Čech-de Rham Cohomology of a Refinement of a Principal Bundle

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Abstract

In this paper we shall study the cohomology of the various spaces appearing in the refinement of a differentiable principal bundle defined by a closed subgroup.

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Introduction

Let $p : E \longrightarrow B$ be a differentiable principal bundle and let $\mathcal{N}_q = (G = F_0 \supset F_1 \supset \ldots \supset F_q = \{e\})$ (q is an integer ≥ 2) be a sequence of closed subgroups of G.Let $E_i = E/F_i$, i=0,1,...,q; $F_k^j = F_j/F_k, 0 \leq j < k \leq q$; $G_k^j = F_j/N_{jk}$ (here N_{jk} is the normal closure of F_j in F_k).Finally, let $p_{jk} : E_k \longrightarrow E_j$ be the canonical map.

D.I. PAPUC ([5]) proved that $p_{jk} : E_k \longrightarrow E_j$ is a differentiable fibre bundle with fibre F_k^j and structure group G_k^j.

A refinement of a principal bundle $\xi = (E, p, B, G)$ is the well-known structure determined by a closed subgroup F_j of G constitued by three bundles $(\xi; \xi_{oj}, \xi_{jq})$.

The paper consists of three sections. The first section contains some preliminaries about the tissues associated to a principal bundle. Also, some examples of tissues and refinements are given.

The second section one contains the construction of the Čech-de Rham complex of an open cover of a manifold (see, [3]).

In the third section we shall study the Čech-de Rham cohomology of a refinement of a principal fibre bundle, whenever the base space of tissue has a finite good cover. Some main results concerning the cohomology of the spaces appearing in this structure are established.

Throughout in this note all spaces are finite-dimensional real differentiable manifolds, without boundary of C^∞ classes and all maps are C^∞.

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1 Refinements of a differentiable principal bundle

Let $(\xi,\mathcal{N}_{\text{II}})$ be a pair consisting of a differentiable principal Steenrod bundle $\xi = (E,p.B,G;\mathcal{A})$ and $\mathcal{N}_{\text{II}} = (\mathcal{G} = \mathcal{F}_{\prime} \supset \mathcal{F}_{\infty} \supset ... \supset \mathcal{F}_{\text{II}-\infty} \supset \mathcal{F}_{\text{II}} = \{\rceil\})$ a sequence of closed subgroups of the structure group G.

We consider the Steenrod bundles $\xi_{jk} = (E_k, p_{jk}, E_j, F_k^j, G_k^j; \mathcal{A}_{|||})$ for $0 \leq j < k \leq q$ determined by ξ and $\mathcal{N}_{\text{II}}$, where $E_j = E/F_j, F_k^j = F_j/F_k, G_k^j = F_j/N_{jk}.N_{jk}$ being the largest normal subgroup of F_j included in F_k and p_{jk} is the canonical map, (see [5],p.372).

The Steenrod tissue $[\xi,\mathcal{N}_{\text{II}}]$ associated to the pair $(\xi,\mathcal{N}_q)$ is the set of all fibre bundles ξ_{jk} for $0 \leq j < k \leq q$. We have that $\xi_{oq} = \xi$ and moreover every fibre bundle ξ_{jq},$0 < j < q$ is a principal one.

The triple $(\xi = \xi_{oq}; \xi_{0j}, \xi_{jq})$, for $0 < j < q$ is called, via [5], the *refinement of* ξ *defined by* F_j.

Example 1. a) *The tissue associated to bundle of tangent linear frames.* Let M be a manifold of dimension n. A *k-tangent linear frame* u_k at a point $x \in M$, where $1 \leq k \leq n$ is a linear independent system $u_k = (X_1, X_2, ..., X_k)$ of the tangent space $T_x(M)$. Let $L_k(M)$ be the set of all k-tangent linear frames u_k at all points of M, and let p be the mapping of $L_k(M)$ onto M which maps a k-tangent linear frame u_k at x into x. The general linear group GL(n;R) acts on $L_k(M)$ on the right as follows. If $a = (a_i^j) \in GL(n;R)$ and $u_k = (X_1, X_2, ..., X_k)$ is a k-tangent linear frame at x then u.a is, by definition, the k-tangent linear frame $(Y_1, Y_2, ..., Y_k)$ at x defined by $Y_i = \sum\limits_{j=1}^{j=k} a_i^j X_j$. It is clear that $GL(n;R)$ acts freely on $L_k(M)$ and $p(u_k) = p(v_k)$ iff v=u.a for some $a \in GL(n;R)$. It is known (see [4]) that $(L_k(M), p, M, GL(n;R))$ is a principal fibre bundle and it is denoted by $L_k(M)$. We call $L_k(M)$ *the bundle of k-tangent linear frames over* M.

