Pseudo-Riemannian Osserman Manifolds

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Abstract

We survey some recent results concerning the Osserman conjecture: if the eigenvalues of the Jacobi operator are constant, need the manifold be locally rank one symmetric? The conjecture is known to hold in the Lorentzian setting and in certain cases to hold in the Riemannian setting. If the manifold has signature (2,2) and if the Jacobi operator is diagonalizable, the conjecture is known to hold; if the manifold has signature (2,2) and if the Jacobi operator is not diagonalizable, there are counter-examples to the conjecture.

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§0 Introduction and notational conventions

Let M be a n–dimensional pseudo–Riemannian manifold of signature (p, q). Denote the metric tensor by $\langle\cdot,\cdot\rangle$. The case $p = 0$ or $q = 0$ is the Riemannian case; the case $p = 1$ or $q = 1$ is the Lorentzian setting. We shall also be interested in case of neutral Einstein manifolds of dimension 4; this is the case $(p, q) = (2, 2)$, see Borowiec et. al. [12], Guillemin and Strenberg [24], and Law [29] for other related results.

Let $S^\epsilon(p) := \{X \in T_pM \mid \langle X, X\rangle = \epsilon 1\}$ be the set of all unit spacelike ($\epsilon = +$) or timelike ($\epsilon = -$) tangent vectors at $p \in M$. Let $S^\epsilon(M) = \cup_p S^\epsilon(p)$. Let $X \in S^\epsilon_p$. Since X is not a null vector, we have $X \oplus X^\perp = T_pM$. The Jacobi operator $\mathcal{K}_X : Y \mapsto R(Y, X)X$ induces a symmetric endomorphism of the vector space $T_X(S^\epsilon_p) = X^\perp = \{Y \in T_pM \mid \langle X, Y\rangle = 0\}$. This operator is important in the study of Riemannian manifolds. It is equally important in the study of pseudo–Riemannian manifolds. For example, the family of free falling particles along a geodesic γ in a Lorentz manifold is described by the normal variational Jacobi vector field V along γ. The Jacobi operator plays the role of the tidal force and V satisfies Newton's second law: $V'' = R_{V\gamma'}(\gamma') = \mathcal{K}_{\gamma'}V$.

We say that M is spacelike ($\epsilon = +$) or timelike ($\epsilon = -$) *Osserman* if the eigenvalues of $\mathcal{K}_X$ are constant on $S^\epsilon(M)$; we say that M is spacelike or timelike *Osserman* at a point $p \in M$ if the eigenvalues of $\mathcal{K}_X$ are constant on $S^\epsilon(p)$. There are manifolds of signature $(0, 4)$ which are Osserman at each point $p \in M$ but which are not Osserman; the eigenvalues can change from point to point, see Gilkey, Swann, and Vanhecke [23]

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for details. Let $X \in S^\epsilon(p)$. If $p \geq 2$ and if $q \geq 2$, the induced metric on $T_X S^\epsilon(p)$ is not definite and thus $\mathcal{K}_X$ need not be diagonalizable. We say that M is spacelike or timelike algebraic-Osserman at a point p of M if the minimal polynomial of $\mathcal{K}_X$ is constant on $S^\epsilon(p)$; similarly we say that M is spacelike or timelike algebraic–Osserman if the minimal polynomial of $\mathcal{K}_X$ is constant on $S^\epsilon(M)$. This fixes the algebraic structure of $\mathcal{K}_{\mathcal{X}}$. There are manifolds of signature $(2,2)$ which are Osserman but which are not algebraic Osserman, see §3 for details. More generally, we say that M is spacelike or timelike Jordan–Osserman at p if the Jordan form of $\mathcal{K}_X$ is independent of $X \in S^\epsilon_p$. M is pointwise spacelike or timelike Jordan–Osserman if M is spacelike or timelike Jordan-Osserman at each $p \in M$. Note that if M is 4 dimensional, then the Jordan-Osserman condition is equivalent with the algebraic-Osserman condition.

