Three-Dimensional Slant Submanifolds of K–Contact Manifolds

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Abstract

The intrinsic geometry of 3–dimensional non anti–invariant slant submanifolds of K–contact manifolds is studied. It is proved that these submanifolds are Riemannian manifolds of quasi constant sectional curvature. Moreover explicit formulae are given for sectional curvatures, Ricci tensor, and scalar curvature of a slant submanifold. A necessary condition is proved for the minimality of a 3–dimensional slant submanifold of a Sasakian manifold of constant φ–sectional curvature. Finally it is proved that every 3–dimensional slant submanifold of a K-contact manifold is locally homogeneous provided it has constant horizontal curvature with respect to the induced metric.

Mathematics Subject Classifications: 53C25, 53C42, 53C30
Key Words: slant submanifolds–contact manifolds

Introduction

Let M be an almost contact metric manifold with structure (φ, ξ, η, g) (our standard reference for contact geometry is [2]). By a *slant* submanifold of M, we mean an immersed submanifold N such that for any $x \in N$ and any $X \in T_xN$ linearly independent on ξ, the angle between φX and T_xN is a constant $\vartheta \in [0, \frac{\pi}{2}]$, called the *slant angle* of N in M and denoted by $sla(N)$. General properties of slant submanifolds according to this definition are studied by the author in [8].

The results given here apply to non anti–invariant three–dimensional slant submanifolds in K–contact manifolds, that is contact metric manifolds whose characteristic vector field is Killing (see [2]). In particular in section 2 we study the intrinsic geometry of a slant submanifold as a Riemannian manifold endowed with the induced metric. This is made using some fundamental properties of the so–called Riemannian manifolds of quasi–constant sectional curvatures (QC–manifolds),which have been studied in [5]. The results used here are recalled in section 1. Indeed, we prove that any 3–dimensional slant submanifold of a K–contact manifold is a QC–manifold with respect to the induced metric. As a consequence, we get explicit formulae for the sectional curvatures, Ricci tensor and scalar curvature of a slant submanifold.

Balkan Journal of Geometry and Its Applications, Vol.3, No.1, 1998, pp. 37-51

Moreover, we find a necessary condition for the minimality of a slant submanifold of a Sasakian manifold of constant φ–sectional curvature. This condition involves the so-called *horizontal curvatures* of the submanifold.

Section 3 deals with three dimensional slant submanifolds having *constant horizontal curvature*. We give a characterization of these slant submanifolds by means of an algebraic property of the Ricci tensor. The main result shows that slant submanifolds of constant horizontal curvature are locally homogeneous Riemannian manifolds with respect to the induced metric.

We also remark that this study is motivated by the existence of examples. In fact, it can be shown that the standard Sasakian space form $\mathbf{R}^{2n+1}$ admits 3–dimensional harmonic foliations whose leaves are all slant submanifolds with constant horizontal curvature (see [10]).

1 Riemannian manifolds of quasi–constant sectional curvatures

In this section we recall some fundamental facts about Riemannian manifolds of quasi–constant sectional curvatures (QC–manifolds), which can be found in [5].

We denote by (M, g, ξ) a Riemannian manifold together with a globally defined unit vector field. Let $\pi \subset T_pM$ be a plane with sectional curvature $K(\pi)$. If ξ_p is orthogonal (resp. belongs) to π, $K(\pi)$ is said to be *horizontal* (resp. *vertical*) and the plane itself is called *horizontal* (resp. *vertical*).

(M, g, ξ) is called a *Riemannian manifold of quasi–constant sectional curvatures (QC–manifold)* if for all planes $\pi \subset T_pM$, $K(\pi)$ depends only on the point p and on the angle ψ_π between ξ_p and π. In this case there exist differentiable functions a, b on M such that for all $p \in M$ and $\pi \subset T_pM$:

$$K(\pi) = a(p) + b(p)\cos^2\psi_\pi. \tag{1.1}$$

Hence in a QC–manifold, all horizontal (resp. vertical) sectional curvatures at the same point p are equal to $a(p)$ (resp. $a(p) + b(p)$).

Let Γ and Ω be the tensor fields on M defined by

$$\begin{aligned}
\Gamma(X,Y,Z,U) &= g(X,Z)g(Y,U) - g(Y,Z)g(X,U)\\
\Omega(X,Y,Z,U) &= g(Y,U)\eta(X)\eta(Z) - g(X,U)\eta(Y)\eta(Z)\\
&+ g(X,Z)\eta(Y)\eta(U) - g(Y,Z)\eta(X)\eta(U).
\end{aligned}$$

The following result gives a useful characterization of QC–manifolds by means of the curvature tensor:

Proposition 1.1 ([5]) *(M, g, ξ) is a Riemannian manifold of quasi–constant sectional curvatures if and only if there exist functions a, b on M such that*

$$R = a\Gamma + b\Omega, \tag{1.2}$$

where R is the curvature tensor of (M, g).

