jk}, C^i_{jk})$. This is v– and h–metrical and two torsions of it vanishes.

Let us come back to the GL–metric (3.1). We cannot derive a nonlinear connection from it. But since it is constructed with $g_{ij}(x,y)$, we may take into consideration the Cartan nonlinear connection $(\overset{\circ}{N}{}^i_j)$ and then all possible nonlinear connections have the form $N^i_j = \overset{\circ}{N}{}^i_j - A^i_j$ with $A^i_j(x,y)$ an arbitrary Finsler tensor field of type $(1,1)$.

Now we replace in the right side of (4.2) the metric g_{ij} by ${}^*g_{ij}$ and the operator δ_j by ${}^s\delta_j = \partial_j - \overset{\circ}{N}{}^k_j\dot\partial_k + A^k_j\dot\partial_k$ and denote the results in the left side by ${}^sF^i_{jk}$ and ${}^sC^i_{jk}$, respectively. Thus we get a Finsler connection ${}^sC\Gamma(N) = (N^i_j, {}^sF^i_{jk}, {}^sC^i_{jk})$ which we call standard metrical connection of GL.

This connection is metrical i.e. ${}^*g_{\underset{s}{ij|k}} = 0$, ${}^*g_{ij}\overset{s}{\big|}_k = 0$ and its $h(hh)$– torsion and $v(vv)$–torsion vanish. It is clear that it depends on A^i_j but if A^i_j is given apriori it is the unique Finsler connection with the above properties. For $A^i_j = 0$ we set ${}^*F := {}^sF$ and ${}^*C := {}^sC$. Thus we have

$$
{}^{s}F^i_{jk} = {}^{*}F^i_{jk} + \frac{1}{2}{}^{*}g^{ih}(A^s_j\dot{\partial}_s{}^{*}g_{hk} + A^s_k\dot{\partial}_s{}^{*}g_{hj} - A^s_h\dot{\partial}_s{}^{*}g_{jk})
$$
$$
{}^{s}C^i_{jk} = {}^{*}C^i_{jk}. \tag{4.3}
$$

The first equation in (4.3) takes also the form

$$
{}^{s}F_{jik} = {}^{*}F_{jik} + {}^{*}C_{kis}A^s_j + {}^{*}C_{jis}A^s_k - {}^{*}g^{ih}A^l_h{}^{*}C_{jkl}.
$$

Remark 4.1. If $({}^{*}g_{ij})$ reduces to an L–metric or to a Finsler metric, (4.3) becomes

$$
{}^{s}F^i_{jk} = {}^{*}F^i_{jk} + C^i_{ks}A^s_j
$$
$$
{}^{s}C^i_{jk} = {}^{*}C^i_{jk}. \tag{4.3$'$}
$$

We notice the following possible choices of A^i_j : $\lambda(x,y)\delta^i_j$, y^iy_j, B^iy_j, y^iB_j, B^iB_j.

By (3.1) we find

$$
\begin{aligned}
{}^{*}F^i_{jk} &= B^i_sF^s_{jk} + \frac{\sigma}{2}\,\overset{*}{g}{}^{ih}[\delta_j(B_hB_k) + \delta_k(B_hB_j) - \delta_h(B_jB_k)]+\\
&+\frac{1}{2}{}^{*}g^{ih}(\sigma_jB_hB_k + \sigma_kB_hB_j - \sigma_hB_jB_k),\\
{}^{*}C^i_{jk} &= B^i_sC^s_{jk} + \frac{\sigma}{2}\,\overset{*}{g}{}^{ih}[\dot{\partial}_j(B_hB_k) + \dot{\partial}_j(B_hB_j) - \dot{\partial}_j(B_jB_k)]+\\
&+\frac{1}{2}{}^{*}g^{ih}(\dot{\sigma}_jB_hB_k + \dot{\sigma}_kB_hB_j - \dot{\sigma}_hB_jB_k), \quad \text{with}
\end{aligned} \tag{4.4}
$$

$$
B^i_s = \dot{\partial}^i_s - {}^{*}\sigma B^iB_s,\ \sigma_k := \delta_k\sigma,\ \dot{\sigma}_k := \dot{\partial}_k\sigma,\ {}^{*}\sigma = \sigma/(1+\sigma B^2). \tag{4.4$'$}
$$

Now, ${}^{s}F^i_{jk}$ and ${}^{s}C^i_{jk}$ are easily deduced from (4.3).

