Generalized Invexities and Global Minimum Properties

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Abstract

New definitions for all types of invexity and generalized invexity of the arbitrary real functions are given. Direct implications between the invexity and generalized invexity types are established. Moreover, it is shown that for a (strict) invex, (strict) pseudoinvex or (strict) quasiinvex function every (strict) local minimum point is one of (strict) global.

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0. Introduction

The concept of convexity plays a very important role in the optimization theory. Various convex models and methods of the convex programming are used also in the study of the Riemannian manifolds [10]. The convex, pseudoconvex and quasiconvex functions [1] knew various generalizations. We quote in the following some of these.

Let A be a nonempty open set of R^n and $f : A \to R$ be a differentiable function on A. Hanson [5] generalized the differentiable convex, pseudoconvex and quasiconvex functions respectively, by the following definitions.

Definition I. The differentiable function f is *invex* on A if there is a vector function $\eta : A \times A \to R^n$ such that

$$\forall x, u \in A : f(x) - f(u) \geq \eta'(x, u)\nabla f(u).$$

($'$ is the sign of transposition and ∇f is the gradient of f).

Definition II. The differentiable function f is *pseudoinvex* on A if there is a vector function $\eta : A \times A \to R^n$ such that

$$\forall x, u \in A : \eta'(x, u)\nabla f(u) \geq 0 \Rightarrow f(x) \geq f(u).$$

Definition III. The differentiable function f is *quasiinvex* on A if there is a vector function $\eta : A \times A \to R^n$ such that

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$$\forall x, u \in A : f(x) \le f(u) \Rightarrow \eta'(x,u)\nabla f(u) \ge 0.$$

Later the types of invexity, pseudoinvexity and quasiinvexity have been introduced, specially in the differentiable case, by Jeyakumar [6] and Preda [8]. Craven [3] introduced the invexity notion for the Lipschitz functions. In the nonsmooth case Giorgi and Mititelu [4] and Mititelu and Stancu-Minasian [7] defined the nonsmooth invex, pseudoinvex and quasiinvex functions using the upper Dini and Clarke directional derivatives, respectively. So, let f be the same function f and vectors $u \in A$ and $v \in R^n$. The symbol

$$f'_+(u;v) = \lim_{\lambda \downarrow 0} \sup \frac{f(u+\lambda v) - f(u)}{\lambda} \quad (u + \lambda v \in A)$$

is called *the upper Dini* directional derivative of f at the point u in the direction v [4]. Let ρ be a real number. Then the definitions of all types of the invexity, pseudoinvexity and quasiinvexity in respect to f, given by Giorgi and Mititelu, are the following:
Definition 1′ (Invexities). The function f is said to be ρ-*invex* on **A** (shortly ρI_+), if there exist vector functions $\eta, \theta : A \times A \to R^n$ such that

$$(\rho I_+) \quad \forall x, u \in A : f(x) - f(u) \ge f'_+(u; \eta(x,u)) + \rho \|\theta(x,u)\|^2.$$

If

(1a) $\rho > 0$ the function f is called *strongly invex*;
(1b) $\rho = 0$ the function f is called *invex*;
(1c) $\rho < 0$ the function f is called *weakly invex*;
(1d) $\forall x \in A,\ x \ne u : f(x) - f(u) > f'_+(u;\eta(x,u))$ the function f is called *strictly invex*.

Definition 2′ (Pseudoinvexities). The function f is said to be ρ-*pseudoinvex* on $A(\rho PI_+)$ if there exist vector functions $\eta, \theta : A \times A \to R^n$ such that

$$(\rho PI_+) \quad \forall x, u \in A : f'_+(u; \eta(x,u)) + \rho \|\theta(x,u)\|^2 \ge 0 \Rightarrow f(x) \ge f(u).$$

If:

(2a) $\rho > 0$ the function f is called *strongly pseudoinvex*;
(2b) $\rho = 0$ the function f is called *pseudoinvex*;
(2c) $\rho < 0$ the function f is called *weakly pseudoinvex*;
(2d) $\forall x, u \in A,\ x \ne u : f'_+(u;\eta(x,u)) \ge 0 \Rightarrow f(x) > f(u)$ the function f is called *strictly pseudoinvex*.