The functions a, b are those in formula (1.1). In particular, if (1.1) or (1.2) holds, it follows that (M, g) has constant sectional curvature if and only if $b = 0$.

We say (M, g, ξ) *has constant vertical curvature* $k \in \mathbf{R}$, if for all $p \in M$ and $\pi \subset T_pM$ vertical plane section, $K(\pi) = k$. Analogously we speak of manifolds (M, g, ξ) with *constant horizontal curvature.*

The following proposition easily follows from (1.1):

Proposition 1.2 *Suppose that* (M, g, ξ) *is a QC–manifold and* $k \in \mathbf{R}$. *The following properties are equivalent:*

i) (M, g) *has constant sectional curvature* k.

ii) (M, g, ξ) *has constant horizontal curvature* k *and constant vertical curvature* k.

Finally we denote by η the canonical 1–form defined by:

$$\eta(X) = g(X, \xi).$$

We recall that (see [5]), if (M, g, ξ) is a QC–manifold with $b \neq 0$, such that $dim(M) \geq 4$, then for all $X, Y \in D$ we have

$$d\eta(X, Y) = 0,$$

where D is the distribution orthogonal to ξ, given by

$$D_p = \{X \in T_pM | \eta(X) = 0\}.$$

2 Three-dimensional slant submanifolds of K–contact manifolds

We start this section recalling some definitions. Let M be a manifold of dimension $2m+1$, $m \geq 1$, and λ a nonzero real number. By a *λ–homotetic contact metric structure* on M we mean any almost contact metric structure (φ, ξ, η, g) such that

$$d\eta = \lambda \Phi$$

where Φ is the associated fundamental 2–form defined by

$$\Phi(X, Y) = g(X, \varphi Y).$$

A *λ–homotetic Sasakian structure* on M is a λ–homotetic contact metric structure which is *normal* in the usual sense, that is:

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0.$$

Obviously if $\lambda = 1$ these notions are exactly those of contact metric structure and Sasakian structure. The following proposition holds (see the theorem at page 73 of [2]):

Proposition 1.1. *A λ–homotetic contact metric structure on M is λ–homotetic Sasakian if and only if*

$$(\nabla_X\varphi)Y = \lambda\{g(X,Y)\xi - \eta(Y)X\}.$$

Now let N be a slant submanifold of odd dimension of an almost contact metric manifold M with structure (φ,ξ,η,g). Let ϑ be the slant angle of N in M. Supposing that N is not anti–invariant, that is $\vartheta \neq \frac{\pi}{2}$, it is known (see [8]) that ξ is tangent to N. Moreover N inherits an almost contact metric structure $(\bar{\varphi},\xi,\eta,g)$ with

$$\bar{\varphi} = \frac{1}{\cos\vartheta}P,$$

where P is the tensor field of type (1,1) on N define by the decomposition

$$\varphi X = PX + FX$$

with PX (resp. FX) tangent (resp. normal) component of φX with respect to N, for any vector field X tangent to N. We remark that the tensor field $Q = P^2$ satisfies

$$QX = \cos^2\vartheta(-X + \eta(X)\xi). \tag{2.1}$$

The following lemma is useful for the local study of 3–dimensional slant submanifolds:
Lemma 2.1. *Let N be a 3–dimensional slant submanifold of an almost contact metric manifold M. Suppose that N is not anti–invariant. If $p \in N$, then in a neighborhood of p there exist vector fields e_1, e_2 tangent to N, such that $\{\xi, e_1, e_2\}$ is a local orthonormal frame satisfying*

$$Pe_1 = \cos\vartheta e_2,\ \ Pe_2 = -\cos\vartheta e_1. \tag{2.2}$$

Proof. Since N has dimension 3, ξ is tangent to N. Hence by the Gram–Schmidt process one gets a local orthonormal frame

$$\{\xi_{|U}, \epsilon_1, \epsilon_2\}$$

defined in a neighborhood U of p. Then the vector fields

$$e_1 = \epsilon_1,\ \ e_2 = \frac{1}{\cos\vartheta}Pe_1$$

have the required properties.