Remark 4.2. The matrix B^i_s is invertible. Its inverse is $(B^{-1})^s_k = \delta^s_k + \sigma B^sB_k$. As such from (4.4) we can find F and C as depending on ${}^{*}F$ and ${}^{*}C$.

In order to evaluate the torsions and curvatures of ${}^{*}C\Gamma({}^{c}N)$ it is more convenient to put (4.4) into the form

$$
{}^{*}F^i_{jk} = F^i_{jk} + \Lambda^i_{jk},
$$
$$
{}^{*}C^i_{jk} = C^i_{jk} + \overset{\circ}{\Lambda}{}^i_{jk}, \text{ for} \tag{4.5}
$$

$$
\begin{aligned}
\Lambda^i_{jk} &= \frac{1}{2}{}^{*}g^{ih}[\delta_k(\sigma B_jB_h) + \delta_j(\sigma B_hB_k) - \delta_h(\sigma B_jB_k)]+\\
&-{}^{*}\sigma B^iB^hF_{jhk}\\
\overset{\circ}{\Lambda}{}^i_{jk} &= \frac{1}{2}{}^{*}g^{ih}[\dot{\partial}_k(\sigma B_jB_h) + \dot{\partial}_j(\sigma B_hB_k) - \dot{\partial}_h(\sigma B_jB_k)]+\\
&-{}^{*}\sigma B^iB^hC_{ijk}.
\end{aligned} \tag{4.5$'$}
$$

The torsions of ${}^{*}C\Gamma({}^{c}N)$ are as follows.

$$(4.6)\qquad \begin{aligned} &{}^*T^i_{jk} = 0,\ {}^*R^i_{jk} = R^i_{jk},\ {}^*S^i_{jk} = 0 \\ &{}^*P^i_{jk} = P^i_{jk} - \Lambda^i_{kj} \text{ and } {}^*C^i_{jk} \text{ from (4.5).} \end{aligned}$$

As for the curvatures we have

$$(4.7)\qquad {}^*S_j{}^i{}_{kh} = S_j{}^i{}_{kh} + \overset{o}{\Lambda}{}_j{}^i{}_{kh} + (C^s_{jk}\overset{o}{\Lambda}{}^i_{sh} + \overset{o}{\Lambda}{}^s_{jk}C^i_{sh} - (k/h))$$

$$(4.7)'\qquad \overset{o}{\Lambda}{}_j{}^i{}_{kh} = \dot\partial_h \overset{o}{\Lambda}{}^i_{jk} + \overset{o}{\Lambda}{}^s_{jk}\overset{o}{\Lambda}{}^i_{sh} - (k/h),$$

where $-(k/h)$ means the substraction of the preceeding terms with k replaced by h.

$$(4.8)\qquad \begin{aligned} {}^*P_j{}^i{}_{kh} = P_j{}^i{}_{kh} + \dot\partial_h\Lambda^i_{jk} - \overset{o}{\Lambda}{}^i_{jh|k} - C^i_{jh\|k} - \overset{o}{\Lambda}{}^i_{jh\|k} + \\ +\dot\partial_k C^i_{jh} + \dot\partial_k \overset{o}{\Lambda}{}^i_{jh} - C^i_{js}\Lambda^s_{hk} + \overset{o}{\Lambda}{}^i_{js}P^s_{hk} - \overset{o}{\Lambda}{}^i_{js}\Lambda^s_{kh}, \end{aligned}$$

where $\|k$ denotes a covariant derivative constructed with Λ^i_{jk}.

