Tensor and Spinor Equivalence on Generalized Metric Tangent Bundles

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Abstract

The relation between spinor of $SL(2, C)$ group and tensors in the framework of lagrange spaces is studied. A geometrical extension to generalized metric tangent bundles is developed by means of spinor. Also, the spinorial equation of causality for the unique solution of the null-cone in the Finsler or Lagrange space is given explicitely.

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1 Introduction

The theory of spinors on pseudo–Riemannian spaces has been recognized by many authors, e.g. [1], [2], [3] for the important role it has played from the mathematical and physical point of view.

The spinors that we are dealing with here, are associated with the group $SL(2, \mathbf{C})$. In particular $SL(2, \mathbf{C})$ acts on $\mathbf{C}^2$. Each element of $\mathbf{C}^2$ represents a two–component spinor. This group is the covering group of the Lorentz group in which the tensors are described [2]. The correspondence between spinors and tensors is achieved by means of mixed quantities initially introduced by Infeld and Van der Waerden.

The correspondence of tensors and spinors establishes a homomoerhism between the Lorentz group and the covering group $SL(2, \mathbf{C})$.

In the following, we give some important relations between spinors and tensors on a general manifold of metric $g_{\mu\nu}$.

Let $\sigma : S \otimes \overline{S} \to V^4$ be a homomorphism between spinor spaces $S, \overline{S}$ and four–vectors belonging to the V^4 space, then the components of σ, which are called the *Pauli–spinor matrices*, are given by

$$\sigma^0_{AB'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1_{AB'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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$$\sigma^2_{AB'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma^3_{AB'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

The hermitian spinorial equivalent notation of $\sigma^{\mu}_{AB'}$ is given by $\sigma^{\mu}_{AB'} = \overline{\sigma}^{\mu}_{BA'} = \overline{\sigma}^{\mu}_{B'A}$. Greek letters $\mu, \nu, \ldots$ represent the usual space–time indices taking the values $0, 1, 2, 3$ and the Roman capital indices $A, B, A^{'}, B^{'}$ are the spinor indices taking the values $0, 1$. The tensor indices are raised and lowered by means of the metric tensor, whereas the raising and lowering of spinor indices is given by the *spinor metric tensors* $\varepsilon_{\mathbf{AC}}, \varepsilon_{\mathbf{B'C'}}$ which are of skew–symmetric form. Thus, for two spinors $\xi^A, n^{A'}$ we have the relations,

$$\begin{aligned} \xi^A &= \varepsilon^{AB}\xi_B, \quad n^{A'} = \varepsilon^{A'B'}n_{B'} \\ \xi_A &= \xi^B\varepsilon_{BA}, \quad n_{A'} = n^{B'}\varepsilon_{B'A'}; \end{aligned} \tag{2}$$

moreover we have,

$$\xi^A n_A = \xi^A n^B \varepsilon_{BA} = -\xi^A \varepsilon_{AB} n^B = -\xi_B n^B.$$

For a real vector V_μ its spinor equivalent is

$$V_{AB'} = V_\mu \sigma^{\mu}_{AB'}, \tag{3a}$$

where $\sigma^{\mu}_{AB'}$ are given by the relation (1). Also, the following formulas are satisfied,

$$\begin{aligned} \sigma^{\mu}_{AB'}\sigma^{\nu AB'} &= g^{\mu\nu}, \\ \sigma^{\mu}_{AB'}\sigma^{AB'}_{\nu} &= \delta^{\mu}_{\nu}. \end{aligned} \tag{3b}$$

The spinor equivalent of a tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \sigma^{AB'}_{\mu}\sigma^{CD'}_{\nu}T_{AB'CD'} \tag{4}$$

and the tensor corresponding to the spinor $T_{AB'CD'}$ is,

$$T_{AB'CD'} = \sigma^{\mu}_{AB'}\sigma^{\nu}_{CD'}T_{\mu\nu}. \tag{5}$$

The relationship between the matrices σ^μ and the geometric tensor $g_{\mu\nu}$, as well as its spinor equivalent are

$$\begin{aligned} g_{\mu\nu}\sigma^{\mu}_{AB'}\sigma^{\nu}_{CD'} &= \varepsilon_{AC}\varepsilon_{B'D'} \\ g_{AB'CD'} = \sigma^{\mu}_{AB'}\sigma^{\nu}_{CD'}g_{\mu\nu} &= \varepsilon_{AC}\varepsilon_{B'D'} \\ g^{AB'CD'} = \sigma^{AB'}_{\mu}\sigma^{CD'}_{\nu}g^{\mu\nu} &= \varepsilon^{AC}\varepsilon^{B'D'}. \end{aligned} \tag{6}$$

The complex conjugation of the spinor $S_{AB'}$ is

$$\overline{S_{AB'}} = \overline{S}_{A'B}. \tag{7}$$

Furthermore, for a real vector V_μ the spinor hermitian equivalence yields $\overline{V}_{B'A} = V_{AB'}$. If a vector y^k is a null–vector,

(8) $$y^k y_k = g_{k\lambda} y^k y^\lambda = 0,$$

then its spinor equivalent will take the form

(9) $$y^k = \sigma^k_{AB'} \theta^A \overline{\theta}^{B'},$$

where, $\theta^A, \overline{\theta}^{B'}$ represents the two–component spinors of $SL(2, \mathbf{C})$ group.

