

On Derivations of Nilpotent Lie Algebras

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Abstract

The aim of the present paper is to prove that the nilpotent Lie algebras of dimension six and seven in general are not the radical of any other Lie algebra.

Mathematics Subject Classification: 17B30

Key Words: nilpotent Lie algebra, derivation, simple algebra, semi-simple Lie algebra, inner derivation, center and radical.

1

Let g be a Lie algebra over a field k of characteristic zero. It is interesting to study the relation between this Lie algebra and its Lie algebra of derivations and as well as with other Lie algebras obtained via the given Lie algebra.

The purpose of this paper is to prove that any nilpotent Lie algebra over a field k of characteristic zero of dimension six and seven, except those which are characteristically nilpotent, can not be the radical of a Lie algebra g having the property $D(g) = I(g) + C(g)$. This is an extension of the work([4]).

The whole paper contains four paragraphs each of them is analyzed as follows. The first paragraph is the introduction. Basic elements of Lie algebras are given in the second paragraph. In the third paragraph are studied the nilpotent Lie algebras of dimension six concerning some properties of this Lie algebras. Finally, the last paragraph faces the same problem for the nilpotent Lie algebras of dimension seven.

2

We shall give some basic notions concerning of this paper.

Let g be a Lie algebra over the field k of characteristic zero of dimension n . It is known that from this algebra we can form the following sequences of ideals of g :

$$(2.1) \quad C^0g = g, \quad C^1g = [g, g], \dots, \quad C^qg = [g, C^{q-1}g], \dots$$

which is called *descending central sequence*.

$$(2.2) \quad C^0g = 0, \quad C^1g = \text{centre}(g), \dots, \quad C^qg = \text{centre}(g/C^{q-1}g)$$

which is called *increasing central sequence* and

$$(2.3) \quad D^0g = g, \quad D^1g = [g, g], \dots, \quad D^qg = [D^{q-1}g, D^{q-1}g], \dots$$

which is called *derived sequence*.

If there exists an integer $q \geq 2$ such that $C^qg = \{0\}$, then the Lie algebra g is called *nilpotent of nilpotency q* .

A linear mapping f on g is called *derivation*, if it satisfies the relation:

$$f[x, y] = [fx, y] + [x, fy], \quad \forall x, y \in g.$$

The set of all derivations f on g is denoted by $D(g)$, that is

$$D(g) = \{f / f : g \rightarrow g, f \text{ linear and } f[x, y] = [fx, y] + [x, fy]\}$$

The following mapping:

$$ad_x : g \rightarrow g, \quad ad_x : y \rightarrow ad_x y = [x, y]$$

is a derivation which is called *inner derivation*. The set of all inner derivations is denoted by $I(g)$ which is an ideal of the Lie algebra of derivations $D(g)$. The other derivations on g , which are not inner, are called outer, which are denoted by $D_i(g)$. It is known that

$$D(g) = I(g) \oplus D_i(g)$$

We must notice that $D_i(g)$ is an ideal of $D(g)$.

The Lie algebra g is called *characteristically nilpotent* if the Lie algebra $D(g)$ is nilpotent.

A Lie algebra g is said to be *semi-simple* if does not contain any non-zero abelian ideal.

If g is semi-simple, then

$$D(g) = I(g).$$

The classification of Lie algebras of finite dimension over a field k of characteristic zero is reduced to Levi's theorem

$$g = S \oplus R,$$

where R is radical of g , which is the maximal solvable ideal of g and S a semi-simple subalgebra of g .

Let S be a semi-simple Lie algebra, since $S = [S, S]$ the trace of S is zero.

The center $Z(g)$ of a Lie algebra g is an ideal of g , such that

$$Z(g) = \{z \in g, [x, z] = 0 \text{ for all } x \in g\}.$$

A Lie algebra g is called *abelian* if and only if $Z(g) = g$.

A nilpotent Lie algebra g is called *quasi-cyclic* provided g has a subspace U such that

$$g = U \oplus [g, g]$$

and such that g is the direct sum of the subspaces U^i where $U^1 = U$ and $U^i = [U, U^{i-1}]$ for $i \geq 2$.

We denote by $C(g)$ the set of all endomorphisms of g , which map g into the center Z of g and $[g, g]$ into $\{0\}$. It can be proved that $C(g)$ is a subalgebra of $D(g)$. Each element of $C(g)$ is called central derivation.

The Lie algebra of derivations $D(g)$ of g can be written as

$$D(g) = I(g) + C(g).$$

It is known that $I(g) \cap C(g)$ is not always empty.