Let M be a Riemannian manifold. If M is locally a rank one symmetric space or is locally flat, then M is locally a 2 point homogeneous space. This means that local isometries of M act transitively on the unit sphere bundle $S^+(M)$. Conversely, any manifold which is locally a 2 point homogeneous space is locally a rank one symmetric space or is locally flat. For such a manifold, the eigenvalues of the Jacobi operator $\mathcal{K}_X$ are constant on $S^+(M)$. Osserman [32] conjectured that the converse holds; we restate his conjecture as follows:

0.1. Conjecture. *If a Riemannian manifold M is Osserman, then M is locally a 2 point homogeneous space.*

This paper is devoted to the study of the Osserman conjecture and its generalizations to pseudo Riemannian metrics. In §1, we review the known results in the Riemannian setting and in §2 we review the known results in the Lorentzian; these are the cases $p = 0$ or $q = 0$, $p = 1$ or $q = 1$. In these cases, $\mathcal{K}_X$ is diagonalizable so the multiplicity of the eigenvalues of $\mathcal{K}_X$ determines the algebraic structure of this operator. In §3, we present some examples of Riemannian manifolds of signature $(2,2)$ which are Osserman but not algebraic Osserman and which are algebraic Osserman but not locally symmetric. In §4, we give normal forms for the Jacobi operators which can occur and in Theorem 4.2 we give the basic characterization result for algebraic-Osserman manifolds of signature $(2,2)$. Sections §5 and §6 outline the proof of Theorem 4.2. We conclude in §7 with some open problems. There is a large history on this subject. In addition to the papers we shall cite later, we refer to the following papers for additional related work: M.Dajczer and K.Nomizu [16], A.I.Malcev [30], Singer [36], and A.G.Walker [42]. It is a pleasant task to thank D.M.Alekseevsky, G.Hall, O.Kowalski, L.Vanhecke and other colleagues for helpful discussions on this subject.

1 The Osserman conjecture for Riemannian manifolds

Suppose $p = 0$ (or equivalently $q = 0$) so that M is a Riemannian manifold. Chi [13] proved Conjecture 0.1when $n \equiv 2 \bmod 4$, and when $n = 4$. We sketch his proof in the case $n \neq 4$ as follows. If n is odd, all the eigenvalues of $\mathcal{K}_X$ are equal; if $n = 4k + 2$, then either all the eigenvalues of $\mathcal{K}_X$ are equal or there is one eigenvalue λ_1 of multiplicity 1 and the remaining eigenvalue λ_2 has multiplicity $n - 2$. This argument uses results of Adams [1] from algebraic topology. If all the eigenvalues are equal, then M has constant sectional curvature. If $\lambda_1 \neq \lambda_2$, Chi shows that there exists an almost complex structure J on M so that $\mathcal{K}_X JX = \lambda_1 JX$. Chi then uses the Bianchi identities to show J is parallel and deduces that M is modeled on complex projective space or its negative curvature dual. The case $n = 4$ is more complicated and uses special properties of 4 dimensional geometry. We also refer to [14], [15] for other work by Chi on related questions.

Chi classified the curvature tensors which can arise if a manifold is Osserman at a single point and if $n \not\equiv 0$ mod 4. Gilkey [22] has constructed germs of Riemannian metrics on $\mathbf{R}^{4k}$ which are Osserman at 0 but which have curvature tensors which are not based on those of a rank 1 symmetric space; in particular, there can be more than 2 eigenvalues. These examples show that the algebraic classification of the curvature tensors which can arise from a manifold which is Osserman at a single point can be quite complicated if $n \equiv 0$ mod 4.

There are other conditions which are closely related to the Osserman condition which have been studied. For example, Berndt and Vanhecke [4] introduced the notion of $\mathcal{C}$ space as a Riemannian manifold for which the Jacobi operators have constant eigenvalues along every geodesic. Ivanov and Petrova [26] studied conformally flat 4 manifolds such that the eigenspaces of the Jacobi operator are parallel along geodesics. Ivanova [25] and Ivanov and Petrova [27] studied similar questions for the skew–symmetric curvature operator. Stanilov [37] studied also geometry of 4 dimensional Osserman manifolds.