We denote by $\bar{\Phi}$ the fundamental 2–form of $(\bar{\varphi},\xi,\eta,g)$. If the structure of the ambient manifold M is contact metric, then (cfr. [8], theorem 3.6)

$$d\eta = \cos\vartheta\bar{\Phi},$$

hence N is a $\cos\vartheta$–homotetic contact metric manifold.

In the following we shall prove that if $dim(N) = 3$ and the ambient manifold is K–contact, then $(\bar{\varphi},\xi,\eta,g)$ is a $\cos\vartheta$–homotetic Sasakian structure.

First of all we recall that supposing M is K–contact, then for any vector field X tangent to N ([8], lemma 4.1):

$$PX = -\nabla_X \xi, \tag{2.3}$$

where ∇ is the Levi–Civita connection on N. Moreover the following theorem holds

Theorem 2.1. ([8]) *Let N be an immersed submanifold of a K-contact manifold M with structure (φ, ξ, η, g), and suppose that ξ is tangent to N. Then if $\vartheta \in [0, \frac{\pi}{2}]$, the following properties are equivalent:*

a) N is slant in M with slant angle ϑ
b) (N, g, ξ) has constant vertical curvature $\cos^2\vartheta$.

Now we prove

Theorem 2.2. *Let N be a 3–dimensional slant submanifold of a K-contact manifold M with structure (φ, ξ, η, g). Suppose that $\vartheta = sla(N) \neq \frac{\pi}{2}$. Then for any X, Y vvector fields tangent to N*

$$(\nabla_X P)Y = \cos^2\vartheta\{g(X,Y)\xi - \eta(Y)X\}. \tag{2.4}$$

It follows that the $\cos\vartheta$–homotetic contact metric structure induced on N is $\cos\vartheta$–homotetic Sasakian.

Proof. Let X, Y be vector fields tangent to N. Let $p \in N$ and $\{e_1, e_2\}$ the orthonormal frame on N defined in a neighborhood U of p, given by lemma 3.2. We shall prove that the vector fields on the two sides of formula (2.4) coincide on U. Put $\xi_{|U} = e_o$, and let w_i^j be the structural 1–forms defined by

$$\nabla_X e_i = \sum_{j=0}^{2} w_i^j(X) e_j.$$

Obviously

$$w_j^i = -w_i^j, \ w_i^i = 0.$$

Notice that by virtue of (2.3)

$$(\nabla_X P)e_o = \nabla_X P e_o - P(\nabla_X e_o) = QX.$$

Moreover, using (2.2) we get

$$\begin{aligned}(\nabla_X P)e_1 &= \nabla_X(\cos\vartheta e_2) - P(w_1^0(X)e_o + w_1^1(X)e_1 + w_1^2(X)e_2) \\ &= \cos\vartheta(w_2^0(X)e_o + w_2^1(X)e_1) + \cos\vartheta w_1^2(X)e_1 = \\ &= \cos\vartheta w_2^0(X)e_0\end{aligned}$$

and analogously

$$(\nabla_X P)e_2 = -\cos\vartheta w_1^0(X)e_0.$$

Moreover, writing

$$Y = \eta(Y)e_o + g(Y, e_1)e_1 + g(Y, e_2)e_2$$

and using the formulas above it follows

$$(2.5) \qquad (\nabla_X P)Y = \eta(Y)QX + \{\cos\vartheta w_2^0(X)g(Y,e_1) - \cos\vartheta w_1^0(X)g(Y,e_2)\}e_o.$$

Finally, notice that by virtue of (2.3), we have

$$w_2^0(X) = g(\nabla_X e_2, \xi) = -g(e_2, \nabla_X \xi) = g(e_2, PX)$$

and

$$w_1^0(X) = g(e_1, PX).$$

By substituting in (2.5), since P is skew–symmetric we get

$$\begin{aligned} (\nabla_X P)Y &= \eta(Y)QX + \cos\vartheta\{g(PX,e_2)g(Y,e_1) - g(PX,e_1)g(Y,e_2)\}\xi \\ &= \eta(Y)QX + \cos^2\vartheta\{g(X,e_1)g(Y,e_1) + g(X,e_2)g(Y,e_2)\}\xi \\ &= \eta(Y)QX + \cos^2\vartheta\{g(X,Y) - \eta(X)\eta(Y)\}\xi \\ &= \cos^2\vartheta\{-\eta(Y)X + \eta(Y)\eta(X)\xi + g(X,Y)\xi - \eta(Y)\eta(X)\xi \end{aligned}$$

and (2.4) is proved.

Finally, for the induced structure $(\bar\varphi, \xi, \eta, g)$ on N, it follows that

$$(\nabla_X \bar\varphi)Y = \cos\vartheta\{g(X,Y)\xi - \eta(Y)X\}.$$

Hence by virtue of proposition 1.1 this structure is $\cos\vartheta$–homotetic Sasakian.