Definition 3′ (Quasiinvexities). The function f is said to be ρ-*quasiinvex* on A if vector functions $\eta, \theta : A \times A \to R^n$ exist such that

$$(\rho QI_+) \quad \forall x, u \in A : f(x) \le f(u) \Rightarrow f'_+(u;\eta(x,u)) + \rho \|\theta(x,u)\|^2 \le 0.$$

If:

(3a) $\rho > 0$ the function f is called *strongly quasiinvex*;
(3b) $\rho = 0$ the function f is called *quasiinvex*;
(3c) $\rho < 0$ the function f is called *weakly quasiinvex*;
(3d) $\forall x, u \in A,\ x \ne u : \forall \lambda \in (0,1) : f(x) \le f(u) \Rightarrow f(u + \lambda \eta(x,u)) < f(u)$ the function f is called *strictly quasiinvex*.

(3e) $\forall x, u \in A,\ x \neq u,\ \forall \lambda \in (0,1) : f(x) < f(u) \Rightarrow f(u + \lambda\eta(x,u)) < f(u)$ the function f is called *semistrictly quasiinvex.*

Remark 1. Some of Definitions $1'$-$3'$ have sense when function $f'_+(u_j)$ is finite.

The purpose of this paper is to give definitions for the nonsmooth invex, pseudoinvex and quasiinvex functions without use directional derivatives. The new definitions are defined at a point. The direct implications between all types of invexity, pseudoinvexity and quasiinvexity are given. Moreover, some properties of global minimum of the nonsmooth invex, pseudoinvex and quasiinvex functions are established.

1 Invexity and generalized invex types

We propose for all types of invexity and incavity definitions as follows.

Definitions 1. A(Invexities at a point). The functions f is said to be *ρ-invex* at a point $u \in A(\rho I)$ if there exist vector functions $\eta, \theta : A \times A \to R^n$ such that

$$(\rho I) \quad \forall x \in A,\ \forall \lambda \in [0,1] : f(u + \lambda\eta(x,u)) \leq f(u) + \lambda[f(x) - f(u)] - \rho\lambda\|\theta(x,u)\|^2.$$

If

(1a) $\rho > 0$ the function f is called *strongly invex* at $u(SgI)$;

(1b) $\rho = 0$ the function f is called *invex* at $u(I)$;

(1c) $\rho < 0$ the function f is called *weakly invex* at $u(WI)$;

(1d) $\forall x \in A,\ x \neq u, \forall \lambda \in (0,1) : f(u + \lambda\eta(x,u)) < f(u) + \lambda[f(x) - f(u)]$, the function f is called *strictly invex* at $u(SI)$.

B (Incavities at a point). The function f is said to be *ρ- incave* at u if the function $-f$ is ρ-invex at u. If $\rho > 0,\ \rho = 0$ or $\rho < 0$ f is called *strongly incave, incave* or *weakly incave* at u. The function f is said to be *strictly incave* at u if the function $-f$ is strictly invex at u.

In this section we suppose that $f'_+(u_i)$ is finite (see Remark 1).