DR-spaces are a class of noncompact harmonic nonsymmetric spaces constructed by E. Damek and R. Ricci [17]. Szabo [39] proved that DR-spaces do not provide counter-examples for the Osserman conjecture. Tricerri and Vanhecke [40] gave a rather short proof of this fact. Boeckx [8] proved that semisymmetric spaces which have volume–preserving geodesic symmetries or which have constant eigenvalues of the Jacobi operator $\mathcal{K}_{\gamma'}$ along geodesic γ are locally symmetric. Moreover he showed that semisymmetric globally Osserman spaces are locally isometric to a two point homogeneous space. Let $\lambda(\lambda^3 + \sigma_1(m,X)\lambda^2 + \sigma_2(m,X)\lambda + \sigma_3(m,X)) = 0$ be the characteristic equation of the Jacobi operator, $m \in M$. Stanilov and Videv [38] proved that (M^4, g) is already a pointwise Osserman space if only σ_1 and σ_3 are independent of X.

2 The Osserman conjecture for Lorentzian manifolds

The Lorentzian case was first studied by Garcia–Rio et al. [19, 20]. They showed that timelike Osserman manifolds has constant sectional curvature. They also showed that if $n \leq 4$, that spacelike Osserman manifolds has constant sectional curvature. The restriction that $n \leq 4$ was removed by later work of [5]; these authors also gave a different proof of the result of Garcia–Rio et. al. [20] concerning timelike Osserman manifolds. We refer to [5] for the proof of the

Theorem 2.2. *The following conditions are equivalent for a Lorentzian manifold.*

(1) M is spacelike Osserman at a point $p \in M$.

(2) M is timelike Osserman at a point $p \in M$.

(3) M has constant sectional curvature at a point $p \in M$.

Here is a brief sketch of the proof. Let M be a n dimensional Lorentzian manifold and let $E_1, \ldots, E_{m-1}, E_m$ a pseudo orthonormal base of type $(+, \ldots, +, -)$. If M is Osserman at a point $p \in M$, then $\mathrm{tr}\mathcal{K}_X^2$ is independent of X, where X is a timelike (or spacelike) unit vector. With a bit of work, one can use this observation to show that $\sum_{i,j}\{(R_{1imj} + R_{mi1j})^2 + (R_{1i1j} + R_{mimj})^2\} = 0$. This shows that we have the identities $R_{1imj} + R_{mi1j} = 0$ and $R_{1i1j} + R_{mimj} = 0$; it then follows that M has constant sectional curvature at p. If one uses a similar argument in the Riemannian case, one ends up with a similar identity which is not coercive; a positive definite sum is not obtained for signature (p, q) if $p > 1$ and $q > 1$ either as will be evident presently; this argument works only for Lorentzian signature! Note that there exists a Schur type lemma for pseudo–Riemannian manifolds; see O'Neill [31], p. 231, and exercise 3.21, p. 96. Thus if a connected Lorentzian manifold M is spacelike or timelike Osserman at every point, then M has *constant* sectional curvature c for some c. We refer to Garcia–Rio and Kupeli [20] for a discussion of the null Osserman condition.

3 Examples of Osserman manifolds of signature $(2, 2)$

In this section, we present examples of manifolds M of signature $(2, 2)$ which are Osserman; 0 is a triple eigenvalue of $\mathcal{K}_X$ for all non–null vectors X. Some of these examples are not locally symmetric. In some of the examples, the minimal polynomial varies from point to point so they are not algebraic Osserman. Such manifolds are Einstein, see [6, Proposition 2.1] for details.

Example 3.1. Firstly, the existence of a manifold M of signature (2,2) for which the Osserman conjecture fails was proved in [6]. This manifold is a locally symmetric manifold of rank 2 which has signature (2,2); 0 is a triple eigenvalue of $\mathcal{K}_X$ for any timelike or spacelike unit vector X. The construction used a result of Wu [44]. This manifold admits an interesting integrable para–quaternionic structure and a parallel dual (neutral) structure N, $N^2 = 0$ and $\nabla N = 0$; we refer to Rozenfield [34] for further details. This provides a counter example to the Osserman conjecture.