Corollary 2.1. *Every K–contact structure on a three-dimensional manifold is Sasakian.*

Proof. Let M be a K–contact manifold of dimension 3 with structure (φ, ξ, η, g). It is obvious that the immersed submanifold (M, i), where i is the identity, is an invariant submanifold of this K–contact manifold. On the other hand, it is clear that the induced structure $(\bar\varphi, \xi, \eta, g)$ on M coincides with (φ, ξ, η, g). By the theorem above this structure must be 1–homotetic Sasakian, which means that it is Sasakian.

Theorem 2.2 will be used in order to study the intrinsic geometry of three-dimensional slant submanifolds of K–contact manifolds. First we prove

Proposition 2.1. *Let M be a λ–homotetic Sasakian manifold with structure (φ, ξ, η, g). Then (M, g, ξ) is a QC–manifold if and only if either $dim(M) = 3$ or (M, g) has constant sectional curvature.*

Proof. Suppose that $dim(M) = 3$. For any point p of M, let D_p be the orthogonal complement of ξ_p with respect to g.

We can define a function $a : M \longrightarrow \mathbf{R}$ as follows

$$\forall p \in M \qquad a(p) = K(D_p).$$

We shall prove that for any plane $\pi \subset T_pM$ the corresponding sectional curvature is given by

$$(2.6) \qquad K(\pi) = a(p) + (\lambda^2 - a(p))\cos^2\psi_\pi,$$

where ψ_π is the angle between ξ_p and π. This implies that (M, g, ξ) is a QC–manifold. Put

$$(2.7) \qquad \eta^\star = \lambda\eta, \;\; \xi^\star = \frac{1}{\lambda}\xi, \;\; g^\star = \lambda^2 g.$$

Since the Levi–Civita connections of g and $g^\star$ coincide, by using proposition 1.1 it is easy to verify that $(\varphi, \xi^\star, \eta^\star, g^\star)$ is a Sasakian structure on M.
Let $\pi \subset T_pM$ be a plane; then we have

$$K(\pi) = \lambda^2 K^\star(\pi), \tag{2.8}$$

where we denote by $K^\star(\pi)$ the sectional curvature of π with respect to the metric $g^\star$. Now fix a basis $\{X^\star, Y^\star\}$ of π orthonormal with respect to $g^\star$. Notice that if the vectors $X^\star, Y^\star, \xi^\star$ are linearly dependent then we have $K^\star(\pi) = 1$ hence $K(\pi) = \lambda^2$ and (2.6) is satisfied since obviously $\psi_\pi = 0$. Thus we can suppose that $X^\star, Y^\star, \xi$ are linearly independent so that we can write

$$X^\star = \eta^\star(X^\star)\xi^\star + \alpha Z, \; Y^\star = \eta^\star(Y^\star)\xi^\star + \beta W$$

with $\alpha, \beta \in \mathbf{R}$ and W, Z unit vectors orthogonal to $\xi^\star$ with respect to $g^\star$. Observe that being $dim(M) = 3$, $Span\{Z, W\}$ coincides with the orthogonal complement of $\xi^\star$ in T_pM with respect to $g^\star$. On the other hand (2.7) obviously implies that this orthogonal complement is D_p.
Moreover, the following formula holds (see [2] page 96)

$$K^\star(\pi) = \eta^\star(X^\star)^2 + \eta^\star(Y^\star)^2 + \{1 - \eta^\star(X^\star)^2 - \eta^\star(Y^\star)^2\}K^\star(Z, W). \tag{2.9}$$

Finally the vectors

$$X = \lambda X^\star, \qquad Y = \lambda Y^\star$$

make up an orthonormal basis of π with respect to g.
Hence by using (2.7), (2.8) formula (2.9) can be rewritten as

$$\frac{1}{\lambda^2}K(\pi) = \eta(X)^2 + \eta(Y)^2 + \{1 - \eta(X)^2 - \eta(Y)^2\}\frac{1}{\lambda^2}K(D_p). \tag{2.10}$$

Since

$$\cos^2 \psi_\pi = \eta(X)^2 + \eta(Y)^2$$

substituting in (2.10) we get

$$K(\pi) = \lambda^2 \cos^2 \psi_\pi + (1 - \cos^2 \psi_\pi)a(p),$$

which is exactly (2.6).