Theorem 1.1. *Definitions $1'$ with u fixed and 1A are equivalent.*

Proof. If u is fixed in Definition $1'$ then from relation (ρI_+) we obtain:

$$(\rho I)_+ \qquad x \in A : f(x) - f(u) \geq f'_+(u; \eta(x,u)) + \rho\|\theta(x,u)\|^2.$$

Function $f'_+(u;\cdot)$ is positively homogeneous and then, by multiplication of $(\rho I)_+$ with an arbitrary scalar $t \geq 0$, we obtain

$$x \in A : tf(x) - tf(u) \geq f'_+(u; t\eta(x,u)) + t\rho\|\theta(x,u)\|^2 \quad \text{for all} \quad t \geq 0,$$

$$x \in A : tf(x) - tf(u) \geq \limsup_{\mu \downarrow 0} \frac{f(u + \mu t\eta(x,u)) - f(u)}{\mu} + t\rho\|\theta(x,u)\|^2$$

for all $t \geq 0$ and $\bar{\mu} > 0$,

$$x \in A : tf(x) - tf(u) \geq \frac{f(u + \bar{\mu} t\eta(x,u)) - f(u)}{\bar{\mu}} + t\rho\|\theta(x,u)\|^2 \quad \text{and even}$$

$$(1) \qquad x \in A,\ t \geq 0 : \bar{\mu} tf(x) - \bar{\mu} tf(u) \geq f(u + \bar{\mu} t\eta(x,u)) - f(u) + \bar{\mu} t\rho\|\theta(x,u)\|^2.$$

Noting $\lambda = \bar{\mu} t$ in (1) it results (ρI) from Definition $1A$.

Conversely, for an arbitrary $\lambda > 0$ in the relation (ρI) by Definition 1A we obtain

$$(2) \qquad \frac{f(u+\mu\eta(x,u)) - f(u)}{\lambda} \leq f(x) - f(u) - \rho\|\theta(x,u)\|^2$$

and taking upper limit by $\lambda \downarrow 0$ in (2) we obatin

$$f'_+(u;\eta(x,u)) \leq f(x) - f(u) - \rho\|\theta(x,u)\|^2,$$

which is (ρI_+) by Definition $1'$.

Theorem 1.2. *Between the invexity types the following direct implications hold:*

(a) Strongly invex (SgI) and $(x \neq u \Rightarrow \theta(x,u) \neq 0) \Rightarrow$ Strictly invex (SI);

(b) Strictly invex (SI) $\Rightarrow$ Invex (I) $\Rightarrow$ Weakly invex (WI).

Proof. See above Theorem 1.1 and Théorème 2.1 from [4].

For the pseudoinvexity and pseudoincavity types at a point we propose definitions as follows.

Definition 2. A (Pseudoinvexities at a point). The function f is said to be *ρ-pseudoinvex* at $u \in A$ (ρPI) if there exist vector functions $\eta, \theta : A \times A \to R^n$ such that

$$(\rho PI) \quad \forall x \in A;\ \forall \lambda \in [0,1] : f(u+\lambda\eta(x,u)) + \rho\lambda\|\theta(x,u)\|^2 \geq f(u) \Rightarrow f(x) \geq f(u).$$

If

(2a) $\rho > 0$ the function f is called *strongly pseudoinvex* at u (Sg PI);

(2b) $\rho = 0$ the function f is called *pseudoinvex* au u (PI);

(2c) $\rho < 0$ the function f is called *weakly pseudoinvex* au u (WPI);

(2d) $\forall x \in A,\ x \neq u, \forall \lambda \in (0,1) : f(u+\lambda\eta(x,u)) \geq f(u) \Rightarrow f(x) > f(u)$, the function f is called *strictly pseudoinvex* at u (SPI).

B (Pseudoincavities at a point). The function f is said to be ρ- *pseudoincave* at u if the function $-f$ is ρ-pseudoinvex at u. If $\rho > 0,\ \rho = 0$ or $\rho < 0$ then f is called *strongly pseudoincave, pseudoincave* or *weakly pseudoincave* at u. The function f is said to be *strictly pseudoincave* at u if the function $-f$ is strictly pseudoinvex at u.

In what follows is necessary the next lemma.