Example 3.2. The following metric on $\mathbf{R}^4$ of signature $(2, 2)$ was constructed by Rakić [33]. The existence of such manifold was proved in [6] and it is a counter-example to the Osserman conjecture. Let

$$\begin{aligned} 6g &= u_2^2 du_1 \otimes du_1 + u_1^2 du_2 \otimes du_2 - u_1 u_2 [du_1 \otimes du_2 + du_2 \otimes du_1] \\ &- 3[du_1 \otimes du_4 + du_4 \otimes du_1 + du_2 \otimes du_3 + du_3 \otimes du_2]. \end{aligned}$$

Example 3.3. Garcia–Rio et. al. [21] constructed the following family of metrics on $\mathbf{R}^4$ of signature $(2, 2)$. If $\partial f_1/\partial u_2 + \partial f_2/\partial u_1 = 0$, let

$$\begin{aligned} g_{(f_1, f_2)} &= u_3 f(u_1, u_2) du_1 \otimes du_1 + u_4 f_2(u_1, u_2) du_2 \otimes du_2 \\ &+ a[du_1 \otimes du_2 + du_2 \otimes du_1] \\ &+ b[du_1 \otimes du_3 + du_3 \otimes du_1 + du_2 \otimes du_4 + du_4 \otimes du_2]; \end{aligned}$$

0 is a triple eigenvalue of $\mathcal{K}_X$ for any non null vector X. The minimal polynomial is either λ^2 or λ^3; there are examples when the minimal polynomials change degree

from point to point. The functions f_1 and f_2 can be chosen such that the metric is not locally symmetric. There are metrics with similar properties of signature (p, q) for any $p \geq 2$ and $q \geq 2$, see Garcia–Rio et. al. [21] for details.

Example 3.4. Bonome et. al. [10] constructed Osserman manifolds with indefinite Kähler metrics of nonnegative or nonpositive holomorphic sectional curvature which are not locally symmetric.

Example 3.5. Ruse et. al. [35, p. 211] constructed a metric which is simple harmonic and which is neither symmetric nor recurrent, but it is Osserman:

$$\begin{aligned} g &= u_2u_3du_1 \otimes du_1 - u_1u_4du_2 \otimes du_2 \\ &+ [du_1 \otimes du_3 + du_3 \otimes du_1 + du_2 \otimes du_4 + du_4 \otimes du_2]. \end{aligned}$$

4 Characterization of Osserman manifolds of signature (2,2)

For the remainder of this paper, we restrict to the special case that M has signature $(2, 2)$ and assume that M is spacelike algebraic-Osserman or timelike algebraic-Osserman. We distinguish 4 different cases depending on the algebraic form of the endomorphism $\mathcal{K}_{\mathcal{X}}$ of $\mathbf{R}^3$.

Definition 4.1.

(1) We say M is *type Ia* if $\mathcal{K}_X$ is diagonalizable.

(2) We say M is *type Ib* if the characteristic polynomial of the Jacobi operator has a complex root.

(3) We say that M is *type II* if $\mathcal{K}_X$ is not diagonalizable and if the minimal polynomial has a double root α.

(4) We say that M is *type III* if $\mathcal{K}_X$ is not diagonalizable and if the minimal polynomial has a triple root α.

There exists a suitable basis so the matrix $\mathcal{K}_X$ has the following form:

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \text{(I-a)}, \begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix} \text{(I-b) } \beta \neq 0, \begin{pmatrix} \alpha & 0 & \frac{\sqrt{2}}{2} \\ 0 & \alpha & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \alpha \end{pmatrix} (III),$$

$$\begin{pmatrix} \alpha - \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \alpha + \frac{1}{2} & 0 \\ 0 & 0 & \beta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -\alpha + \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\alpha - \frac{1}{2} & 0 \\ 0 & 0 & \beta \end{pmatrix} (II).$$

Theorem 4.2. *Let M be a 4–dimensional pseudo–Riemannian manifold of signature $(2, 2)$. Then the following conditions are equivalent:*

(1) M is timelike algebraic-Osserman.

(2) M is spacelike algebraic-Osserman.

(3) The universal covering space $\tilde{M}$ of M is one of the following manifolds

(a) $\tilde{M}$ is a manifold of constant sectional curvature;

(b) $\tilde{M}$ is a Kähler manifold of constant holomorphic sectional curvature;

(c) $\tilde{M}$ is a para-complex manifold of constant para–holomorphic sectional curvature;

(d) Jacobi operator of $\tilde{M}$ is nondiagonalizable, its characteristic polynomial has triple zero α and its curvature is given by Lemma 6.3.