In particular, when k=n, then $L_n(M) = L(M)$ is called the *bundle of tangent linear frames over M.*

The *tangent bundle* $T(M)$ over M is the bundle $(T(M), \pi, M, R^n, GL(n;R))$ associated with the bundle of tangent linear frames L(M) over M with the standard fibre R^n.

We consider the pair $(\xi,\mathcal{G})$, where $\xi = (L_n(M), p.M, GL(n;R))$ is the principal fibre bundle of tangent linear frames over M and $\mathcal{G}$ is the following sequence

$$\mathcal{G} = (GL(n;R) = G_n \supset G_{n-1} \supset ... \supset G_1 \supset G_0 = \{e\}),$$

where $G_k = \{a = (a_j^i) \in G_n | a_j^i = \delta_j^i, j = \overline{1, n-k}, i = \overline{1,n}\}$.

It is known that the quotient manifold G_n/G_{n-k} is diffeomorphic with the Stiefel manifold $V_{n,k}$ of all systems formed by k linear independent vectors of R^n.

We construct the tissue associated to pair $(\xi,\mathcal{G})$. We have $[\xi,\mathcal{G}] = \{\xi_{jk} | 0 \leq j < k \leq n\}$, where $\tilde{\xi}_{jk} = (L_n(M)/G_{n-k}, \tilde{p}_{jk}, L_n(M)/G_{n-j}, G_{n-j}/G_{n-k}, G_{n-j})$, since the largest normal subgroup of G_n included in G_m is G_0.

For all $1 \leq j \leq n-1$ there exist a diffeomorphism $\varphi_j : L_n(M)/G_{n-j} \longrightarrow L_j(M)$ such that $\varphi_j \circ \tilde{p}_{jk} = p_{jk} \circ \varphi_j$, where $p_{jk} : L_k(M) \longrightarrow L_j(M)$ is the canonical projection.

Using the above diffeomorphism, the fibre bundle $\tilde{\xi}_{jk}$ can be replaced by the fibre bundle $\xi_{jk} = (L_k(M), p_{jk}, L_j(M), G_{n-j}/G_{n-k}, G_{n-j})$.

Hence, the tissue associated to the pair $(\xi, \mathcal{G})$ is $[\xi, \mathcal{G}] = \{\xi_{jk} | 0 \leq j < k \leq n\}$. We have $\xi_{0n} = \xi$ and $\xi_{jn}(0 < j < n)$ is a principal fibre bundle with the structure group G_{n-j}.

If $0 \leq j < k \leq n$, then ξ_{jk} is the bundle associated to ξ_{jn} with the fibre type G_{n-j}/G_{n-k} and G_{n-j} as structure group.

The refinement of ξ defined by G_{n-k} is the following $(\xi_{0n} = \xi; \xi_{0,n-k}, \xi_{n-k,n})$, where

$$\xi_{0,n-k} = (L_{n-k}(M), p_{0,n-k}, M, V_{n,n-k}, G_n)$$

is the bundle associated to ξ_{on} with the Stiefel manifold $V_{n,n-k}$ as fibre type and G_n as structure group;

$$\xi_{n-k,n} = (L_n(M), p_{n-k,n}, L_{n-k}(M), G_k)$$

is the principal fibre bundle with G_k as the structure group.

Applying the general properties of the tissus associated to principal differentiable fibre bundle, we have the main results:

Let $L_n(M)$ be the principal bundle of tangent linear frames over a n-dimensional manifold M. The structure group $GL(n; R)$ of $L_n(M)$ can be reduced to the group G_{n-k}, $1 \leq k < n$ iff the fibre bundle $\xi_{0,n-k}$ has a cross section.

b) Let $\xi = (L_n(V_n), p, V_n, GL(n; R))$ be the principal bundle of tangent linear frames to a n-manifold V_n and $\mathcal{N}_2 = (GL(n; R) \supset D(n, R) \supset E)$ a sequence of $GL(n; R)$, where $D(n; R) = \{(a\delta_i^j) | a \in R\}$ is the diagonal subgroup and $E = \{(\delta_i^j)\}$.