$$(4.9)\qquad {}^*R_j{}^i{}_{kh} = R_j{}^i{}_{kh} + \Lambda_j{}^i{}_{kh} + (F^s_{jk}\Lambda^i_{sh} + \Lambda^s_{jk}F^i_{sh} - (k/h)) + \overset{o}{\Lambda}{}^s_{js}R^s_{kh},$$

where

$$(4.9)'\qquad \Lambda_j{}^i{}_{kh} = \delta_h\Lambda^i_{jk} + \Lambda^s_{jk}\Lambda^i_{sh} - (k/h).$$

5 Parallel and concurrent Finsler vector fields

Let $B^i(x,y)$ be a Finsler vector field and $F\Gamma(N)$ be a Finsler connection. Then it is said that (B^i) is parallel if

$$(5.1)\qquad B^i_{|k} = 0,\ B^i\big|_k = 0$$

and (B^i) is concurrent if

$$(5.2)\qquad B^i_{|k} = -\delta^i_k,\ B^i\big|_k = 0.$$

It is our purpose to confirm the correctness of these definitions from the viewpoint of the almost Kählerian model of a Finsler space (see [9], ch.VII for details on this model). A different confirmation of these definitions is given in [8] using the principal Finsler bundle model due to M. Matsumoto. The giving of N is equivalent to the decomposition

$$(5.3)\qquad T_uTM = H_uTM \oplus V_uTM,\ u \in TM \text{ (Whitney' sum)}.$$

Accordingly we have two projectors h and v and an almost product structure P such that if we put $X = hX + vX$ for a vector field X on TM, then

$$(5.5)\qquad P(hX) = hX,\ P(vX) = -vX.$$

The set of Finsler connections is in a one–to–one correspondence with the set of linear connections on TM which verify

(5.6) $$D_X P = 0,\ D_X F = 0 \text{ for any vector field } X \text{ on } TM.$$

By the very definition, a vector field B on TM is parallel with respect to D if

(5.7) $$D_X B = 0,$$

and is concurrent if

(5.8) $$D_X B = -X, \text{ for any vector field } X \text{ on } TM.$$

Let $(\delta_i, \dot{\partial}_i)$ be the usual adapted basis for the decomposition (5.3). The above mentioned one–to–one correspondence is established by

(5.9) $$\begin{aligned} D_{\delta_k}\delta_j &= L^i_{jk}\delta_i, \quad D_{\dot{\partial}_k}\delta_j = V^i_{jk}\delta_i,\\ D_{\delta_k}\dot{\partial}_j &= L^i_{jk}\dot{\partial}_i, \quad D_{\dot{\partial}_k}\dot{\partial}_j = V^i_{jk}\dot{\partial}_i, \end{aligned}$$

for $D \leftrightarrow F\Gamma(N) = (N^i_j, L^i_{jk}, V^i_{jk})$.

It is obvious that (5.7) is equivalent to

(5.7)$'$ $$D_{\delta_k}B = 0,\ D_{\dot{\partial}_k}B = 0,$$

and (5.8) is equivalent to

(5.8)$'$ $$D_{\delta_k}B = -\delta_k,\ D_{\dot{\partial}_k}B = -\dot{\partial}_k.$$

Let now be $B = B^i(x,y)\delta_i + \hat{B}^i(x,y)\dot{\partial}_i$. Then (5.7)$'$ is equivalent by virtue of (5.9) with

(5.7)$''$ $$B^i_{|k} = 0,\ B^i\big|_k = 0,\ \hat{B}^i_{|k} = 0,\ \hat{B}^i\big|_k = 0.$$

One may associate to $B^i(x,y)$ at least the following three vector fields on $TM : B^i\delta_i$, $B^i\dot{\partial}_i$, $B^i\delta_i + B^i\dot{\partial}_i$ and it is obvious by (5.7)$''$ that $B^i(x,y)$ is parallel in the sense of (5.1) if and only if at least one from these vector fields on TM is parallel with respect to D. Thus (5.1) is in agreement with the usual definition of parallelism.