Lemma. *For all $x \in A$ and $\rho \in R$ the following equivalence holds*

$$(3) \qquad f'_+(u;\eta(x,u)) + \rho\|\theta(x,u)\|^2 \geq 0 \Leftrightarrow$$

$$(4) \qquad f(u+\lambda\eta(x,u)) + \rho\lambda\|\theta(x,u)\|^2 \geq f(u), \quad \forall \lambda \in (0,1).$$

Proof. Function $f'_+(u;\cdot)$ is positively homogeneous and then for all $t \geq 0$ from (3) it results

$$f'_+(u;t\eta(x,u)) + t\rho\|\theta(x,u)\|^2 \geq f(u)$$

or equivalently

$$\limsup_{\mu\downarrow 0} \frac{f(u+\mu t\eta(x,u)) - f(u)}{\bar{\mu}} + t\rho\|\theta(x,u)\|^2 \geq 0, \quad \forall t \geq 0.$$

Then there is a $\bar{\mu} > 0$ such that

$$\frac{f(u+\bar{\mu}t\eta(x,u))-f(u)}{\bar{\mu}}+t\rho\|\theta(x,u)\|^2\geq 0,\quad \forall t\geq 0$$

or equivalently,

$$f(u+\bar{\mu}t\eta(x,u))-f(u)+\bar{\mu}t\rho\|\theta(x,u)\|^2\geq 0,\quad \forall t\geq 0$$

We denote $\lambda=\bar{\mu}t$ in this inequality and one obtaine (4).
Conversely, by (4) for $\lambda>0$, we obtain

$$\frac{f(u+\lambda\eta(x,u))-f(u)}{\lambda}+\rho\|\theta(x,u)\|^2\geq 0$$

and taking upper limit by $\lambda\downarrow 0$ in this inequality one obtains inequality (3).
Theorem 1.3. *Definitions 2′ with u fixed and 2A are equivalent.*
Proof. One uses the above Lemma.
Theorem 1.4. *Between the types of pseudoinvexity the following direct implications hold:*

(a) Strongly pseudoinvex (Sg PI) and injective ⇒ Strictly pseudoinvex (SPI);

(b) Strictly pseudoinvex (SPI) ⇒ Pseudoinvex (P) ⇒ Weakly pseudoinvex (WPI).

Proof. See Théorème 2.3 from [4] and the above Lemma.
Theorem 1.5.. *If the function f is ρ-invex at $u\in A$ then f is ρ-pseudoinvex at u. Moreover, if f is strictly invex at u then f is strictly pseudoinvex at u.*
Proof. If f is ρ-invex at u then we have

$$(5)\qquad x\in A,\ \lambda\in[0,1]: f(u+\lambda\eta(x,u))+\rho\lambda\|\theta(x,u)\|^2-f(u)\leq\lambda[f(x)-f(u)]$$

and if $f(u+\lambda\eta(x,u))+\rho\lambda\|\theta)x,u)\|^2\geq f(u)$ then from (5) it results $f(x)\geq f(u)$. For the quasiinvexity and quasiincavity types at a point we propose definitions as follows.
Definition 3. A (Quasiinvexities at a point). The function f is said to be *ρ-quasiinvex* at $u\in A$ (ρQI) if there are vector functions $\eta,\theta: A\times A\to R^n$ such that

$$(\rho QI)\quad \forall x\in A,\ \forall\lambda\in[0,1]: f(x)\leq f(u)\Rightarrow f(u+\lambda\eta(x,u))+\rho\lambda\|\theta(x,u)\|^2\leq f(u).$$

If

(3a) $\rho>0$ the function f is called *strongly quasiinvex* at u (Sg QI);

(3b) $\rho=0$ the function f is called *quasiinvex* at u (QI);

(3c) $\rho<0$ the function f is called *weakly quasiinvex* at u (WQI);

(3d) $\forall x\in A,\ x\neq u,\ \forall\lambda\in(0,1): f(x)\leq f(u)\Rightarrow f(u+\lambda\eta(x,u))<f(u)$ the function f is called *strictly quasiinvex* at u (SQI);

(3e) $\forall x\in A,\ x\neq u,\ \forall\lambda\in(0,1): f(x)<f(u)\Rightarrow f(u+\lambda\eta(x,u))<f(u)$ the function f is called *semistrictly quasiinvex* at u (SSQI).