Proof. We refer to [6] for complete details regarding the proof of this theorem and will content ourselves in this note with sketching the proof. Since M is Osserman, the quantities $\mathrm{tr}\mathcal{K}_{\mathcal{X}}$, $\mathrm{tr}\mathcal{K}^{\in}_{\mathcal{X}}$, and $\mathrm{tr}\mathcal{K}^{\ni}$ are constant; it now follows that M is Einstein. Furthermore, we can determine the components of the curvature tensor for the four types relative to some appropriate basis. It then follows that the manifold is curvature homogeneous; we refer to Kowalski et. al. [28] for further details. In the next section, we shall complete the proof by studying the 4 cases individually.

5 Proof of Theorem 4.2 in Case Ia and Ib

We begin by recalling some classification results for spaces of constant sectional curvature, for Kähler manifolds of constant holomorphic sectional curvature, and for para–Kähler manifolds of constant paraholomorphic sectional curvature.

5.1 Manifolds with constant sectional curvature. Let M be a pseudo Riemannian manifold of signature (p,q) which has constant sectional curvature c. The curvature tensor of M is given by $R(u,v)w = c\{g(v,w)u - g(u,w)v\}$. If M is complete, connected and simply connected, $M = M(c,p,q)$ is determined by (c,p,q). These manifolds have been classified by Wolf [43]; $M(\pm 1, 2, 2)$ is the pseudo–sphere in $\mathbf{R}^5$ with the appropriate non–Euclidean metric.

5.2 Kähler manifolds of constant holomorphic sectional curvature. The projective space $\mathbf{CP}^2$ is a Riemannian manifold with constant holomorphic sectional curvature $c > 0$; there is a negative curvature dual. In a similar fashion, the indefinite projective space $\mathbf{CP}^n_s(c)$ of signature $(2p, 2n-2p)$ can be constructed, see Baros et. al. [3] for details. The Kähler space form $\mathbf{CP}^n_s(c)$ has constant holomorphic sectional curvature $c \neq 0$ with curvature tensor:

$$\begin{aligned} R(u,v)w &= \frac{c}{4}\{g(v,w)u - g(u,w)v \\ &+ g(Jv,w)Ju - g(Ju,w)Jv - 2g(Ju,v)Jw\}. \end{aligned}$$

Furthermore, every Kähler manifold M of signature $(2s, 2n-2s)$ with constant holomorphic sectional curvature $c \neq 0$ is holomorphically isometric to $\mathbf{CP}^n_s(c)$.

5.3 Para-Kähler manifolds of constant paraholomorphic sectional curvature. The tangent bundle TS^n of the standard sphere can be equipped with a pseudo Riemannian metric g of signature (n,n) and a para–complex structure such that $P^n(B) = (TS^n, g, F)$ is of constant paraholomorphic sectional curvature c, $c \neq 0$. For $n > 1$, $P^n(B)$ is complete, connected and simple connected, see Gadea et. al. [18] for details. Furthermore, every para–Käehler manifold M^{2n} with constant paraholomorphic sectional curvature c is F holomorphically isometric to $P^n(B)$. The curvature tensor of $P^n(B)$ is given by

$$\begin{aligned} R(x,y)z &= \frac{c}{4}\{g(v,w)u - g(u,w)v \\ &- g(Jv,w)Ju + g(Ju,w)Jv + 2g(Ju,v)Jw\}. \end{aligned}$$

We note that indefinite Kähler manifolds with vanishing holomorphic sectional curvature are flat; similarly para–Kähler manifolds with vanishing paraholomorphic sectional curvature are flat. But the complete classification of flat–pseudo Riemannian manifolds is not known, see Wolf [43].

5.4. Type Ia. We prove Theorem 4.2 when the Jacobi operator is diagonalizable by studying the covariant derivatives of the curvature tensor. Let M be algebraic Osserman and have signature (2,2), where $\mathcal{K}_X$ is diagonalizable. We use the second Bianchi identity to show that (M, g) has to be a locally rank one symmetric space or flat. It then follows that M must be a space of constant sectional curvature or be a Kähler manifold of constant holomorphic sectional curvature or be a para–Kähler manifolds of constant paraholomorphic sectional curvature. The argument is similar to given by Chi [13] in the Riemannian case in dimension 4. The crucial point in the proof is if the Jacobi operator $\mathcal{K}_{\mathcal{X}}$ is of type Ia, then the eigenvalues α, β and γ can not be all distinct. If $\alpha = \beta = \gamma$, M has constant curvature. If $\beta = \gamma$, it is possible to construct an integrable complex or paracomplex structure and complete the classification. This yields (a)-(c) in Theorem 4.2. (3).