In the Riemannian space, the covariant derivative of $x-$ dependent spinors will take the form

(10)
$$\begin{aligned}
D_\mu \xi^A &= \frac{\partial \xi^A}{\partial x^\mu} + L^A_{B\mu} \xi^B, \\
D_\mu \overline{\xi}^{A'} &= \frac{\partial \overline{\xi}^{A'}}{\partial x^\mu} + \overline{L}^{A'}_{B'\mu} \overline{\xi}^{B'}, \\
D_\mu \xi_A &= \frac{\partial \xi_A}{\partial x^\mu} - L^B_{A\mu} \xi_B, \\
D_\mu \overline{\xi}_{A'} &= \frac{\partial \overline{\xi}_{A'}}{\partial x^\mu} - \overline{L}^{B'}_{A'\mu} \overline{\xi}_{B'},
\end{aligned}$$

where $\xi^A, \xi_A, \overline{\xi}^{A'}, \overline{\xi}_{A'}$ represents two–component spinors and $L^A_{B\mu}, \overline{L}^{A'}_{B'\mu}$ the spinor affine connections. In the case that we have spinors with two indices, the covariant derivative will be in the form

(11) $$D_\mu \xi^{AB'} = \frac{\partial \xi^{AB'}}{\partial x^\mu} + L^A_{C\mu} \xi^{CB'} + \overline{L}^{B'}_{C'\mu} \xi^{AC'}.$$

Applying this formula to the spinor metric tensors $\varepsilon_{AC}, \varepsilon_{B'C'}$ we get

(12) $$D_\mu \varepsilon_{AB} = \frac{\partial \varepsilon_{AB}}{\partial x^\mu} - L^C_{A\mu} \varepsilon_{CB} - L^C_{B\mu} \varepsilon_{AC}.$$

If

$$D_\mu \varepsilon_{AB} = 0,$$

we shall say that the spinor connection coefficients $L^A_{B\mu}$ are *metrical* together with the relations

(13) $$D_\mu \sigma^\nu_{AB'} = 0, \quad D_\mu \varepsilon^{AB} = 0, \quad D_\mu \varepsilon_{A'B'} = 0, D_\mu \varepsilon^{A'B'} = 0.$$

From the relation (12) we immediately obtain

$$L_{BA\mu} = L_{AB\mu},$$

where we used the relation

$$L_{AB\mu} = L^C_{B\mu} \varepsilon_{CA}.$$

Also from the relation 13 a) we have

(14) $$D_\mu \sigma^\nu_{AB'} = \partial_\mu \sigma^\nu_{AB'} + L^\nu_{\mu\rho} \sigma^\rho_{AB'} - L^C_{A\mu} \sigma^\nu_{CB'} - \overline{L}^{D'}_{B'\mu} \sigma^\nu_{AD'} = 0.$$

2 Generalization of the Equivalent of Two Component–Spinors with Tensors

The above mentioned well–known procedure for $SL(2,\mathbf{C})$ group between spinors and tensors in a pseudo–Riemannian space–time can be applied to more generalized metric spaces or bundles. For example G. Asanov [6] applied this method for Finsler spaces (FS), where the two–component spinors $n(x,y)$ depend on the position and direction variables or $n(x^i, z^\alpha)$, with z^α a scalar for a gauge approach. Concerning this approach some results were given relatively to the gauge covariant derivative of spinors and the Finslerian tetrad. In our present study we give the relation between spinors of $SL(2,\mathbf{C})$ group and tensors in the framework of Lagrange spaces (LS).

The expansion for the covariant derivatives, connections non–linear connections, torsions and curvatures are the main purpose of our approach.

In the following, we shall study the case that the vectors of LS are null–vectors and consequently fulfill the relation (9). In Finsler type space–time the metric tensor $g_{ij}(x,y)$ depends on the position and directional variables, where the vector y may be identified with the frame velocity ([6] ch. t). So, a vector v^i will be called *null* if

$$g_{ij}(x,v)v^i v^j = 0.$$

In this case there is no unique solution for the light–cone [7], [8]. The problem of causality is solved considering the velocity as a parameter and the motion of a particle in Finsler space is described by a pair (x,y). The metric form in such a case will be given by

$$ds^2 = g_{ij}(x,v)dx^i dx^j. \tag{15a}$$

When a particle is moving in the tangent bundle of a Finsler (Lagrange) space–time its line–element will be given by

$$d\sigma^2 = G_{ab}dx^a dx^b = g^{(0)}_{ij}(x,y)dx^i dx^j + g^{(1)}_{\alpha\beta}(x,y)\delta y^\alpha \delta y^\beta, \quad \left(y^\alpha = \frac{dx^\alpha}{dt}\right), \tag{16}$$