Let us make a similar analysis for concurrent Finsler vector fields. By (5.8), B is concurrent on TM if and only if

(5.10) $$B^i_{|k} = -\delta^i_k,\ B^i\Big|_k = 0,\ \hat{B}^i_{|k} = 0,\ \tilde{B}^i\big|_k = -\delta^i_k.$$

Now we assume that D or $F\Gamma(N)$ is of Cartan type, that is,

(5.11) $$y^i_{|k} = 0,\ y^i\big|_k = \delta^i_k.$$

The tensors $y^i_{|k}$ and $y^i\big|_k$ are called h–deflection and v–deflection tensors, respectively. The equations (5.11) hold for all four remarkable connections in Finsler spaces.

If moreover we assume that $\hat{B}^i$ is positively homogeneous of degree 1, a natural assumption in Finslerian setting, writing $\hat{B}^i\big|_k = -\delta^i_k$ in the form $\dot{\partial}_k\hat{B}^i + V^i_{jk}\hat{B}^j = -\delta^i_k$ and contracting it by y^k it results using (5.11) that $y^k\dot{\partial}_k\hat{B}^i = -y^i$. Thus by the Euler theorem, $\hat{B}^i = -y^i$ and then $\hat{B}^i_{|k} = 0$ reduces to $y^i_{|k} = 0$ i.e. the first equation in (5.11). Concluding, if we associate to the Finsler vector field $B^i(x,y)$ the vector field $B = B^i(x,y)\delta_i - y^i\dot{\partial}_i$ on TM, we find that $(B^i(x,y))$ is concurrent in the sense of (5.2) if and only if B is concurrent by the new definition of concurrence on any manifold. In other words, the condition (5.2) is in agreement with the notion of concurrence for vector fields.

6 The metric ${}^*g_{ij}$ with $B^i(x,y)$ a concurrent Finsler vector field

In this section we are dealing with the GL–metric ${}^*g_{ij}$ given by (3.1) for $B^i(x,y)$ a concurrent Finsler vector field with respect to the Cartan connection $C\Gamma$ of F^n i.e.

$$B^i_{|j} = -\delta^i_j,\ B^i\big|_j = 0. \tag{6.1}$$

First we notice some results on concurrent Finsler vector fields due to M. Matsumoto and K. Eguchi [8].

If $B^i(x,y)$ is concurrent we have with respect to $C\Gamma$:

(6.2) $B_{i|j} = -g_{ij},\ B_i\big|_j = 0,$

(6.3) $B^h R_{hijk} = 0,\ B^h P_{hijk} + C_{ijk} = 0,\ B^h S_{hijk} = 0,$

(6.4) $B^i C_{ijk} = C^s_{jk} B_s = 0,$

(6.5) $B^i = B^i(x)$ and $B_i = B_i(x)$ i.e. B^i and B_i are functions on position only,

(6.6) $\partial_i B_j = \partial_j B_i = F^s_{ij} B_s - g_{ij},\ \partial_k B^i = -\delta^i_k - F^i_{sk} B^k.$

In these circumstancies a direct calculation yields

$$\begin{aligned} \Lambda^i_{jk} &= \frac{{}^*\sigma}{2\sigma} B^i(\sigma_k B_j + \sigma_j B_k + \sigma(B^s\sigma_s)B_j B_k - 2\sigma g_{jk}) - \frac{1}{2}\sigma^i B_j B_k \\ \overset{\circ}{\Lambda}{}^i_{jk} &= \frac{{}^*\sigma}{2\sigma} B^i(\dot\sigma_k B_j + \dot\sigma_j B_k + \sigma(B^s\dot\sigma_s)B_j B_k - \frac{1}{2}\dot\sigma^i B_j B_k, \text{ where} \end{aligned} \tag{6.7}$$

$$\sigma_k := \delta_k\sigma,\ \dot\sigma_k := \dot\partial_k\sigma,\ \sigma^i = g^{ik}\sigma_k,\ \dot\sigma^i = g^{ik}\dot\sigma_k. \tag{6.7$'$}$$