B (Quasiincavities at a point). The function f is said to be *ρ- quasiincave* at u if the function $-f$ is ρ-quasiinvex at u. If $\rho>0,\ \rho=0$ or $\rho<0$ then f is called *strongly quasiincave, quasiincave* or *weakly quasiincave* at u. The function f is said to be *strictly quasiincave* at u if the function $-f$ is strictly quasiinvex at u and f is said to be *semistrictly quasiincave* at u if $-f$ is semistrictly quasiinvex at u.
Theorem 1.6. *Definitions 3′ with u fixed and 3A are equivalent.*
Proof. One uses the above Lemma.

Theorem 1.7. *Between the types of quasiinvexity the next direct implications hold at u:*

(a) Strongly quasiinvex (Sg QI) and $(x \neq u \Rightarrow \theta(x,u) \neq 0) \Rightarrow$ Strictly quasiinvex (SQI);

(b) Strictly quasiinvex (SQI) $\Rightarrow$ Semistrictly quasiinvex (SSQI);

(c) Semistrictly quasiinvex (SSQI) and lower semicontinuous on A and $\eta(\cdot;u)$ bounded on $A \Rightarrow$ Quasiinvex (QI);

(d) Quasiinvex (QI) $\Rightarrow$ Weakly quasiinvex (WQI).

Proof. For (a) and (d) one use the Lemma and Théorème 2.7 by [4].

(b) Is obvious.

(c) We must show that

$$(QI) \quad \forall x \in A,\ \forall \lambda \in [0,1] : f(x) \leq f(u) \Rightarrow f(u + \lambda\eta(x,u)) \leq f(u).$$

Because f is (SSQI) at u and all $\lambda > 0$, $f(u + \lambda\eta(x,u)) < f(u)$ implies $f(u + \lambda\eta(x,u)) \leq f(u)$, it results that f is (QI) at u.
We now have to prove that

$$\forall x \in A,\ \forall \lambda \in [0,1] : f(x) = f(u) \Rightarrow f(u + \lambda\eta(x,u)) \leq f(u). \tag{6}$$

Assume by reductio ad absurdum that (6) is not true. Then

$$\exists t \in A,\ \exists \bar{\lambda} \in (0,1] : f(t) = f(u) \quad \text{and} \quad f(u + \bar{\lambda}\eta(t,u)) > f(u).$$

We denote $\bar{x} = u + \bar{\lambda}\eta(t,u)$ and also

$$f(\bar{x}) - f(u) = a (a > 0). \tag{7}$$

Because f is lower semicontinuous at x it results that for any $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that for any $x \in A$ for which $\|x - \bar{x}\| < \delta_\varepsilon$ one has $f(x) > f(\bar{x}) - \varepsilon$. In particular for $x = u$ one gets that $\|u - \bar{x}\| < \delta_\varepsilon$ implies

$$f(u) > f(\bar{x}) - \varepsilon. \tag{8}$$

Choosing $\varepsilon = a$ from (7) and (8) one obtain $a > a$, which is contradictory.
In this proof we suppose that $\|u - \bar{x}\| < \delta_\varepsilon$, which is equivalent to $\|\eta(t,u)\| < \delta_\varepsilon/\bar{\lambda}$. From this it follows that the function $\eta(\cdot,u)$ must be bounded on A.