where the indices i,j and α,β taking the values $1,2,3,4$ and

$$\delta y^\alpha = dy^\alpha + \mathcal{N}^\alpha_j dx^j.$$

Thus we have

Theorem 2.1. *The null–geodesic condition (15) is satisfied for a particle which is moving in the tangent bundle of Finsler space–time of metric $d\sigma^2$ (rel. 16) with the assumption, the velocity v is taken as a parameter of the absolute parallelism*

$$\delta y^\alpha = 0. \tag{17}$$

The previous treatment of null-vectors in Finsler spaces can also be considered for Lagrange spaces involving Lagrangians which are not homogeneous [9], [8]. The introduction of spinors $\theta,\overline{\theta}$ of the covering group $SL(2,\mathbf{C})$ in the metric tensor $g(x,\theta,\overline{\theta})$ under the correspondence between spinors and tensors in LS,

$$(x,y) \to (x, V_{AB'}) \to (x, \theta^A, \overline{\theta}^{A'})$$

preserves the anisotropy of space with torsions. In this case all objects depend on the position and spinors, e.g. the Pauli matrices $\tilde{\sigma}^i_{AA'}(x,\theta,\overline{\theta})$. Such an approach can be developed for a second–order spinor bundle applying the method analogous to [4]. In virtue of relation (8), a null vector in spinor form can be characterized by

$$g_{AA'BB'}\theta^A\overline{\theta}^{A'}\theta^B\overline{\theta}^{B'} = \tilde{\sigma}^i_{AA'}\tilde{\sigma}^j_{BB'}g_{ij}\theta^A\overline{\theta}^{A'}\theta^B\overline{\theta}^{B'} = 0. \tag{18}$$

Proposition 2.2. *In a tangent bundle of metric (Finsler, Lagrange)*

$$G = g_{ij}(x,y)dx^i dx^j + h_{ab}(x,y)\delta y^a \delta y^b,$$

if the vector y is a null, then the corresponding spinor metric of the bundle will be given in the form

$$G = g_{AA'BB'}d\theta^A d\overline{\theta}^{A'} d\theta^B d\overline{\theta}^{B'} + h_{AA'BB'}\delta(\theta^B\overline{\theta}^{B'})\delta(\theta^A\overline{\theta}^{A'}) \tag{19}$$

or equivalently

$$G = g_{AA'BB'}d\theta^A d\overline{\theta}^{A'} d\theta^B d\overline{\theta}^{B'} + h_{AA'BB'}\delta y^{AA'}\delta y^{BB'},$$

where $y^{AA'} = \theta^A\overline{\theta}^{A'}$, when y is null vector (cf. [2]).
Proof. The relation (19) is obvious by virtue of (6) and (9).
Remark. A generalized spinor can be considered as the square root of a Finsler (Lagrange) null vector.

3 Adapted Frames and Linear Connections

In the general case of a LS, the spinor equivalent to the metric tensor

$$g_{ij} = \frac{\partial^2 L}{\partial y^i \partial y^j} \quad , \quad L = \frac{1}{2}F^2$$

is given by

$$g_{ij} = \tilde{\sigma}^{AA'}_i \tilde{\sigma}^{BB'}_j g_{AA'BB'}. \tag{20}$$

The corresponding Lagrangian will be $\overline{L} : M \times \mathbf{C}^2 \times \mathbf{C}^2 \to \mathbf{R}$, with the property $\overline{L}(x,\theta,\overline{\theta}) = L(x,y)$, where L represents the Lagrangian in a Lagrange space. We can adopt the spinor equivalent form of the adapted frames and their duals in a LS,

$$\left(\frac{\delta}{\delta x^\mu}, \frac{\partial}{\partial y^i}\right) \to \left(\frac{\delta}{\delta x^\mu}, \frac{\partial}{\partial \theta^A}, \frac{\partial}{\partial \overline{\theta}^{A'}}\right), \quad (dx^\mu, \delta y^i) \to (dx^\mu, \delta\theta^A, \delta\overline{\theta}^{A'})$$

as well as the spinor counterpart of the non–linear connection $\mathcal{N}^i_\mu$ of a LS,

$$\mathcal{N}^i_\mu \to (N^A_\mu, \overline{N}^{A'}_\mu).$$

The geometrical objects $\delta\theta^A, \delta\overline{\theta}^{A'}$ are given by

$$\delta\theta^A = d\theta^A + N^A_\mu dx^\mu, \quad \delta\overline{\theta}^{A'} = d\overline{\theta}^{A'} + \overline{N}^{A'}_\mu dx^\mu. \tag{21}$$

In virtue of (3), the bases $\partial_\mu, \partial_{AA'}$ are related as follows

$$\partial_\mu = \widetilde{\sigma}^{AA'}_\mu \partial_{AA'}, \tag{22}$$

where $\partial_\mu = \dfrac{\partial}{\partial x^\mu}$ and $\partial_{AA'} = \dfrac{\partial}{\partial\theta^A}\dfrac{\partial}{\partial\overline{\theta}^{A'}}$.