To prove the converse, suppose that (M, g, ξ) is a QC–manifold and that (M, g) has not constant sectional curvature. We prove by contradiction that it must be $dim(M) = 3$. Assume that $dim(M) \geq 4$. Then by virtue of the remarks at the end of section §1, for all X, Y vector fields orthogonal to ξ we would have

$$d\eta(X, Y) = 0$$

which is impossible, since $\lambda \neq 0$.
Theorem 2.3. *Let N be a non anti–invariant slant submanifold of a K-contact manifold M with structure (φ, ξ, η, g). Let ϑ be the slant angle of N in M.*
Then (N, g, ξ) is a Riemannian manifold of quasi–constant sectional curvatures if and only if either $dim(N) = 3$ or (N, g) has constant sectional curvature $\cos^2\vartheta$. Moreover if $dim(N) = 3$, the following properties hold:

I. *For any $p \in N$ denote by ${\xi_p}^{\perp}$ the orthogonal complement of ξ_p in T_pN and let $a : N \longrightarrow \mathbf{R}$ the function given by*

$$a(p) = K({\xi_p}^{\perp}). \tag{2.11}$$

Then for any plane $\pi \subset T_pN$:

$$K(\pi) = a(p) + (\cos^2\vartheta - a(p))\cos^2\psi_\pi, \tag{2.12}$$

where ψ_π is the angle between ξ_p and π.

II. *The Ricci tensor of N has the following expression*

$$S(X,Y) = (a + \cos^2\vartheta)g(X,Y) + (\cos^2\vartheta - a)\eta(X)\eta(Y) \tag{2.13}$$

III. *The scalar curvature of N is given by*

$$r = 2a + 4\cos^2\vartheta. \tag{2.14}$$

Proof. Supposed that N has dimension 3, by virtue of theorem 2.2 the induced structure $(\bar\varphi, \xi, \eta, g)$ on N is $\cos\vartheta$–homotetic Sasakian and this implies by virtue of proposition 2.1 that (N, g, ξ) is QC.
To prove the converse, suppose that (N, g, ξ) is a QC–manifold. First notice that if (N, g) has constant sectional curvature, then the constant sectional curvature must be $\cos^2\vartheta$ by virtue of theorem 2.1. If (N, g) has not constant sectional curvature, we prove by contradiction that $dim\,(N) = 3$. Assume that $dim(N) \geq 4$. As in the proof of proposition 2.1, it would follow that

$$d\eta(X,Y) = 0$$

for all vector fields tangent to N and orthogonal to ξ. Hence, since the induced structure $(\bar\varphi, \xi, \eta, g)$ is $\cos\vartheta$–homotetic

$$\cos\vartheta\bar\Phi(X,Y) = 0$$

which is absurd because N is not anti–invariant. We conclude that $dim(N) = 3$.
Now assume $dim(N) = 3$. Formula (2.12) follows from (2.6) putting $\lambda = \cos\vartheta$. On the other hand since N is a QC–manifold, (2.12) can also be obtained directly by the general formula (1.1). In fact, notice that in the present case we have $b = \cos^2\vartheta - a$ by virtue of theorem 3.3. Moreover using this expression for b, formula (2.1) gives for the curvature tensor R of N:

$$\begin{aligned} R(X,Y,Z,U) \;=\; & a\{g(X,Z)g(Y,U) - g(Y,Z)g(X,U)\} + \\ & + (\cos^2\vartheta - a)\{g(Y,U)\eta(X)\eta(Z) - g(X,U)\eta(Y)\eta(Z) \\ & + g(X,Z)\eta(Y)\eta(U) - g(Y,Z)\eta(X)\eta(U)\}. \end{aligned}$$

In order to prove (2.13), let $p \in N$ and fix an orthonormal basis $\{\xi_p, e_1, e_2\}$ of T_pN. Let $e_o = \xi_p$. Then for all $X, Z \in T_pN$ we get

$$\begin{aligned}
S_p(X,Z) &= \sum_{i=0}^{2} R(X,e_i,Z,e_i)\\
&= a\sum_{i=0}^{2}\{g(X,Z)g(e_i,e_i)-g(e_i,Z)g(X,e_i)\}+\\
&\quad +(\cos^2\vartheta-a)\sum_{i=0}^{2}\{g(e_i,e_i)\eta(X)\eta(Z)-g(X,e_i)\eta(e_i)\eta(Z)\\
&\quad +g(X,Z)\eta(e_i)^2-g(e_i,Z)\eta(X)\eta(e_i)\}\\
&= a\{3g(X,Z)-g(X,Z)\}+\\
&\quad +(\cos^2\vartheta-a)\{3\eta(X)\eta(Z)-\eta(X)\eta(Z)+g(X,Z)-\eta(Z)\eta(X)\}\\
&= (a+\cos^2\vartheta)g(X,Z)+(\cos^2\vartheta-a)\eta(X)\eta(Z)
\end{aligned}$$

and (2.13) is proved. (2.14) soon follows by the definition of scalar curvature:

$$\begin{aligned}
r(p) &= \sum_{i=0}^{2} S_p(e_i,e_i)\\
&= (\cos^2\vartheta+a)\sum_{i=0}^{2} g(e_i,e_i)+(\cos^2\vartheta-a)\sum_{i=0}^{2}\eta(e_i)^2 = 2a+4\cos^2\vartheta.
\end{aligned}$$