Looking at (6.7) we see that the simplest case is given by

$$\sigma_k = 0,\ \dot\sigma_k = 0. \tag{6.8}$$

From (6.8) it results that σ is a constant c. And ${}^*F^2 := {}^*g_{ij}y^iy^j$ takes the form

$${}^*F^2 = F^2 + c\beta^2,\ \beta = B_i(x)y^i. \tag{6.9}$$

Thus, for $c > 0$, ${}^*F$ is a new Finsler function which is obtained from F by a particular β–change.

The case $c = 1$ is studied in [8].

Further on we have

$${}^*F^i_{jk} = F^i_{jk} - {}^*\sigma B^i g_{jk},\ {}^*C^i_{jk} = C^i_{jk}. \tag{6.10}$$

Remark 6.1. The Cartan nonlinear connection of ${}^*F^n = (M, {}^*F)$ is given by $N^i_j = \overset{c}{N}{}^i_j - \overset{*}{\sigma} B^i y_j$ i.e. the difference tensor is $A^i_j = \overset{*}{\sigma} B^i y_j$. Inserting it in $(4.3)'$ we find ${}^sF^i_{jk} = {}^*F^i_{jk}$. Therefore, in the geometry of ${}^*F^n$ we may equally use $\overset{c}{N}{}^i_j$ or N^i_j.

By (6.10) we immediately get

$$ {}^*S_{ijkh} = S_{ijkh}. \tag{6.11} $$

Again by (6.10) but after a long calculation one finds

$$ {}^*R_{ijkh} = R_{ijkh} + {}^*\sigma(g_{ik}g_{jh} - g_{ih}g_{jk}). \tag{6.12} $$

This suggests us to take into consideration the case when F^n is h–isotropic i.e. there exists a constant K such that $R_{ijkh} = K(g_{ik}g_{jh} - g_{ih}g_{jk})$. A contraction of this last equation by B^i gives for $K \neq 0$, $B_k g_{jh} - B_h g_{jk} = 0$ in virtue of (6.3). A new contraction by B^k yields $B^2 g_{jh} = B_j B_h$ which contradicts the condition rank $(g_{ij}) = n > 1$. Thus we have

Theorem 6.1. *If F^n is h–isotropic, then it does not admit any concurrent Finsler vector field.*

The proof of the following two theorems are the same as for $c = 1$ (see Theorems 14 and 15 in [8]).

Theorem 6.2. *If F^n admits a concurrent Finsler vector field, then there is no a Finsler vector field which to be concurrent with respect to ${}^*F$ given by* (6.9).
Theorem 6.3. *If F^n admits a concurrent Finsler vector field and is $R3$–like, then ${}^*F^n = (M, {}^*F)$ with ${}^*F$ from* (6.5) *is also $R3$–like.*

Now we consider a more complicated case

$$ \sigma_k = 0, \ \dot{\sigma}_k \neq 0. \tag{6.13} $$

Remark 6.2. The equation $\sigma_k := \dfrac{\partial \sigma}{\partial x^h} - \overset{c}{N}{}^s_k \dfrac{\partial \sigma}{\partial y^s} = 0$ is known as Tavakol–Van der Berg equation. A solution of it is for instance $\sigma = aF^2$ for $a \in \mathbb{R}$. For more details see [12].