Theorem 3.1. *In a Lagrange space the spinor equivalent of the adapted basis* $(\delta/\delta x^\mu, \partial/\partial y^\alpha)$ *and its dual* $(dx^\mu, \delta y^\alpha)$ *are given by*

(23)

a) $\dfrac{\delta}{\delta x^\mu} = \widetilde{\sigma}^{AA'}_\mu \partial_A \partial_{A'} - N^A_\mu \partial_A - \overline{N}^{A'}_\mu \partial_{A'}$

b) $\partial_P \widetilde{\sigma}^{AA'}_P = \partial_{AA'}, \quad P = \{i, \alpha\}$

c) $dx^\mu = \widetilde{\sigma}^\mu_{AA'} d\theta^A d\overline{\theta}^{A'}$

d) $\delta y^\alpha = (\overline{\theta}^{A'} d\theta^A + \theta^A d\overline{\theta}^{A'})\widetilde{\sigma}^\alpha_{AA'} + (\overline{\theta}^{A'} N^A_\gamma + \theta^A \overline{N}^{A'}_\gamma)\widetilde{\sigma}^\alpha_{AA'} \widetilde{\sigma}^\gamma_{AA'} d\theta^B d\theta^{\overline{B}'}$.

Proof. The relations (23) are derived from (21) and (22).

Proposition 3.2. *If* y^α, N^α_j *represent a null vector and a non–linear connection in a Lagrange space, then its corresponding spinor representations are given by*

$$dy^\alpha = \widetilde{\sigma}^\alpha_{AA'}(\overline{\theta}^{A'} d\theta^A + \theta^A d\overline{\theta}^{A'}) \quad N^\alpha_j = \widetilde{\sigma}^\alpha_{AA'}(\overline{\theta}^{A'} N^A_j + \theta^A \overline{N}^{A'}_j). \tag{24}$$

Proof. The relation (24) is obvious because of (23 d).

Proposition 3.3. *The null–geodesic equation of spinor equivalence in a* LS *or* FS *is given by*

$$\overline{\theta}^{A'} d\theta^A(\widetilde{\sigma}^\mu_{AA'} N^A_\mu d\overline{\theta}^{A'} + 1) + \theta^A d\overline{\theta}^{A'}(\widetilde{\sigma}^\mu_{AA'} \overline{N}^{A'}_\mu d\theta^A + 1) = 0. \tag{25}$$

Proof. In virtue of relations (18) and (23c,d) we obtain the relation (25).

Affine connections and affine spinor connections are defined in the frames of LS by the following formulas

$$D_{\delta/\delta x^\mu}\left(\frac{\delta}{\delta x^\nu}\right) = L^k_{\nu\mu}\frac{\delta}{\delta x^k}, \quad D_{\delta/\delta x^\mu}\left(\frac{\partial}{\partial\theta^A}\right) = L^B_{A\mu}\frac{\partial}{\partial\theta^B},$$

$$D_{\delta/\delta x^\mu}\left(\frac{\partial}{\partial\overline{\theta}^{A'}}\right) = \overline{L}^{B'}_{A'\mu}\frac{\partial}{\partial\overline{\theta}^{B'}}, \quad D_{\partial/\partial\theta^A}\left(\frac{\delta}{\delta x^\mu}\right) = C^\nu_{\mu A}\frac{\delta}{\delta x^\nu}, \tag{26}$$

$$D_{\partial/\partial\theta^A}\left(\frac{\partial}{\partial\overline{\theta}^{B'}}\right) = C^{C'}_{B'A}\frac{\partial}{\partial\theta^{C'}}, \quad D_{\partial/\partial\overline{\theta}^{A'}}\left(\frac{\partial}{\partial\theta^B}\right) = \overline{C}^C_{BA'}\frac{\partial}{\partial\theta^C},$$

$$D_{\partial/\partial\overline{\theta}^{A'}}\left(\frac{\partial}{\partial\overline{\theta}^{B'}}\right)=\overline{C}^{C'}_{B'\,A'}\frac{\partial}{\partial\overline{\theta}^{C'}},\quad D_{\partial/\partial\theta^{A}}\left(\frac{\partial}{\partial\theta^{B}}\right)=C^{C}_{BA}\frac{\partial}{\partial\theta^{C}},$$

$$D_{\partial/\partial\overline{\theta}^{A'}}\left(\frac{\delta}{\delta x^{\mu}}\right)=\overline{C}^{\nu}_{\mu A'}\frac{\delta}{\delta x^{\nu}}.$$

We can give the covariant derivatives of the higher order generalized spinors $\zeta^{AB'}_{BA'}(x,\theta,\overline{\theta})$,