A Lie algebra g is called T when it is not the radical of any other Lie algebra L having the property

$$D(L) = I(L) + C(L).$$

3

It is known that there is a classification of nilpotent Lie algebras of dimension six over a field C or R . In this paragraph we study some properties of these nilpotent Lie algebras.

Now, we can prove the following theorem.

Theorem 3.1. *The nilpotent Lie algebras over a field \mathbf{k} of characteristic zero of dimension 6 belong to the Lie algebras of type \mathbf{T} .*

Proof. We consider the classification of nilpotent Lie algebras of dimension 6 which is given in ([5]). Also, we consider two cases (I) quasi cyclic (II) no quasi-cyclic.

(I) Let $g_{6,1}$ be the nilpotent Lie algebra described in terms of a basis $x_1, x_2, x_3, x_4, x_5, x_6$ by the following multiplication tables.

$$[x_1, x_2] = x_5, \quad [x_1, x_4] = x_6, \quad [x_2, x_3] = x_6.$$

By the definition of a quasi-cyclic nilpotent Lie algebra given in paragraph 2 we obtain

$$U = U^1 = \{x_1, x_2, x_3, x_4\}$$

$$U^2 = [U, U^1] = \{x_5, x_6\}$$

$$U^3 = [U, U^2] = \{0\}$$

So, the nilpotent Lie algebra $g_{6,1}$ can be written as the direct sum

$$g_{6,1} = U \oplus U^2$$

and by the definition given in paragraph 2, we conclude that $g_{6,1}$ is a quasi-cyclic nilpotent Lie algebra.

According to the results in ([4]) we obtain that $g_{6,1}$ is a T -algebra.

In the same way we prove that the nilpotent Lie algebras

$$g_{6,3}, g_{6,4}, g_{6,6}, g_{6,9}, g_{6,10}, g_{6,14}, g_{6,15}, g_{6,16}, g_{6,18}, g_{6,4}$$

are T -algebras.

(II) Let $g_{6,2}$, be the following nilpotent Lie algebra

$$[x_1, x_2] = x_5, \quad [x_1, x_5] = x_6, \quad [x_3, x_4] = x_6.$$

The Lie algebra of derivations $D(g_{6,2})$ of $g_{6,2}$ is represented by the set of matrices

$$D(g_{6,2}) = \left\{ \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 & 0 & 0 \\ a_{13} & 0 & a_{33} & a_{43} & 0 & 0 \\ a_{14} & 0 & a_{34} & 2a_{11} + a_{22} + a_{33} & 0 & 0 \\ a_{15} & a_{25} & a_{14} & -a_{13} & a_{11} + a_{22} & 0 \\ a_{16} & a_{26} & a_{36} & a_{46} & a_{25} & 2a_{11} + a_{22} \end{bmatrix} \right\}$$

The endomorphism of $D(g_{6,2})$ defined by

$$Dx_1 = x_1, \quad Dx_2 = x_2, \quad Dx_3 = x_3, \quad Dx_4 = 2x_4, \quad Dx_5 = 2x_5, \quad Dx_6 = 3x_6$$

is a derivation of $D(g_{6,2})$ whose trace is not zero.

According to the results in ([4]) we obtain that $g_{6,2}$ is not the radical of a Lie algebra L whose $D(L)$ can be written in the form

$$D(L) = I(L) + C(L).$$

Hence $g_{6,2}$ is a T -algebra.

With the same method we prove that the nilpotent Lie algebras

$$g_{6,5}, g_{6,7}, g_{6,8}, g_{6,11}, g_{6,12}, g_{6,13}, g_{6,17}, g_{6,19}, g_{6,10}$$

are T -algebras.

4

In this section we are concerned with the nilpotent Lie algebras of dimension seven over a field C or R of characteristic zero. Comparing with the previous paragraph in which we studied some properties of the nilpotent Lie algebras of dimension six, in this last paragraph we additionally study some properties of a second category, that of the characteristically nilpotent Lie algebras of dimension seven.

We can prove the following

Theorem 4.1. *The nilpotent Lie algebras over a field \mathbf{k} of characteristic zero of dimension 7 can be classified in two categories. The first consists of the Lie algebras of type \mathbf{T} and the second one of the characteristically nilpotent Lie algebras.*

Proof. We consider the classification of nilpotent Lie algebras of dimension 7 which is given in ([5]). Also, we consider two cases (I) T -algebras (1. quasi-cyclic 2. not quasi-cyclic) (II) characteristically nilpotent Lie algebras.