Now (6.10) is replaced by

$$ \begin{aligned} {}^*F^i_{jk} &= F^i_{jk} - {}^*\sigma B^i g_{jk} \\ {}^*C^i_{jk} &= C^i_{jk} + \frac{{}^*\sigma}{2\sigma} B^i(\dot{\sigma}_k B_j + \dot{\sigma}_j B_k + \sigma(B^s \dot{\sigma}_s) B_j B_k) - \frac{1}{2}\dot{\sigma}^i B_j B_k. \end{aligned} \tag{6.14} $$

The Remark 6.1 is still valid for this case. Precisely, if we ask for the vanishing of the h–deflection of ${}^*F\Gamma(\overset{c}{N})$, then ${}^*N^i_j = \overset{c}{N}{}^i_j - \overset{*}{\sigma} B^i y_j$ and so ${}^sF\Gamma(\overset{*}{N})$ coincides with ${}^*F\Gamma(\overset{c}{N})$.

Now we notice

$$ {}^*C_j = C_j + \frac{{}^*\sigma B^2}{2\sigma}\dot{\sigma}_j, \ C_j := C^i_{ji}, \tag{6.15} $$

$$ {}^*C_{jik} = C_{jik} + \frac{1}{2}(\dot{\sigma}_k B_i B_j + \dot{\sigma}_j B_i B_k - \dot{\sigma}_i B_j B_k). \tag{6.16} $$

A long calculation yields

(6.17)
$$\begin{aligned}{}^*R_{jskh} &= R_{jskh} + {}^*\sigma(g_{jk}g_{sh} - g_{jh}g_{sk}) + \\ &+ \frac{{}^*\sigma}{\sigma}B_s(\partial_k\sigma \cdot g_{jh} - \partial_h\sigma \cdot g_{jk}) + \frac{1}{2}B_jB_sR^q_{kh}\dot{\sigma}_q.\end{aligned}$$

Let us assume that F^n is a locally Minkowski space. Then $R_j{}^i{}_{kh} = 0$ and $C^i_{jk|h} = 0$. In a local chart in which g_{ij} do not depend on x we have $\overset{c}{N}{}^i_j = 0$ and so $\partial_k\sigma = \overset{c}{N}{}^p_j\dot{\sigma}_p = 0$ i.e. σ does not depend on x.

The equation (6.17) reduces to

(6.18)
$${}^*R_{jskh} = {}^*\sigma(g_{jk}g_{sh} - g_{jh}g_{sk}).$$

It takes also the form

(6.18)$'$
$$\begin{aligned}&{}^*R_{jskh} = {}^*\sigma({}^*g_{jk}{}^*g_{sh} - {}^*g_{jh}{}^*g_{sk}) + \sigma{}^*\sigma(B_jB_{hsk} + B_sB_{kjh}) \text{ for} \\ &B_{hsk} := B_hg_{sk} - B_kg_{sh}.\end{aligned}$$

We notice that B_{hsk} is never vanishing since otherwise a contraction by B^h gives a contradiction with rank $(g_{ij}) = n > 1$.

7 A Beil metric for a Finsler space with (α, β)–metric

Here we consider again the Beil metric described in Remark 3.5. Let F^n be a Finsler space with an (α, β)–metric. A natural Beil metric is then

(7.1)
$${}^*g_{ij}(x, y) = a_{ij}(x) + \sigma(x, y)b_i(x)b_j(x).$$

Let γ^i_{jk} be the Christoffel symbols for $a_{ij}(x)$. Then $\overset{c}{N}{}^i_j = \gamma^i_{jk}y^k =: \gamma^i_{j0}$ and the triple $\Gamma = (\gamma^i_{j0}, \gamma^i_{jk}, 0)$ may be thought of as a Finsler connection.