5.5. Type Ib. We show that the Jacobi operator of an algebraic–Osserman manifold can not be of type Ib, see [6] for the proof of the following theorem:

5.6. Theorem. *Let M be a timelike or spacelike algebraic–Osserman manifold of signature* $(2, 2)$. Then $\mathcal{K}_X$ is not of type Ib.

6 Proof of Theorem 4.2 in case II and III

In this section, we assume M is a timelike or spacelike algebraic Osserman manifold of signature $(2, 2)$ such that $\mathcal{K}_X$ is not diagonalizable and has all real eigenvalues. This case is of the special interest; there exists a non-flat algebraic Osserman manifold which is not locally rank one symmetric.

If the minimal polynomial of $\mathcal{K}_X$ has a double root, then $\beta = 4\alpha$ or $\beta = \alpha$. Furthermore if $\beta = 4\alpha$, then $\beta = 4\alpha = 0$. Suppose $\alpha \neq 0$. Then M admits a parallel (integrable) null 2-plane distribution. We refer to Ruse et. al. [35] for the proof of the

6.1.Theorem. *Let M^4 admit a parallel 2–dimensional null plane. There exist coordinates on M such that $ds^2 = f du_1^2 + 2s du_1 du_2 + g du_2^2 + du_1 du_3 + du_2 du_4$ for suitably chosen functions f, s, and g.*

6.2 $\mathcal{K}_X$ has a triple zero. The only remained possibility is that $\mathcal{K}_X$ has a triple zero and the minimal polynomial is second or third order. In this case, it is possible to determine the components of the curvature tensor and to prove the existence of some null distributions. We work with a null basis $\{F_i\}$ and curvature tensors Q, P, and S, where

(1) $F_1 := (E_1 - E_4)/\sqrt{2}$, $F_2 := (E_2 + E_3)/\sqrt{2}$,
$F_3 := (E_2 - E_3)/\sqrt{2}$, $F_4 = (E_1 + E_4)/\sqrt{2}$,

and $\{E_i\}$ is a pseudo-orthonormal basis, so the matrix $\mathcal{K}_X$ has the form (II) or (III).

(2) $Q(U, V)W = c(g(V, W)U - g(U, W)V)$; Q is the curvature tensor of the constant sectional curvature metric

(3) $P := (F_3 \wedge F_4) \vee (F_3 \wedge F_4)$.

(4) $S := (F_1 \wedge F_3) \vee ((F_1 \wedge F_4) - (F_2 \wedge F_3))$.

We refer to [6] for the proof of the following Lemma; all curvature tensors are computed with respect to the basis F_i.

6.3. Lemma. *Let M be Jordan Osserman of signature* $(2, 2)$.

(1) If the minimal polynomial of $\mathcal{K}_X$ *has a double root* α*, then* $R = -2P + \alpha Q$, $R_{1441} = R_{2332} = R_{1243} = R_{1342} = \alpha$ *and* $R_{4334} = 2$.

(2) If the minimal polynomial of $\mathcal{K}_X$ *has a triple root* α*, then* $R = \sqrt{2}S - \alpha Q$, $R_{1441} = R_{2332} = R_{1243} = R_{1342} = -\alpha$*, and* $R_{1332} = R_{1314} = \sqrt{2}$.

This lemma shows these curvature tensors are similar to the curvature tensor of a constant sectional curvature metric when $\alpha \neq 0$. For manifolds in this class the holonomy algebra is full, i.e., $h = so(2,2)$; these manifolds admit an autoparallel integrable null plane field, we refer to [6], Proposition 7.8 for details. Note that in §3 we showed that there are manifolds in this class when $\alpha = 0$.