(27)

$$\begin{aligned}
\nabla_{\mu}\zeta^{AB'}_{BA'} &= \frac{\delta\zeta^{AB'\dots}_{BA'\dots}}{\delta x^{\mu}}+L^{A}_{C\mu}\zeta^{CB'\dots}_{BA'\dots}+\overline{L}^{B'}_{C'\,\mu}\zeta^{AC'\dots}_{BA'\dots}-L^{C}_{B\mu}\zeta^{AB'\dots}_{CA'\dots}-\overline{L}^{C'}_{\mu A'}\zeta^{AB'\dots}_{BC'\dots}\\
\nabla_{E}\zeta^{AB'\dots}_{BA'\dots} &= \frac{\partial\zeta^{AB'\dots}_{BA'\dots}}{\partial\theta^{E}}+C^{A}_{CE}\zeta^{CB'\dots}_{BA'\dots}+\overline{C}^{B'}_{C'\,E}\zeta^{AC'\dots}_{BA'\dots}-C^{C}_{BE}\zeta^{AB'}_{CA'}-\overline{C}^{C'}_{EA'}\zeta^{AB'}_{BC'}\\
\nabla_{Z'}\zeta^{AB'\dots}_{BA'\dots} &= \frac{\partial\zeta^{AB'\dots}_{BA'\dots}}{\partial\overline{\theta}^{Z'}}+\overline{C}^{A}_{CZ'}\zeta^{CB'\dots}_{BA'\dots}+\overline{C}^{B'}_{C'Z'}\zeta^{AC'\dots}_{BA'\dots}-\overline{C}^{C'}_{Z'A'}\zeta^{AB'\dots}_{BC'\dots}.
\end{aligned}$$

Proposition 3.4. *If the connections defined by the relations (26) are of the Cartan–type, then the spinor equivalent relations are given by*

(28)

$$\begin{aligned}
\overline{\theta}^{A'}\frac{\delta\theta^{A}}{\delta x^{k}}+L^{A}_{Ck}\theta^{C}\overline{\theta}^{A'}+\theta^{A}\frac{\delta\overline{\theta}^{A'}}{\delta x^{k}}+\overline{L}^{A'}_{C'\,k}\overline{\theta}^{C'}\theta^{A} &= 0,\\
(\widetilde{\sigma}^{AA'}_{\beta})^{-1}(\overline{\theta}^{A'}\nabla_{E}\theta^{A}+\theta^{A}\nabla_{E}\overline{\theta}^{A'}) &= 1,\\
(\widetilde{\sigma}^{AA'}_{\gamma})^{-1}(\overline{\theta}^{A'}\nabla_{Z}\theta^{A}+\theta^{A}\nabla_{Z'}\overline{\theta}^{A'}) &= 1.
\end{aligned}$$

Proof. Applying the relations (27) to a null vector y with the Cartan–type properties $y^{\alpha}_{|k}=0$ and $y^{\alpha}|_{\beta}=\delta^{\alpha}_{\beta}$ [5] [6], and taking into account the (3a), (9) we obtain the relations (30). (As we have mentioned previously the y–covariant derivative has corresponded to the spinor covariant derivatives).

4 Torsions and Curvatures

The spinor torsions corresponding to the torsions of LS are given by an analogous method to that one we derived in [4] for a deformed bundle. The torsion tensor field T of a D–connection is given by $T(X,Y)=D_XY-D_YX-[X,Y]$.

Relatively to an adapted frame we have the relations

(29)
$$
\begin{aligned}
a)\quad & T\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta x^\lambda}\right)=T^{\mu}_{\lambda k}\frac{\delta}{\delta x^\mu}+T^{A}_{\lambda k}\frac{\partial}{\partial\theta^A}+\overline{T}^{A'}_{\lambda k}\frac{\partial}{\partial\overline{\theta}^{A'}}\\
b)\quad & T\left(\frac{\partial}{\partial\theta^A},\frac{\delta}{\delta x^\mu}\right)=T^{\nu}_{\mu A}\frac{\delta}{\delta x^\nu}+T^{B}_{\mu A}\frac{\partial}{\partial\theta^B}+\overline{T}^{B'}_{\mu A}\frac{\partial}{\partial\overline{\theta}^{B'}}\\
c)\quad & T\left(\frac{\partial}{\partial\overline{\theta}^{A'}},\frac{\delta}{\delta x^\mu}\right)=T^{\nu}_{\mu A'}\frac{\delta}{\delta x^\nu}+T^{B}_{\mu A'}\frac{\partial}{\partial\theta^B}+\overline{T}^{B'}_{\mu A'}\frac{\partial}{\partial\overline{\theta}^{B'}}\\
d)\quad & T\left(\frac{\partial}{\partial\theta^A},\frac{\partial}{\partial\overline{\theta}^B}\right)=T^{\mu}_{BA}\frac{\delta}{\delta x^\mu}+T^{C}_{BA}\frac{\partial}{\partial\theta^C}+\overline{T}^{C'}_{BA}\frac{\partial}{\partial\theta^{C'}}\\
e)\quad & T\left(\frac{\partial}{\partial\theta^A},\frac{\partial}{\partial\overline{\theta}^{B'}}\right)=T^{\mu}_{B'A}\frac{\delta}{\delta x^\mu}+T^{C}_{B'A}\frac{\partial}{\partial\theta^C}+\overline{T}^{C'}_{B'A}\frac{\partial}{\partial\overline{\theta}^{C'}}\\
f)\quad & T\left(\frac{\partial}{\partial\overline{\theta}^{A'}},\frac{\partial}{\partial\overline{\theta}^B}\right)=T^{\mu}_{BA'}\frac{\delta}{\delta x^\mu}+T^{C}_{BA'}\frac{\partial}{\partial\theta^C}+\overline{T}^{C'}_{BA'}\frac{\partial}{\partial\overline{\theta}^{C'}}\\
g)\quad & T\left(\frac{\partial}{\partial\overline{\theta}^{A'}},\frac{\partial}{\partial\overline{\theta}^{B'}}\right)=T^{\mu}_{B'A'}\frac{\delta}{\delta x^\mu}+T^{C}_{B'A'}\frac{\partial}{\partial\theta^C}+\overline{T}^{C'}_{B'A'}\frac{\partial}{\partial\overline{\theta}^{C'}}.
\end{aligned}
$$