Corollary 2.2. *Let (N,g) a Riemannian manifold of dimension 3 and let $\vartheta\in[0,\frac{\pi}{2}]$. Suppose that N admits two ϑ–slant isometric immersions into two K-contact manifolds M_1, M_2, whose characteristic vector fields ξ_1, ξ_2 satisfy*

$$g(\xi_1,\xi_2)=0.$$

Then N has constant sectional curvature $\cos^2\vartheta$.
Proof. By the theorem just proved, (N,g,ξ_1) and (N,g,ξ_2) are both QC–manifolds. In particular theorem 2.1 implies that they have both constant vertical curvature $\cos^2\vartheta$. Now, if $p\in N$, by the hypothesis, the plane orthogonal to ξ_{1p} in T_pN is a vertical plane of (N,g,ξ_2), hence its sectional curvature is $\cos^2\vartheta$. This shows that (N,g,ξ_1) has also constant horizontal curvature $\cos^2\vartheta$ and the assertion follows from proposition 1.2.

Remark. In all that follows if N is a slant submanifold of dimension 3 of a K–contact manifold M, we denote by $\xi_p^\perp$ the orthogonal complement of ξ_p in T_pN, while a denotes the function $a: N\longrightarrow \mathbf{R}$ defined by (2.11).

Moreover the expressions 'slant submanifold' and 'slant immersion' will always mean 'non anti–invariant slant submanifold 'and ' slant immersion with slant angle different from $\frac{\pi}{2}$' respectively.
Lemma 2.2. *Let N be a slant submanifold of dimension 3 of a Sasakian manifold M with constant φ–sectional curvature k. For any $p\in N$, the sectional curvature in M of the plane ${\xi_p}^\perp$ equals*

$$\frac{1}{4}\{(k+3)+3\cos^2\vartheta(k-1)\}.$$

Proof. It is known that in a Sasakian manifold the following formula holds (cfr [2] page 95):

$$\begin{aligned} K(X,Y) &= \frac{1}{8}\{3(1+\cos\zeta)^2H(X+\varphi Y)+3(1-\cos\zeta)^2H(X-\varphi Y) \\ &\quad -H(X+Y)-H(X-Y)-H(X)-H(Y)+6\sin^2\zeta\} \end{aligned}$$

where $\{X,Y\}$ are orthonormal tangent vectors in a point $p\in M$, which are both orthogonal to ξ_p, $\zeta\in[0,\pi]$ is the angle between X e φY, and $H(X)=K(X,\varphi X)$.
Now let p a point of the slant submanifold N and let X be a unit vector in $\xi_p^\perp$. Since

$$g(PX,PX)=-g(X,QX)=\cos^2\vartheta$$

it follows that, putting $Y=\dfrac{1}{\cos\vartheta}PX$, we get an orthonormal basis $\{X,Y\}$ of $\xi_p^\perp$. Moreover notice that

$$g(X,\varphi Y)=g(X,PY)=g(X,\frac{1}{\cos\vartheta}QX)=-\cos\vartheta.$$

By the above remark it follows that the sectional curvature of the plane ${\xi_p}^\perp$ in the ambient manifold M equals

$$\frac{1}{8}\{3(1-\cos\vartheta)^2k+3(1+\cos\vartheta)^2k-4k+6(1-\cos^2\vartheta)\}$$

that is

$$\frac{1}{4}\{3k(1+\cos^2\vartheta)-2k+3-3\cos^2\vartheta)\}.$$

Theorem 2.4. *Let M be a Sasakian manifold with constant φ–sectional curvature k. A necessary condition for a slant 3–dimensional submanifold N of M to be minimal is*

$$a\leq\frac{1}{4}\{(k+3)+3\cos^2\vartheta(k-1)\}.$$

Proof. Suppose N is minimal and let $p\in N$. Fix an orthonormal basis $\{\xi_p,e_1,e_2\}$ of T_pN. By the above lemma and Gauss' equation we have

$$a(p)=\frac{1}{4}\{(k+3)+3\cos^2\vartheta(k-1)\}+g(\alpha(e_1,e_1),\alpha(e_2,e_2))-\|\alpha(e_1,e_2)\|^2. \tag{2.15}$$