Since, $D(n; R)$ is a normal subgroup of $GL(n; R)$, it follows that the refinement of ξ defined by $D(n; R)$, denoted by $(\xi'_{02} = \xi; \xi'_{01}, \xi'_{12})$, is formed from the following three principal bundles: ξ, $\xi'_{01} = (t(V_n), p_{01}, V_n, GP(n-1; R))$ is the principal bundle of tangent directions to V_n and $\xi'_{12} = (L_n(V_n), p_{12}, t(V_n), D(n; R))$, where $GP(n-1; R)$ is the (n-1)-dimensional real projective group.

2 Čech-de Rham complex of an open cover

In the sequel, we denote by Ω^* the algebra over $\mathbf{R}$ generated by $dx_1, dx_2, ..., dx_n$ with the relations

$$(dx_j)^2 = 0; dx_i dx_j = -dx_j dx_i \; for \; i \neq j,$$

where $x_1, x_2, ..., x_n$ are the coordinates on $\mathbf{R}^n$.

For any open subset U of $\mathbf{R}^n$, the C^∞ *differential* q-*forms on* U are elements of $\Omega^q(U) = \{C^\infty \text{ functions on} U\} \otimes_{\mathbf{R}} \Omega^*$, i.e., if $\omega \in \Omega^q(U)$ then $\omega = \sum f_{i_1 i_2 ... i_q} dx_{i_1} ... dx_{i_q}$, where $f_{i_1 ... i_q}$ are C^∞ functions.

There is a differential operator $d : \Omega^q(U) \longrightarrow \Omega^{q+1}(U)$ defined as follows :

i) if $f \in \Omega^0(U)$, then $df = \sum \frac{\partial f}{\partial x_i} dx_i$;

ii) if $\omega = \sum f_{i_1 ... i_q} dx_{i_1} ... dx_{i_q}$, then $d\omega = \sum df_{i_1 ... i_q} dx_{i_1} ... dx_{i_q}$.

The complex $\Omega^*(U) = \bigoplus_{q=0}^{n} \Omega^q(U)$ together with the differential operator d is called the *de Rham complex on U*. The kernel of d is called the *closed forms* and the image of d , the *exact forms*.

The q-th *de Rham cohomology of* U is the vector space

$$H^q_{DR}(U) = \{closed\ q-forms\}/\{exact\ q-forms\}.$$

We also write $H^q(U)$ for q-th de Rham cohomology of U.

Let $\mathcal{U}$ be an open cover$\{U, V\}$ of a manifold M. There is a sequence of inclusions of open sets

$$M \longleftarrow U \coprod V \underset{\overset{\partial_1}{\longleftarrow}}{\overset{\partial_0}{\longleftarrow}} U \cap V,$$

where $U \coprod V$ is the disjoint union of U and V and ∂_0, ∂_1 are the inclusions of $U \cap V$ in V and in U, respectively.

Applying the contravariant functor Ω^*, we get a sequence of restrictions of forms

$$\Omega^*(M) \longrightarrow \Omega^*(U) \oplus \Omega^*(V) \underset{\overset{\partial_1^*}{\longrightarrow}}{\overset{\partial_0^*}{\longrightarrow}} \Omega^*(U \cap V) \longrightarrow O,$$

where by restriction of a form to a submanifold we mean its image under the pullback map induced by the inclusions.

By taking the difference of the last two maps, we obtain the *Mayer-Vietoris (short) exact sequence*

$$(1) \qquad O \longrightarrow \Omega^*(M) \longrightarrow \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{\delta} \Omega^*(U \cap V) \longrightarrow O,$$

where $\delta(\omega, \tau) = \tau - \omega$.

The Mayer-Vietoris (1) gives rise to a *long exact sequence in cohomology*

$$(2) \qquad H^q(M) \xrightarrow{r} H^q(U) \oplus H^q(V) \xrightarrow{\delta} H^q(U \cap V) \xrightarrow{d^*} H^{q+1}(M) \longrightarrow \ldots,$$

where d^* is the *coboundary operator* given by

$$(3) \qquad d^*([\omega]) = \begin{cases} [-d(\rho_V \omega)] & on\ U \\ [d(\rho_U \omega)] & on\ V, \end{cases}$$

where (ρ_U, ρ_V) is a partition of unity subordinate to cover $\mathcal{U}$ and $[\omega]$ denotes the cohomology class of the form ω.