(I₁) Let $g_{7,1.17}$ be the following nilpotent Lie algebra of dimension seven,

$$\begin{aligned} [x_1, x_2] &= x_3, & [x_1, x_3] &= x_4, & [x_1, x_4] &= x_6, & [x_1, x_6] &= x_7, \\ [x_2, x_3] &= x_5, & [x_2, x_5] &= x_6, & [x_2, x_6] &= x_7, & [x_3, x_4] &= -x_7, \\ [x_3, x_5] &= x_7. \end{aligned}$$

We have

$$\begin{aligned} U &= U^1 = \{x_1, x_2\} \\ U^2 &= [U, U^1] = \{x_3\} \\ U^3 &= [U, U^2] = \{x_4, x_5\} \\ U^4 &= [U, U^3] = \{x_6\} \\ U^5 &= [U, U^4] = \{x_7\} \\ U^6 &= [U, U^5] = \{0\}. \end{aligned}$$

Hence the nilpotent Lie algebra $g_{7,1.17}$ can be written as the direct sum

$$g_{7,1.17} = U^1 \oplus U^2 \oplus U^3 \oplus U^4 \oplus U^5$$

and by the definition given in paragraph 2, we conclude that $g_{6,1.17}$ is a quasi-cyclic nilpotent Lie algebra.

According to the results in ([4]) we obtain that $g_{6,1}$ is a T -algebra.

In the same way we prove that the nilpotent Lie algebras $g_{7,1.19}$, $g_{7,2.3}$, $g_{7,2.4}$, $g_{7,2.5}$, $g_{7,2.6}$, $g_{7,2.7}$, $g_{7,2.8}$, $g_{7,2.9}$, $g_{7,2.12}$, $g_{7,2.16}$, $g_{7,2.26}$, $g_{7,2.34}$, $g_{7,2.35}$, $g_{7,2.43}$, $g_{7,2.44}$, $g_{7,3.1(i^\lambda)}$, $g_{7,3.1(ii)}$, $g_{7,3.2}$, $g_{7,3.3}$, $g_{7,3.4}$, $g_{7,3.6}$, $g_{7,3.10}$, $g_{7,3.11}$, $g_{7,3.12}$, $g_{7,3.16}$, $g_{7,3.19}$, $g_{7,3.20}$, $g_{7,3.21}$, $g_{7,3.22}$, $g_{7,3.23}$, $g_{7,3.24}$, $g_{7,4.2}$, $g_{7,4.3}$, $g_{7,4.4}$, $g_{7,4.5}$ are T -algebras.

(I₂) Let $g_{7,1.01(i)}$ be the nilpotent Lie algebra described in terms of a basis $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ by the following multiplication tables

$$\begin{aligned} [x_1, x_2] &= x_4, & [x_1, x_4] &= x_5, & [x_1, x_5] &= x_6, & [x_1, x_6] &= x_7, \\ [x_2, x_3] &= x_5 + x_7, & [x_3, x_4] &= -x_6, & [x_3, x_5] &= -x_7. \end{aligned}$$

The Lie algebra of derivations $D(g_{7,1.01(i)})$ of $g_{7,1.01(i)}$ is represented by the set of matrices

$$D(g_{7,1.01(i)}) = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{14} & a_{24} & -a_{12} & a_{22} & 0 & 0 & 0 & 0 \\ a_{15} & a_{25} & -a_{14} & a_{24} & a_{22} & 0 & 0 & 0 \\ a_{16} & a_{26} & -(a_{12} + a_{15}) & a_{25} & a_{24} & a_{22} & 0 & 0 \\ a_{17} & a_{27} & a_{37} & a_{26} & a_{25} & a_{24} & a_{22} & 0 \end{bmatrix} \right\}$$

The endomorphism of $D(g_{7,1.01(i)})$ defined by

$$Dx_1 = 0, \quad Dx_2 = x_2, \quad Dx_3 = 0, \quad Dx_4 = x_4,$$

$$Dx_5 = x_5, \quad Dx_6 = x_6, \quad Dx_7 = x_7,$$

is a derivation of $D(g_{7,1.01(i)})$ whose trace is not zero.

As a consequence of the results in ([4]) we have that $\mathfrak{g}_{7,1.01(i)}$ is not the radical of a Lie algebra L , whose the Lie algebra of derivations $D(L)$ can be written as

$$D(L) = I(L) + C(L).$$

From the above we conclude that $\mathfrak{g}_{7,1.01(i)}$ is a T -algebra.

In the same way we prove that the following nilpotent Lie algebras are T -algebras.