The torsion (29 a)) can be written in the form

(30)
$$
D_{\delta/\delta x^\mu}\frac{\delta}{\delta x^\nu}-D_{\delta/\delta x^\nu}\frac{\delta}{\delta x^\mu}-\left[\frac{\delta}{\delta x^\mu},\frac{\delta}{\delta x^\nu}\right]=L^{\lambda}_{\nu\mu}\frac{\delta}{\delta x^\lambda}-L^{\lambda}_{\mu\nu}\frac{\delta}{\delta x^\lambda}-R^{A}_{\mu\nu}\frac{\partial}{\partial\theta^A}-V^{A'}_{\mu\nu}\frac{\partial}{\partial\overline{\theta}^{A'}},
$$

where the brackets have the form

(31)
$$
[\delta/\delta x^\mu,\delta/\delta x^\nu]=R^{A}_{\mu\nu}\frac{\partial}{\partial\theta^A}+V^{A'}_{\mu\nu}\frac{\partial}{\partial\overline{\theta}^{A'}},
$$

and $\delta/\delta x^\mu, R^A_{\mu\nu}, V^{A'}_{\mu\nu}$ are given by

$$
\frac{\delta}{\delta x^k}=\frac{\partial}{\partial x^k}-\mathcal{N}^A_k\frac{\partial}{\partial\theta^A}-\overline{N}^{A'}_k\frac{\partial}{\partial\overline{\theta}^{A'}}\quad,\quad R^A_{\mu\nu}=\frac{\delta N^A_\mu}{\delta x^\nu}-\frac{\delta\mathcal{N}^A_\nu}{\delta x^\mu}
$$

$$
V^{A'}_{\mu\nu}=\frac{\delta N^{A'}}{\delta x^\nu}-\frac{\delta N^{A'}}{\delta x^\mu}.
$$

The terms $R^A_{\mu\nu}, V^{A'}_{\mu\nu}$ represents *the spinor–curvatures of non–linear connections* $N^A_\nu, N^{A'}_\mu$. In virtue of the relations (29.a), (30), (31) we obtain

(32)
$$
T^{\lambda}_{\mu\nu}=L^{\lambda}_{\mu\nu}-L^{\lambda}_{\nu\mu}\quad,\quad T^{A}_{\nu\mu}=-R^{A}_{\mu\nu}\quad,\quad \overline{T}^{A'}_{\nu\mu}=-V^{A'}_{\mu\nu}.
$$

Similarly from the relations (31b–31g), comparing with the torsion in the following form,

$$T\left(\frac{\delta}{\delta Y^P},\frac{\delta}{\delta Y^Q}\right)=D_{\delta/\delta Y^P}\frac{\delta}{\delta Y^Q}-D_{\delta/\delta Y^Q}\frac{\delta}{\delta Y^P}-\left[\frac{\delta}{\delta Y^P}-\frac{\delta}{\delta Y^Q}\right] \tag{33}$$

we can obtain the relations

$$\begin{aligned}
&T^{\lambda}_{A\mu}=C^{\lambda}_{A\mu},\quad T^{A}_{B\mu}=\frac{\partial N^{A}_{\mu}}{\partial\theta^{B}}-L^{A}_{B\mu}\\
&T^{A'}_{A\mu}=-\widetilde{Y}^{A'}_{A\mu},\quad T^{\lambda}_{AB}=T^{\lambda}_{AA'}\\
&T^{\ell}_{AB}=C^{\ell}_{AB}-C^{\ell}_{BA},\quad \overline{T}^{A'}_{AB}=-R^{A'}_{AB},\quad T^{\lambda}_{\mu A'}=-\overline{C}^{\lambda}_{A'\mu}\\
&T^{A}_{\mu A'}=-\frac{\partial N^{A}_{\mu}}{\partial\overline{\theta}^{A'}},\quad T^{A'}_{\mu B'}=C^{A'}_{B'\mu}-P^{A'}_{\mu B}\\
&T^{B}_{AA'}=-C^{B}_{AA'},\quad T^{A'}_{AB'}=C^{A'}_{AB'}-\frac{\partial C^{A'}_{A}}{\partial\overline{\theta}^{B'}},
\end{aligned} \tag{34}$$

where we have put

$$\frac{\delta}{\delta Y^P}=\left\{\frac{\partial}{\partial\theta^A},\frac{\partial}{\partial\overline{\theta}^{A'}}\right\},\frac{\delta}{\delta Y^Q}=\left\{\frac{\delta}{\delta x^{\mu}},\frac{\partial}{\partial\overline{\theta}^{\Delta}}\right\},\quad \Delta=B,B' \text{ and } C^{A'}_{A}=C^{A'}_{BA}\theta^{B}.$$