7 Open problems

The results presented in the previous sections show that the Osserman conjecture is closely related to other conjectures and that there are much more examples of Osserman manifolds in pseudo-Riemannian geometry. There are still many open problems in this area; we present a few as follows:

7.1. Question.Do there exist Osserman manifolds with nondiagonalizable Jacobi operators which are not Ricci flat?

7.2. Question. Is it necessary for timelike (spacelike) Osserman manifold to be locally homogeneous? Note that if one supposes only that the characteristic polynomial of the Jacobi operator is constant, then it does not follow that the manifolds are locally homogeneous. Further details concerning the relationship between the Osserman conjecture and the local homogeneity in the Riemannian setting can be found in [41].

The following question is a very natural one; an affirmative answer was given in [11] and [21] under some additional assumptions.

7.3. Question. Is a timelike (spacelike) Osserman manifold with a diagonalizable Jacobi operator necessarily either a locally rank 1 symmetric space or a flat space?

Motivated by results of Gilkey (see [22]) in the Riemannian setting, one can formulate the following problem.

7.4. Question. Do there exist manifolds M^n such that M^n is pointwise Jordan-Osserman but so that M^n is not Jordan Osserman; i.e., is it possible to find a manifold where the Jordan form of $\mathcal{K}_X$ changes from point to point?

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References

[1] J.Adams, Vector fields on spheres Annals of Math. vol 75, 1962, 603–632.

[2] W.Ambrose and I. M.Singer, *A theorem on holonomy*, Trans. AMS, vol 75, 1953, 428-443.

[3] M.Baros and A.Romero, *Indefinite Kähler manifolds*, Math. Ann., vol 261, 1982, 55–62.

[4] J.Berndt and L.Vanhecke, *Two naturally generalizations of locally symmetric spaces*, Diff. Geom. and Appl., vol 2, 1992, 57–80.

[5] N.Blažić, NḂokan and PĠilkey, *A Note on Osserman Lorentzian manifolds*, Bull. London Math. Soc., vol 29, 1997, 227–230.

[6] N.Blažić, NḂokan and ZṘakić, *Characterization of 4-dimensional Osserman pseudo–Riemannian manifolds*, preprints, 1995, 1997.

[7] N.Blažić, NḂokan and ZṘakić, *On the geometry of Osserman manifolds with nonlinear minimal polynomial for Jacobi operator*, preprint, 1997.

[8] E.Boeckx, *Einstein–like semi–symmetric spaces*, Arch. Math. Brno, vol 29, 235–240, 1993.

[9] E.Boeckx, O.Kowalski and L.Vanhecke, *Riemannian manifolds of conullity two*, World Scientific Publ., Singapore, 1996.

[10] A.Bonome, E.Garcia–Rio, L.Hervella, R.Vásquez–Lorenzo, *Nonsymmetric Osserman indefinite Kähler manifolds*, preprint, 1997.

[11] A.Bonome, R.Castro, E.Garcia–Rio, L.Hervella and R.Vasquez–Lorenzo, *On Osserman semi-Riemannian manifolds*, preprint, 1997.

[12] A.Borowiec, M.Ferraris, M.Francaviglia, I.Volovich, *Almost complex and almost product Einstein manifolds from a variational principle*, preprint, (1996).

[13] Q.S.Chi, *A curvature characterization of certain locally rank–one symmetric spaces*, J. Diff. Geom., vol 28, 1988, 187–202.

[14] Q.S.Chi, *Curvature characterization and classification of rank–one symmetric spaces*, Pac. J. Math., vol 150, no. 1, 1991, 31–42.

[15] Q.S.Chi, *Quaternionic Kähler manifolds and a curvature characterization of two–point homogeneous spaces*, Illinois J. Math., vol 35, no. 3, 1991, 408–419.

[16] M.Dajczer and K.Nomizu, *On sectional curvature of indefinite metrics II*, Math. Ann., vol 247, 1980, 279–282.

[17] E.Damek and F.Ricci, *A class of nonsymmetric harmonic Riemannian spaces*, Bull. Amer. Math. Soc. 27, 1992, 139–142.

[18] P.M.Gadea and J.M. Masqué, *Classification of non–flat para–Kählerian space forms*, Houston J. Math. vol 21, (1), 1995, 89–94.