We have

Theorem 7.1. *If $b_i(x)$ is parallel and σ is covariant constant with respect to Γ, then Γ is like Chern–Rund connection for $({}^*g_{ij})$.*

Proof. Let $_{;k}$ denote the h–covariant derivative with respect to Γ. Notice that v–covariant derivative is just the derivative with respect to y. Our hypothesis read

(7.2)
$$b_{i;k} = 0,\ \delta_k\sigma = 0,\ \delta_k = \partial_k - \gamma^s_{k0}\dot{\partial}_s.$$

Then we easily get

(7.3)
$$\begin{aligned}{}^*g_{i;jk} &= (\delta_k\sigma)b_ib_j = 0 \\ {}^*g_{ij,k} &= (\dot{\partial}_k\sigma)b_ib_j = 2{}^*C_{ikj}.\end{aligned}$$

Thus Γ is h–metrical and no metrical for ${}^*g_{ij}$. Hence it is similar to the Chern–Rund connection from Finsler geometry. **q.e.d.**

The Chern–Rund connection is a remarkable one in Finsler geometry ([1]). Notice that its h–deflection vanishes.

From now on we assume $b_{i;k} = 0$ and $\delta_k \sigma = 0$.

A direct calculation yields

$$\begin{aligned} {}^*F^i_{jk} &= \gamma^i_{jk}, \\ {}^*C^i_{jk} &= \frac{\overset{*}{\sigma}}{2\sigma} b^i(\dot{\sigma}_k b_j + \dot{\sigma}_j b_k + \sigma(b^h \dot{\sigma}_h) b_j b_k) - \frac{1}{2}\dot{\sigma}^i b_j b_k. \end{aligned} \tag{7.4}$$

The first equation in (7.4) is important in many respects. For instance using it we find the h–curvature of ${}^*F\Gamma(\overset{c}{N})$ in the form

$$ {}^*R_h{}^i{}_{jh} = \gamma_h{}^i{}_{jh} + \overset{\circ}{\Lambda}_h{}^i{}_s R^s_{jk}, \tag{7.5}$$

where $\gamma_h{}^i{}_{jh}$ is the curvature tensor of $a_{ij}(x)$ and $R^i_{jk} = \gamma_0{}^i{}_{jk}$. Here, as before, the index 0 indicates the contraction by y. Consequently, (7.5) takes the form

$$ {}^*R_h{}^i{}_{jk} = (\delta^i_s \delta^r_h + \overset{\circ}{\Lambda}{}^i{}_{hs} y^r)\gamma_r{}^s{}_{jk}. \tag{7.6}$$

From Ricci identities we find $\gamma_i{}^s{}_{jk} b_s = 0$ and from (7.5) we deduce

$$ {}^*R_{hijk} = \gamma_{hijk} + \frac{1}{2} b_h b_i \gamma_0{}^s{}_{jk} \dot{\sigma}_s. \tag{7.7}$$

As for Ricci curvatures one finds

$$ {}^*R_{ij} = r_{ij}, \tag{7.8}$$

where r_{ij} is the Ricci curvature for $(a_{ij}(x))$. From here it results

$$ {}^*R = r, \tag{7.9}$$

where ${}^*R$ and r are the scalar curvatures for $({}^*g_{ij})$ and $(a_{ij}(x))$, respectively.

So, the h–Einstein tensor field of ${}^*g_{ij}$ i.e. ${}^*E_{ij} = {}^*R_{ij} - \frac{1}{2}{}^*R\,{}^*g_{ij}$ is related to the Einstein tensor E_{ij} of $a_{ij}(x)$ by

$$ {}^*E_{ij} = E_{ij} + \frac{\sigma r}{2} b_i b_j. \tag{7.10}$$

Consequently, the h–Einstein equation for GL i.e. ${}^*E_{ij} = \kappa {}^*\tau_{ij}$ with $\kappa \in \mathbb{R}$ reduces to

$$ r_{ij} - \frac{r}{2} a_{ij} = \kappa \tau_{ij}, \tag{7.11}$$

where

$$ \tau_{ij} = {}^*\tau_{ij} - \frac{\sigma r}{2\kappa} b_i b_j. \tag{7.12}$$

The equation (7.11) is the Einstein equation for $(M, a_{ij}(x))$ but with the energy–momentum tensor influenced by a field described by b_i. In the the unified theory of R.G. Beil the term $b_i b_j$ in (7.12) is a "matter term" which could be the energy density of the self-field of a charged object.

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