We observe that the long exact sequence in cohomology allows one to compute in many cases the cohomology of M from the cohomology of the open subsets U and V.

Instead of a cover with two open sets as in the usual Mayer-Vietoris sequence, consider the open cover $\mathcal{U} = \{U_\alpha | \alpha \in J\}$ of M, where the index set J is a countable ordered set. Denote the pairwise intersections $U_\alpha \cap U_\beta$ by $U_{\alpha\beta}$ (when $\alpha < \beta$), triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$ by $U_{\alpha\beta\gamma}$(when $\alpha < \beta < \gamma$), etc.

There is a sequence of inclusions of open sets

$$M \longleftarrow \coprod U_{\alpha_0} \underset{\overset{\partial_1}{\longleftarrow}}{\overset{\partial_0}{\longleftarrow}} \coprod U_{\alpha_0\alpha_1} \begin{array}{c} \overset{\partial_0}{\longleftarrow} \\ \overset{\partial_1}{\longleftarrow} \\ \overset{\partial_2}{\longleftarrow} \end{array} \coprod U_{\alpha_0\alpha_1\alpha_2} \begin{array}{c} \longleftarrow \\ \cdots \\ \longleftarrow \end{array},$$

where ∂_i is the inclusion which „ignores" the i-th open set, for example, $\partial_0 : U_{\alpha_0\alpha_1\alpha_2} \longrightarrow U_{\alpha_1\alpha_2}$.

This sequence of inclusions of open sets induces a sequence of restrictions of forms

$$\Omega^*(M) \xrightarrow{r} \Pi\Omega^*(U_{\alpha_0}) \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} \Pi\Omega^*(U_{\alpha_0\alpha_1}) \overset{\delta_0}{\underset{\delta_2}{\overset{\delta_1}{\longrightarrow}}} \Pi\Omega^*(U_{\alpha_0\alpha_1\alpha_2}) \longrightarrow \dots ,$$

where δ_0, for instance, is induced from the inclusion $\partial_0 : \coprod U_{\alpha\beta\gamma} \longrightarrow \coprod U_{\beta\gamma}$ and therefore is the restriction $\delta_0; \Pi\Omega^*(U_{\beta\gamma}) \longrightarrow \Pi\Omega^*(U_{\alpha\beta\gamma})$.

We define the *difference operator* δ:$\Pi\Omega^*(U_{\alpha_0\alpha_1}) \longrightarrow \Pi\Omega^*(U_{\alpha_0\alpha_1\alpha_2})$ to be the alternating difference $\delta_0 - \delta_1 + \delta_2$.

The following sequence

$$O \longrightarrow \Omega^*(M) \xrightarrow{r} \Pi\Omega^*(U_{\alpha_0}) \xrightarrow{\delta} \Pi\Omega^*(U_{\alpha_0\alpha_1}) \xrightarrow{\delta} \Pi\Omega^*(U_{\alpha_0\alpha_1\alpha_2}) \xrightarrow{\delta} \dots \tag{4}$$

is exact and it is called the *generalized Mayer-Vietoris sequence.*

If $\mathcal{U}$=$\{U_\alpha | \alpha \in J\}$ is an open cover of M, consider the double complex

$$C^*(\mathcal{U}, \Omega^*) = \bigoplus_{p,q\geq 0} K^{p,q} = \bigoplus_{p,q\geq 0} C^p(\mathcal{U}, \Omega^q),$$

where $C^p(\mathcal{U}, \Omega^q) = \Pi\Omega^q(U_{\alpha_0\alpha_1\dots\alpha_p})$, i.e., $K^{p,q}$ consists of the „ p- cochains of the cover $\mathcal{U}$ with values in the q- forms".

For example: $K^{0,q} = C^0(\mathcal{U}, \Omega^q) = \Pi\Omega^q(U_{\alpha_0})$, $K^{1,q} = C^1(\mathcal{U}, \Omega^q) = \Pi\Omega^q(U_{\alpha_0\alpha_1})$.