So, we obtain the following:

Proposition 4.1. *In the adapted basis of a generalized metric tangent bundle the spinor equivalent of coefficients of the torsion T of a D–connection, are given by the relations (32)–(34).*

Proposition 4.2. *D–connection has no torsion if and only if all terms of the relation (34) are equal to zero.*

The curvature tensor field R of a D–connection has the form $R(X,Y)Z=[D_X,D_Y]Z-D_{[X,Y]}Z\ \forall X,Y,Z\in\mathcal{X}(\mathcal{TM})$. The coefficients of the curvature tensor and the corresponding spinor–curvature tensors in spinor bundle are given by

(35)

$$R^{k}_{\lambda\nu\mu} = \frac{\delta L^{k}_{\lambda\nu}}{\delta x^{\mu}} - \frac{\delta L^{k}_{\lambda\mu}}{\delta x^{\nu}} + L^{\rho}_{\lambda\nu} L^{k}_{\rho\mu} - L^{\rho}_{\lambda\mu} L^{k}_{\rho\nu} - R^{A}_{\mu\nu} C^{k}_{A\lambda} - V^{A'}_{\mu\nu} \overline{C}^{k}{}_{A'\lambda}$$

$$R^{B}_{A\nu\mu} = \frac{\delta L^{B}_{\nu A}}{\delta x^{\mu}} - \frac{\delta L^{B}_{\mu A}}{\delta x^{\nu}} + L^{\rho}_{A\nu} L^{B}_{\rho\mu} - L^{\rho}_{A\mu} L^{B}_{\rho\nu} - R^{\rho}_{\mu\nu} L^{B}_{A\rho} - V^{A'}_{\mu\nu} \overline{C}^{B}_{A'A}$$

$$R^{B'}_{A'\nu\mu} = \frac{\delta \overline{L}^{B'}{}_{\nu A'}}{\delta x^{\mu}} - \frac{\delta \overline{L}^{B'}_{\mu A'}}{\delta x^{\nu}} + \overline{L}^{B'}_{\nu A'} L_{\mu} - L^{B'}_{A'\nu} L_{\mu} - R^{A}_{\mu\nu} C^{B'}_{A'A} - V^{D'}_{\mu\nu} \overline{C}^{B'}_{D'A'}$$

$$P^{k}_{\nu\mu A} = \frac{\delta L^{k}_{\nu\mu}}{\delta \theta^{A}} - \frac{\delta C^{k}_{A\nu}}{\partial x^{\mu}} + L^{\lambda}_{\nu\mu} C^{k}_{A\lambda} - C^{\lambda}_{A\nu} L^{k}_{\lambda\mu} + \frac{\partial N^{E}_{\mu}}{\partial \theta^{A}} C^{k}_{E\nu} + \widetilde{Y}^{A'k}_{\mu A A'\nu}$$

$$P^{\ell}_{AB\mu} = \frac{\partial L^{\ell}_{A\mu}}{\partial \theta^{B}} - \frac{\delta C^{\ell}_{AB}}{\delta x^{\mu}} + L^{k}_{A\mu} C^{\ell}_{kB} - C^{k}_{AB} L^{\ell}_{k\mu} + \frac{\partial N^{n}_{\mu}}{\partial \theta^{A}} C^{\ell}_{Bn} + \widetilde{Y}^{A'}_{\mu A} C^{\ell}_{A'B}$$

$$P^{B'}_{A'A\mu} = \frac{\partial L^{B'}_{A'\mu}}{\partial \theta^{A}} - \frac{\delta C^{B'}_{A'A}}{\delta x^{\mu}} + L^{B}_{A'\mu} C^{B'}_{BA} - C^{B}_{A'A} L^{B'}_{B\mu} + \frac{\delta N^{E}_{\mu}}{\partial \theta^{A}} C^{B'}_{A'E} + \widetilde{Y}^{E'}_{\mu A} \overline{C}^{B'}_{E'A'}$$

$$S^{k}_{\mu AB} = \frac{\partial C^{k}_{\mu A}}{\partial \theta^{B}} - \frac{\partial C^{k}_{\mu B}}{\partial \theta^{A}} + C^{\lambda}_{\mu A} C^{k}_{\lambda B} - C^{\lambda}_{\mu B} C^{k}_{\lambda A} - R^{A'}_{AB} C^{k}_{A'\mu}$$

$$S^{m}_{\ell AB} = \frac{\partial C^{m}_{\ell A}}{\partial \theta^{B}} - \frac{\partial C^{m}_{\ell B}}{\partial \theta^{A}} + C^{n}_{\ell A} C^{m}_{nB} - C^{n}_{\ell B} C^{m}_{nA} - R^{A'}_{AB} \overline{C}^{m}_{A'\ell}$$