On the other hand the minimality of N implies

$$\alpha(\xi_p,\xi_p)+\alpha(e_1,e_1)+\alpha(e_2,e_2)=0.$$

But since ξ is Killing, $\alpha(\xi_p,\xi_p)=0$, hence we get

$$\alpha(e_1,e_1)=-\alpha(e_2,e_2).$$

Now formula (2.15) can be rewritten as

$$a(p) = \frac{1}{4}\{(k+3) + 3\cos^2\vartheta(k-1)\} - \|\alpha(e_1, e_1)\|^2 - \|\alpha(e_1, e_2)\|^2$$

and the assertion follows.

Remark. If M has constant φ–sectional curvature $k = -3$, the necessary condition for the minimality of a 3–dimensional slant submanifold becomes:

$$a \leq -3\cos^2\vartheta.$$

There exist examples of three-dimensional minimal slant submanifold of the Sasakian space form $\mathbf{R}^{2n+1}$, for which $a = -3\cos^2\vartheta$ identically (see [10]).

On the other hand, when $k = 1$ the above condition becomes

$$a \leq 1.$$

The standard 3–sphere S^3 is an example of totally geodesic invariant submanifold of the Sasakian space form S^{2n+1}, satisfying $a = 1$.

3 Local homogeneity of slant submanifolds with constant horizontal curvature

Let N be a non anti–invariant, 3–dimensional slant submanifold of a K–contact manifold M with structure (φ, ξ, η, g). According to the definition given in section 2, to say that (N, g, ξ) has constant horizontal curvature $c \in \mathbf{R}$ is equivalent to saying that the function $a : N \longrightarrow \mathbf{R}$ in theorem 2.3 is constant and equals c.

We shall prove that in this case N is a locally homogeneous Riemannian manifold with respect to the induced metric.

We first prove some results about the Ricci tensor of N.

Proposition 3.1 *Let N be a 3–dimensional slant submanifold of a K–contact manifold M with structure (φ, ξ, η, g). The covariant derivative of the Ricci tensor S of N is given by:*

$$\begin{aligned}(\nabla_X S)(Y, Z) &= X(a)(g(Y, Z) - \eta(Y)\eta(Z)) + \\ &\quad + \cos\vartheta(\cos^2\vartheta - a)\{\bar{\Phi}(X, Y)\eta(Z) + \bar{\Phi}(X, Z)\eta(Y)\},\end{aligned}$$

where $\bar{\Phi}$ denotes the fundamental 2–form of the induced structure $(\bar{\varphi}, \xi, \eta, g)$ on N. In particular, if N has constant horizontal curvature c, we have

$$(\nabla_X S)(Y, Z) = \cos\vartheta(\cos^2\vartheta - c)\{\bar{\Phi}(X, Y)\eta(Z) + \bar{\Phi}(X, Z)\eta(Y)\}.$$

Proof. By using (2.13) we get

$$
\begin{aligned}
(\nabla_X S)(Y,Z) &= \nabla_X S(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z) \\
&= \nabla_X((a+\cos^2\vartheta)g(Y,Z)) + \nabla_X((\cos^2\vartheta - a)\eta(Y)\eta(Z)) \\
&- (a+\cos^2\vartheta)g(\nabla_X Y, Z) - (\cos^2\vartheta - a)\eta(\nabla_X Y)\eta(Z) \\
&- (a+\cos^2\vartheta)g(Y, \nabla_X Z) - (\cos^2\vartheta - a)\eta(\nabla_X Z)\eta(Y) \\
&= X(a)(g(Y,Z) - \eta(Y)\eta(Z)) + (\cos^2\vartheta - a)\{\nabla_X(\eta(Y)\eta(Z)) \\
&- \eta(\nabla_X Y)\eta(Z) - \eta(\nabla_X Z)\eta(Y)
\end{aligned}
$$

On the other hand

$$
\nabla_X(\eta(Y)\eta(Z)) = (g(\nabla_X Y, \xi) + g(Y, \nabla_X \xi))\eta(Z) + (g(\nabla_X Z, \xi) + g(Z, \nabla_X \xi))\eta(Y)
$$

hence

$$
(*) \qquad
\begin{aligned}
(\nabla_X S)(Y,Z) &= X(a)(g(Y,Z) - \eta(Y)\eta(Z)) + \\
&+ (\cos^2\vartheta - a)\{g(Y, \nabla_X \xi)\eta(Z) + g(Z, \nabla_X \xi)\eta(Y)\}.
\end{aligned}
$$