The double complex is equipped with the following two differential operators δ and d, where $\delta : C^p(\mathcal{U}, \Omega^q) \longrightarrow C^{p+1}(\mathcal{U}, \Omega^q)$ is the *difference operator* and $d : C^p(\mathcal{U}, \Omega^q) \longrightarrow C^p(\mathcal{U}, \Omega^{q+1})$ is the *exterior derivative.*

We have the following two sequences

$$O \longrightarrow \Omega^q(M) \xrightarrow{r} K^{p,q} \xrightarrow{\delta} K^{p+1,q} \longrightarrow \dots \tag{5}$$

and

$$K^{p,0} \xrightarrow{d} K^{p,1} \xrightarrow{d} \dots \longrightarrow K^{p,q} \xrightarrow{d} K^{p,q+1} \longrightarrow \dots . \tag{6}$$

The double graded complex $C^*(\mathcal{U}, \Omega^*) = \bigoplus_{p,q\geq 0} C^p(\mathcal{U}, \Omega^q)$ is called the *Čech-de Rham complex of the cover* $\mathcal{U}$ of M and an element of the Čech-de Rham complex is called *a Čech-de Rham cochain.*

Given the doubly graded complex $K^{*,*}$ with commuting operators d and δ, one can associate a singly graded complex K^*, where $K^* = \bigoplus_{p+q=n} K^{p,q}$ and defining the differential operator D by $D = \delta + (-1)^p.d$, on $K^{p,q}$.

In the sequel we will use the same symbol $C^*(\mathcal{U}, \Omega^*)$ to denote the double complex and its associated single complex.

The double graded complex $C^*(\mathcal{U}, \Omega^*)$ computes the de Rham cohomology of M, i.e.

$$H_D\{C^*(\mathcal{U}, \Omega^*)\} \cong H^*_{DR}(M). \tag{7}$$

We have

$$H^n_{DR}(M) = \bigoplus_{p+q=n} H_D\{C^*(\mathcal{U},\Omega^*)\}.$$

Let $\mathcal{U} = \{U_\alpha|\alpha \in J\}$ be a *good cover* of M (i.e., all finite intersections $U_{\alpha_0} \cap U_{\alpha_1} \cap ... \cap U_{\alpha_p}$ are diffeomorphic to $\mathbf{R}^n$) and we denote by $H^*(\mathcal{U},\mathbf{R})$ the *Čech cohomology of the cover* $\mathcal{U}$.

If $\mathcal{U}$ is a good cover of M, then the double complex $C^*(\mathcal{U},\Omega^*)$ computes the Čech cohomology of the cover $\mathcal{U}$ of M, i.e.

$$H^*(\mathcal{U},\mathbf{R}) \cong H_D\{C^*(\mathcal{U},\Omega^*)\}. \tag{8}$$

Therefore, if $\mathcal{U}$ is a good cover of the manifold M, then there is an isomorphism between the de Rham cohomology of M and the Čech cohomology of the good cover $\mathcal{U}$ of M, i.e.

$$H^*_{DR}(M) \cong H^*(\mathcal{U},\mathbf{R}). \tag{9}$$

This result provides us with a way of computing the de Rham cohomology by means of combinatorics.

3 Čech-de Rham cohomology of a refinement

Theorem 1. *Let $[\xi,\mathcal{N}_q]$ be a totally trivial tissue (i.e. the bundles ξ_{jq} are product bundles for each $0 \le j < q$) associated to pair $(\xi,\mathcal{N}_q)$. Then the following assertions hold*

$$(i) \qquad H^*_{DR}(E_j) \cong H^*_{DR}(B) \otimes H^*_{DR}(F^0_1) \otimes ... \otimes H^*_{DR}(F^{j-1}_j)$$

$$(ii) \qquad H^*_{DR}(F^0_j) \cong H^*_{DR}(F^0_1) \otimes H^*_{DR}(F^1_2) \otimes ... \otimes H^*_{DR}(F^{j-1}_j)$$

for all $0 \le j \le q$.