$$S^{B'}_{A'AB} = \frac{\partial C^{B'}_{A'A}}{\partial \theta^{B}} - \frac{\partial C^{B'}_{A'B}}{\partial \theta^{A}} + C^{D'}_{A'A} C^{B'}_{D'B} - C^{D'}_{A'B} C^{B'}_{D'B} - R^{D'}_{AB} \overline{C}^{B'}_{D'A'}$$

$$I^{k}_{\nu A'\mu} = \frac{\delta \overline{C}^{k}_{A'\nu}}{\delta x^{\mu}} - \frac{\partial L^{k}_{\nu\mu}}{\partial \overline{\theta}^{A'}} + C^{\rho}_{A'\nu} L^{k}_{\rho\mu} - L^{\rho}_{\nu\mu} \overline{C}^{k}_{A'\rho} - \frac{\partial N^{A}_{\mu}}{\partial \overline{\theta}^{A'}} C^{k}_{A\nu} - \overline{L}^{A}_{A'\mu} C^{k}_{A\nu}$$

$$I^{B}_{AA'\mu} = \frac{\delta C^{B}_{A'A}}{\delta x^{\mu}} - \frac{\partial L^{B}_{A\mu}}{\partial \overline{\theta}^{A'}} + C^{\rho}_{A'A} L^{B}_{\rho\mu} - L^{\rho}_{A\mu} C^{B}_{A'\rho} - \frac{\partial N^{\rho}_{\mu}}{\partial \overline{\theta}^{A'}} C^{B}_{A\rho} - \overline{L}^{D'}_{A'\mu} \overline{C}^{B}_{D'A}$$

$$I^{B'}_{A'C'\mu} = \frac{\delta \overline{C}^{B'}_{A'C'}}{\delta x^{\mu}} - \frac{\partial \overline{L}^{B'}_{A'\mu}}{\partial \overline{\theta}^{C'}} + \overline{C}^{B'}_{A'D'} \overline{L}^{D'}_{C'\mu} - \overline{L}^{B'}_{E'\mu} \overline{C}^{E'}_{A'B'} - \frac{\partial N^{\rho}_{\mu}}{\partial \overline{\theta}^{A'}} \overline{L}^{B'}_{C'\rho} - \overline{L}^{D'}_{A'\mu} \overline{C}^{B'}_{D'C'}$$

$$J^{k}_{\nu A'B} = \frac{\partial C^{k}_{A'\nu}}{\partial \theta^{B}} - \frac{\partial C^{k}_{B\nu}}{\partial \overline{\theta}^{A'}} + C^{\rho}_{A'\nu} C^{k}_{B\rho} - C^{\rho}_{B\nu} C^{k}_{A'\rho} - \frac{\partial L^{D'}_{B}}{\partial \overline{\theta}^{A'}} C^{k}_{D'\nu}$$

$$J^{\rho}_{AA'B} = \frac{\partial C^{\rho}_{A'B}}{\partial \theta^{A}} - \frac{\partial C^{\rho}_{AB}}{\partial \overline{\theta}^{A'}} + C^{k}_{A'A} C^{\rho}_{kB} - C^{k}_{AB} C^{\rho}_{A'k} - \frac{\partial L^{D'}_{A}}{\partial \overline{\theta}^{A'}} C^{\rho}_{D'B}$$

$$J^{B'}_{A'C'A} = \frac{\partial \overline{C}^{B'}_{A'C'}}{\partial \theta^{A}} - \frac{\partial C^{B'}_{A'A}}{\partial \overline{\theta}^{A'}} + C^{B'}_{A'D} C^{D}_{C'A} - C^{B'}_{A'E} C^{E}_{C'A} - \frac{\partial L^{D'}_{A}}{\partial \overline{\theta}^{A'}} \overline{C}^{B'}_{C'D'}$$

$$K^{\nu}_{\mu A'B'} = K^{B}_{AA'B'} = K^{B'}_{A'C'D'} = 0.$$

So we have

Theorem 4.1. *The coefficients of the curvatures of a D-connection are given by the relation (35).*

Theorem 4.2. *In a tangent bundle a D–connection has no curvature if and only if all the coefficients (rel. 35) of the curvatures are equal to zero.*

5 Discussion

The advantage of the framework that uses a spinor geometrical representation for the generalized spaces is that it enables the description of particles with spins $1/2, 3/2, \ldots$, in addition to those with spins $0, 1, 2, \ldots$, etc., while the usual tensors can describe only the latter kind of particles. From the mathematical and physical point of view spinors are considered to be more fundamental than tensors. These are associated to the group $SL(2, \mathbf{C})$ which is the covering group of the Lorentz group with which the tensors are associated.

Moreover, with the spinorial equivalence of the null vectors we can describe particles such as photons, neutrinos etc, in a generalized metric space–time.

Also, the spinorial equation of causality for the unique solution of the null–cone in the Lagrange or Finsler space was given explicitly by the proposition (3.3).

Additionally, the gravitational field can be described by virtue of the corresponding spinorial form of the metric tensor equivalent to the spinor bundle. This will be the object of our future study.

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