Now recall that $\nabla_X \xi = \widetilde{\nabla}_X \xi - \alpha(X,\xi)$ and $\varphi X = -\widetilde{\nabla}_X \xi$, so we write

$$
g(Y, \nabla_X \xi) = g(X, \varphi Y) = g(X, PY) = \cos\vartheta g(X, \bar{\varphi} Y) = \cos\vartheta \bar{\Phi}(X,Y).
$$

Substituting this formula in (*) we finally obtain

$$
\begin{aligned}
(\nabla_X S)(Y,Z) &= X(a)(g(Y,Z) - \eta(Y)\eta(Z)) + \\
&+ \cos\vartheta(\cos^2\vartheta - a)\{\bar{\Phi}(X,Y)\eta(Z) + \bar{\Phi}(X,Z)\eta(Y)\}
\end{aligned}
$$

and this proves the assertion.

The following theorem characterizes slant submanifolds with constant horizontal curvature by means of a remarkable property of S:

Theorem 3.1. *Let N be a 3–dimensional slant submanifold of a K–contact manifold M with structure (φ, ξ, η, g). The following properties are equivalent:*

a) (N, g, ξ) has constant horizontal curvature

b) For all vector fields X tangent to N $(\nabla_X S)(X,X) = 0$, where S is the Ricci tensor of N.

Proof. If N has constant horizontal curvature then b) follows immediately by proposition 3.1. Vice-versa, suppose b) holds. A general result (see [1],page 432) insures that any Riemannian manifold whose Ricci tensor satisfies b) must have constant scalar curvature. In the present case, by virtue of (2.14) it follows that a is constant.

Theorem 3.2. *Any connected, 3–dimensional, non anti–invariant slant submanifold N of a K–contact manifold M, with constant horizontal curvature, is a locally homogeneous Riemannian manifold with respect to the induced metric.*

Proof. The proof is based on the theory of homogeneous structures on Riemannian manifold developed by Tricerri and Vanhecke in [11]. In particular we shall prove that N admits a homogeneous structure of type $\mathcal{T}_3$. This suffices to prove our assertion by virtue of theorem 1.10 in [11]. On the other hand, since (N, g) is 3–dimensional, connected and orientable, the problem of finding such a structure is equivalent to finding a tensor field T in N of type (1,2) satisfying (see theorem 6.3 and the proof of theorem 6.4 in [11]):

$$(\nabla_X S)(Y, Z) = -S(T_X Y, Z) - S(Y, T_X Z) \tag{3.2}$$

$$T = \lambda dV, \lambda \in \mathbf{R}. \tag{3.3}$$

In formula (3.2) T is thought as a tensor of type (0,3) in the usual way:

$$T(X, Y, Z) = g(T_X Y, Z)$$

while dV is the volume form with respect to a suitable orientation of N.
Now one easily sees that the orientation of N can be chosen in such a way that

$$dV\,(X, Y, \xi) = -\Phi\,(X, Y). \tag{3.1}$$

Hence we define a tensor T putting

$$T = \cos\vartheta dV.$$

By the above remarks, to prove the theorem we just need to show that this tensor satisfies (3.2).
In fact, by virtue of (2.13) we have

$$\begin{aligned} S(T_X Y, Z) &= (c + \cos^2\vartheta) g(T_X Y, Z) + (\cos^2\vartheta - c)\eta(T_X Y)\eta(Z) = \\ &= \cos\vartheta\{(c + \cos^2\vartheta) dV(X, Y, Z) + (\cos^2\vartheta - c) dV(X, Y, \xi)\eta(Z)\} \end{aligned}$$

$$S(Y, T_X Z) = \cos\vartheta\{(c + \cos^2\vartheta) dV(X, Z, Y) + (\cos^2\vartheta - c) dV(X, Z, \xi)\eta(Y)\}$$

hence

$$S(T_X Y, Z) + S(Y, T_X Z) = \cos\vartheta(\cos^2\vartheta - c)\{dV(X, Y, \xi)\eta(Z) + dV(X, Z, \xi)\eta(Y)\}$$

that is

$$S(T_X Y, Z) + S(Y, T_X Z) = -\cos\vartheta(\cos^2\vartheta - c)\{\bar{\Phi}(X, Y)\eta(Z) + \bar{\Phi}(X, Z)\eta(Y)\}$$

and the assertion follows by virtue of proposition 3.1.

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