$\mathfrak{g}_{7,1.01(i)}$, $\mathfrak{g}_{7,1.01(ii)}$, $\mathfrak{g}_{7,1.02}$, $\mathfrak{g}_{7,1.03}$, $\mathfrak{g}_{7,1.1(i^\lambda)}$, $\mathfrak{g}_{7,1.1(ii)}$, $\mathfrak{g}_{7,1.1(iii)}$, $\mathfrak{g}_{7,1.1(iv)}$, $\mathfrak{g}_{7,1.1(v)}$, $\mathfrak{g}_{7,1.1(vi)}$, $\mathfrak{g}_{7,1.2(i^\lambda)}$, $\mathfrak{g}_{7,1.2(ii)}$, $\mathfrak{g}_{7,1.2(iii)}$, $\mathfrak{g}_{7,1.2(iv)}$, $\mathfrak{g}_{7,1.3(i^\lambda)}$, $\mathfrak{g}_{7,1.3(ii)}$, $\mathfrak{g}_{7,1.3(iii)}$, $\mathfrak{g}_{7,1.3(iv)}$, $\mathfrak{g}_{7,1.3(v)}$, $\mathfrak{g}_{7,1.4}$, $\mathfrak{g}_{7,1.5}$, $\mathfrak{g}_{7,1.6}$, $\mathfrak{g}_{7,1.7}$, $\mathfrak{g}_{7,1.8}$, $\mathfrak{g}_{7,1.9}$, $\mathfrak{g}_{7,1.10}$, $\mathfrak{g}_{7,1.11}$, $\mathfrak{g}_{7,1.12}$, $\mathfrak{g}_{7,1.13}$, $\mathfrak{g}_{7,1.14}$, $\mathfrak{g}_{7,1.15}$, $\mathfrak{g}_{7,1.16}$, $\mathfrak{g}_{7,1.18}$, $\mathfrak{g}_{7,1.20}$, $\mathfrak{g}_{7,1.21}$, $\mathfrak{g}_{7,2.1(i^\lambda)}$, $\mathfrak{g}_{7,2.1(ii)}$, $\mathfrak{g}_{7,2.1(iii)}$, $\mathfrak{g}_{7,2.1(iv)}$, $\mathfrak{g}_{7,2.1(v)}$, $\mathfrak{g}_{7,2.2}$, $\mathfrak{g}_{7,2.10}$, $\mathfrak{g}_{7,2.11}$, $\mathfrak{g}_{7,2.13}$, $\mathfrak{g}_{7,2.14}$, $\mathfrak{g}_{7,2.15}$, $\mathfrak{g}_{7,2.17}$, $\mathfrak{g}_{7,2.18}$, $\mathfrak{g}_{7,2.19}$, $\mathfrak{g}_{7,2.20}$, $\mathfrak{g}_{7,2.21}$, $\mathfrak{g}_{7,2.22}$, $\mathfrak{g}_{7,2.23}$, $\mathfrak{g}_{7,2.24}$, $\mathfrak{g}_{7,2.25}$, $\mathfrak{g}_{7,2.27}$, $\mathfrak{g}_{7,2.28}$, $\mathfrak{g}_{7,2.29}$, $\mathfrak{g}_{7,2.30}$, $\mathfrak{g}_{7,2.31}$, $\mathfrak{g}_{7,2.32}$, $\mathfrak{g}_{7,3.33}$, $\mathfrak{g}_{7,2.36}$, $\mathfrak{g}_{7,2.37}$, $\mathfrak{g}_{7,2.38}$, $\mathfrak{g}_{7,2.39}$, $\mathfrak{g}_{7,2.40}$, $\mathfrak{g}_{7,2.41}$, $\mathfrak{g}_{7,2.42}$, $\mathfrak{g}_{7,2.45}$, $\mathfrak{g}_{7,3.1(iii)}$, $\mathfrak{g}_{7,3.14}$, $\mathfrak{g}_{7,3.15}$, $\mathfrak{g}_{7,3.17}$, $\mathfrak{g}_{7,3.18}$, $\mathfrak{g}_{7,4.1}$.

(II) Finally let

$$\mathfrak{g}_{7,0.1}, \mathfrak{g}_{7,0.2}, \mathfrak{g}_{7,0.3}, \mathfrak{g}_{7,0.4}, \mathfrak{g}_{7,0.5}, \mathfrak{g}_{7,0.6}, \mathfrak{g}_{7,0.7}, \mathfrak{g}_{7,0.8}$$

be nilpotent Lie algebras over a field k of characteristic zero of dimension seven, given in ([5]).

From the fact that each of the Lie algebras $D(\mathfrak{g}_{7,0.i})$, $i = 1, 2, \dots, 8$ of derivations of the Lie algebras $\mathfrak{g}_{7,0.i}$, $i = 1, 2, \dots, 8$ is nilpotent, we conclude that the above eight Lie algebras are characteristically nilpotent.

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