[19] E.Garcia–Rio and D.N. Kupeli, 4 *Dimensional Osserman Lorentzian manifolds, New developments in differential geometry.* Proceedings of the colloquium on differential geometry, Debrecen, Hungary, July 26-30, 1996, Kluwer Academic Publishers, vol 350, 201–211.

[20] E.Garcia–Rio, D.N. Kupeli and M.E.Vázquez–Abal, *On a problem of Osserman in Lorentzian geometry*, Differential Geometry and its Applications, vol 7, 1997, 85–100.

[21] E.Garcia–Rio, M.E.Vázquez–Abal and R.Vázquez–Lorenzo, *Nonsymmetric Osserman pseudo Riemannian manifolds*, preprint, 1996.

[22] P.Gilkey, *Manifolds whose curvature operator has constant eigenvalues at the basepoint*, J. Geom. Anal., vol 4, 1994, 155–158.

[23] P.Gilkey, A.Swann and L.Vanhecke, *Isoparametric geodesic spheres and a conjecture of Osserman concerning the Jacobi operator*, Quart. J. Math. Oxford., vol 46, 1995, 299–320.

[24] V.Guillemin and S.Strenberg, *An ultra–hyperbolic analogue of the Robertson–Kerr theorem*, Lett. Math. Phys., vol 12, 1986, 1–6.

[25] R.Ivanova, *Pointwise constancy of the skew–symmetric curvature operator's characteristical coefficients*, preprint.

[26] S.Ivanov and I.Petrova, *Conformally flat Einstein like* 4 *manifolds and conformally flat Riemannian* 4 *manifolds all of whose Jacobi operators have parallel eigenspaces along every geodesic*, preprint.

[27] S.Ivanov and I.Petrova, *Riemannian manifold in which the skew–symmetric curvature operator has pointwise constant eigenvalues*, to appear Geometria Dedicata.

[28] O.Kowalski, F.Tricerri and L.Vanhecke, Curvature homogeneous Riemannian manifolds, vol 71, J. Math. Pures et Appl., 1992, 471–501.

[29] P.R.Law *Neutral Einstein metrics in four dimensions*, vol 32 (11), J. Math. Phys., 1991, 3039–3042.

[30] A.I.Malcev, *Foundations of linear algebra*, Rus. edit. Nauka, Moskva, 1970, Russian.

[31] B.O'Neill, *Semi–Riemannian geometry with applications to Relativity*, Accademic Press, New York, 1983.

[32] R.Osserman, *Curvature in the eighties*, Amer. Math. Monthly, vol 97, 1990, 731–756.

[33] Z.Rakić, *An example of rank two symmetric Osserman space*, Bull. Australian M.S., to appear.

[34] B. A. Rozenfeld, *Noneuclidean geometries*, GITTL, Moskva, 1955, Russian.

[35] H.S.Ruse, A.G.Walker and T.J.Willmore, *Harmonic spaces*, Cremonese, Rome, 1961.

[36] I.M.Singer, *Infinitesimally homogeneous spaces*, Comm. Pure Appl. Math., vol 13, 1960, 685–697.

[37] G.Stanilov, *On the geometry of the Jacobi operators on* 4*–dimensional Riemannian manifolds*, Tensor, vol 51, 1992, 9–15.

[38] G.Stanilov and V.Videv, *On Osserman conjecture by characteristical coefficients* Algebras Groups Geom, vol12, 1995, 157–163.

[39] Z.I.Szabo, *Spectral theory for operator families on Riemannian manifolds*, Proc. Sympos. Pure Math., vol 54, Part 3, 1993, 615–665.

[40] F.Tricerri and L. Vanhecke, *Geometry of a class of nonsymmetric harmonic manifolds*, Differential geometry and its applications, 1992, 415–426.

[41] L.Vanhecke, *Scalar curvature invariants and local homogeneity*, Rend. Circ. Mat. Palermo, to appear.

[42] A.G.Walker, *Canonical form for a Riemannian space with a parallel field of null planes*, Quart. J. Math., (Oxford), vol (2), 1, 1950, 69–79.

[43] J.A.Wolf, *Spaces of constant curvature*, Univ. California, Berkeley, 1972.

[44] H.Wu, *Holonomy groups of indefinite metrics*, Pac. J. Math., vol 20, (2), 1967, 351–392.

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