Proof. The tissue $[\xi,\mathcal{N}_q]$ being totally trivial it follows that the space $E_j = E/F_j$ is homeomorphic with $B \times F^0_1 \times ... \times F^{j-1}_j$ and the homogeneous space F^0_j is homeomorphic with $F^0_1 \times F^1_2 \times ... \times F^{j-1}_j$ (see,[5],Th.3.) For j=1, the spaces E_1 and $B \times F^0_1$ are homeomorphic and $H^*_{DR}(E_1) = H^*_{DR}(B \times F^0_1)$. But by Künneth's formula, we have $H^*_{DR}(B \times F^0_1) \cong H^*_{DR}(B) \otimes H^*_{DR}(F^0_1)$, and we obtain $H^*_{DR}(E_1) \cong H^*_{DR}(B) \otimes H^*_{DR}(F^0_1)$.

This means

$$H^n_{DR}(E_1) = \bigoplus_{p+q=n} H^p_{DR}(B) \otimes H^q_{DR}(F^0_1).$$

Applying now the induction and the general properties of tensor product, by similar arguments we obtain the isomorphisms (i) and (ii).

In the sequel we suppose that the base space B of the principal bundle $\xi = (E,p,B,G)$ has a finite good cover.

Theorem 2. *Let $\xi = (E,p,B,G)$ be a principal bundle such that B has a finite good cover. If F_j (j fixed) is a closed subgroup of G such that the cohomology of G and*

F_j are finite-dimensional, then for the refinement $(\xi;\xi_{0j},\xi_{jq})$ of ξ defined by F_j the following assertions hold

$$(i) \qquad H^*_{DR}(E)\cong H^*_{DR}(B)\otimes H^*_{DR}(G)$$

$$(ii) \qquad H^*_{DR}(E)\cong H^*_{DR}(E_j)\otimes H^*_{DR}(F_j).$$

Proof. (i) The space of cohomology of E (for every n) being a vector space follows that it has a base, that this there are global cohomology classes $\{e_i|i\in I\}$ on E. If we restrict $\{e_i\}$ to each fiber of ξ imply that $\{e_i\}$ generate the cohomology of the fiber G, and we can extract a base of $H^n_{DR}(G)$, since the cohomology of G is finite-dimensional.Therefore, there are global cohomology classes $e_1,e_2,...,e_r$ on E which when restrict to each fiber freely generate the cohomology of fiber. Hence,the hypothesis of Leray-Hirsch's theorem are satisfied for ξ and we have the isomorphisn (i).

(ii) We suppose that $\mathcal{U}=\{U_i|i=1,2,...,n\}$ is a finite good cover of B. Then $\mathcal{U}^{`}=\{p_{oj}^{-1}(U_i)|i=1,2,...,n\}$ is a finite good cover of E_j. Hence the base space E_j of the bundle ξ_{jq} has a finite good cover. Using now Leray-Hirsch's theorem and the fact that the cohomology of F_j are finite-dimensional the same argument from proof of (i) gives the isomorphism (ii).

Corollary 1. *Let $\xi=(E,p,B,G)$ be a principal bundle such that the base space B and the structure group G are compact spaces. Then for the refinement $(\xi;\xi_{0j},\xi_{jq})$ of ξ defined by a closed subgroup F_j of G, the following assertions hold*

$$(i) \qquad H^*_{DR}(E)\cong H^*_{DR}(B)\otimes H^*_{DR}(G)$$

$$(ii) \qquad H^*_{DR}(E)\cong H^*_{DR}(E_j)\otimes H^*_{DR}(F_j).$$

Proof. The base space B of ξ has a finite good cover since the manifold B is compact. The hypothesis of Theorem 2 are verified since the cohomology of a compact manifold is finite-dimensional. Applying now Theorem 2 we obtain the isomorphisms (i) and (ii).

Theorem 3. *Let $\xi=(E,p,B,G)$ be a principal bundle such that the base space B has a finite good cover. Let F_j (j- fixed) be a closed subgroup of G such that the cohomology of F_j and F_j^0 are finite-dimensional. Then for the refinement $(\xi;\xi_{0j},\xi_{jq})$ of ξ defined by F_j the following assertions hold*

$$(i) \qquad H^*_{DR}(E)\cong H^*_{DR}(E_j)\otimes H^*_{DR}(F_j)$$

$$(ii) \qquad H^*_{DR}(E_j)\cong H^*_{DR}(B)\otimes H^*_{DR}(F_j^0)$$

$$(iii) \qquad H^*_{DR}(E)\cong H^*_{DR}(B)\otimes H^*_{DR}(F_j^0)\otimes H^*_{DR}(F_j).$$

Proof. (i) We have that the base space E_j of ξ_{jq} has a finite good cover and the cohomology of the fibre F_j is finite-dimensional. Applying now Leray-Hirsch's theorem,we obtain the isomorphism (i).

(ii) We have that the base space B of ξ_{0j} has a finite good cover and the cohomology of the fibre F_j^0 is finite-dimensional. We can apply Leray-Hirsch's theorem and we obtain the isomorphism (ii).

(iii) This isomorphism results from (i) and (ii).

Theorem 4. *Let $\xi = (E, p, B, G)$ be a principal bundle such that the base space B has a finite good cover and the structure group G is a connected Lie group. If F is a maximal compact subgroup of G, then for the refinement $(\xi; \xi_{01}, \xi_{12})$ of ξ defined by F the following assertions hold*

$$(i) \qquad H^*_{DR}(E) \cong H^*_{DR}(E/F) \otimes H^*_{DR}(F)$$

$$(ii) \qquad H^*_{DR}(E/F) \cong H^*_{DR}(B) \otimes H^*_{DR}(G/F)$$

$$(iii) \qquad H^*_{DR}(G) \cong H^*_{DR}(F) \otimes H^*_{DR}(G/F)$$

$$(iv) \qquad H^*_{DR}(E) \cong H^*_{DR}(B) \otimes H^*_{DR}(G/F) \otimes H^*_{DR}(F).$$

Proof. Since F is a maximal compact subgroup of G imply, by Ivasawa's theorem, that G is homeomorphic with the direct product of F and a Euclidian space (i.e., G is homeomorphic to $F \times \mathbf{R}^m$). Then $H^*_{DR}(G) = H^*_{DR}(F \times (G/F))$ and using the Küunneth's formula it follows the isomorphism (iii). Since F is compact and G/F is a Euclidean space it follows that the cohomology of F and G/F are finite-dimensional. Hence the hypothesis of Theorem 3 are verified and we obtain the isomorphisms (i) , (ii) and (iv).

Theorem 5. *Let $\xi = (E, p, B, G)$ be a principal bundle such that the base space has a finite good cover and the structure group is a simply connected Lie group. Let F be a normal closed subgroup of G such that the factor group G/F is abelian. If the cohomology of G is finite-dimensional, then for the refinement $(\xi; \xi_{01}, \xi_{12})$ of ξ defined by F the following assertions hold:*

$$(i) \qquad H^*_{DR}(E) \cong H^*_{DR}(B) \otimes H^*_{DR}(G)$$

$$(ii) \qquad H^*_{DR}(E/F) \cong H^*_{DR}(B) \otimes H^*_{DR}(G/F).$$

Proof. (i) We apply the same argument used in the proof of Theorem 1. (i).

(ii) Since F is a normal closed subgroup of G follows that ξ_{01} is a principal bundle having G/F as structure group. But G/F being a simply connected Lie group imply that there is an integer m such that G/F is diffeomorphic with the Euclidian space $\mathbf{R}^m$. Hence, $\xi_{01} = (E/F, p_{01}, B, G/F)$ is a principal bundle for which the fibre is diffeomorphic with a Euclidean space. Then there exists a cross section of ξ_{01} defined on B. Applying Theorem 1 from [7], p.36, it follows that ξ_{01} is a trivial bundle; hence, E/F and $B \times (G/F)$ are diffeomorphic.We have that $H^*_{DR}(E/F) = H^*_{DR}(B \times (G/F))$. Using now the Künneth's formula we obtain (ii).

Example 2. Let $(\xi^{'}_{02} = \xi; \xi^{'}_{01}, \xi^{'}_{12})$ the refinement of

$$\xi = (L_n(V_n), p, V_n, GL(n, R))$$

defined by $F = GL(n,R)$, see Example 1 (b). If the base B of ξ has a finite good cover or is a compact space, then:

$$H^*_{DR}(L_n(V_n)) \cong H^*_{DR}(V_n) \otimes H^*_{DR}(GL(n,R))$$

$$H^*_{DR}(L_n(V_n) \cong H^*_{DR}(t(V_n)) \otimes H^*_{DR}(\mathbf{R}^m).$$

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