Theorem 1.8.. *Between the types of pseudoinvexity and quasiinvexity the next implications hold at u:*

(a) Strongly pseudoinvex (Sg PI) $\Rightarrow$ Strongly quasiinvex (SgQI);

(b) Strictly pseudoinvex (SPI) $\Rightarrow$ Semistrictly quasiinvex (SSQI);

(c) Weakly pseudoinvex (WPI) $\Rightarrow$ Weakly quasiinvex (WQI);

(d) Pseudoinvex (QI) $\Rightarrow$ Semistrictly quasiinvex (SSQI).

Proof. One use the Lemma and Théorème 2.8 by [4].

Direct implications which exist between various types of invexity, pseudoinvexity and quasiinvexity at a point, according to Theorems 1.2, 1.4, 1.5, 1.7 and 1.8 are given in the following block diagram.

weakly invex	$\longrightarrow$	weakly pseudoinvex	$\longrightarrow$	weakly quasiinvex
$\uparrow$		$\uparrow$		$\uparrow$
				quasiinvex
				$\uparrow$
invex	$\longrightarrow$	pseudoinvex	$\longrightarrow$	semistrictly quasiinvex
$\uparrow$		$\uparrow$		$\uparrow$
strictly invex	$\longrightarrow$	strictly pseudoinvex	$\longrightarrow$	strictly quasiinvex
$\uparrow$		$\uparrow$		$\uparrow$
strongly invex	$\longrightarrow$	strongly pseudoinvex	$\longrightarrow$	strongly quasiinvex

Remark 2. Functions 0-invex (case 1A(b)) have been introduced by Ben Israel and Mond in 1986 and its have been denomed "pre-invex functions" by Jeyakumar (see Weir and Mond [11]).

2 Convexity, pseudoconvexity and quasiconvexity types

. In the particular case when A is a nonempty convex set, $\eta(x,u) = x - u$ and $\theta(x,u) = x - u$ we recover the types of convexity, pseudoconvexity and quasiconvexity at a point, as follows:

Definition 2.1. A(Convexities at a point). The function f is said to be *ρ-convex* at $u \in A$ (ρC) if

$$(\rho C) \quad \forall x \in A,\ \forall \lambda \in [0,1] : f(\lambda x + (1-\lambda)u) \leq \lambda f(x) + (1-\lambda)f(u) - \rho\lambda \|x-u\|^2.$$

If

(1′a) $\rho > 0$ the function f is called *strongly convex* at u (SgC);

(1′b) $\rho = 0$ the function f is called *convex* at u (C);

(1′c) $\rho < 0$ the function f is called *weakly convex* at u (WC);

(1′d) $\forall x \in A,\ x \neq u,\ \forall \lambda \in (0,1) : f(\lambda x + (1-\lambda)u) < \lambda f(x) + (1-\lambda)f(u)$ then f is called *strictly convex* at u (SC).

B (Concavities at a point. The function f is said to be *ρ-concave* at u if the function $-f$ is ρ-convex at u. If $\rho > 0$, $\rho = 0$ or $\rho < 0$ then f is called *strongly concave, concave* or *weakly concave* at u. The function f is said to be *strictly concave* at u if the function $-f$ is strictly convex at u.

Definition 2.2. A(Pseudoconvexities at a point). The function f is said to be *ρ-pseudoconvex* at $u \in A$ (ρPC) if

$$(\rho PC) \quad \forall x \in A,\ \forall \lambda \in [0,1] : f(\lambda x + (1-\lambda)u) + \rho\lambda\|x-u\|^2 \geq f(u) \Rightarrow f(x) \geq f(u).$$

If

(2′a) $\rho > 0$ the function f is called *strongly pseudoconvex* at u (SgPC);

(2′b) $\rho = 0$ the function f is called *pseudoconvex* at u (PC);

(2′c) $\rho < 0$ the function f is called *weakly pseudoconvex* at u (WPC);

(2′d) $\forall x \in A,\ x \neq u,\ \forall \lambda in (0,1) : f(\lambda x + (1-\lambda)u) \geq f(u) \Rightarrow f(x) > f(u)$ then f is called *strictly pseudoconvex* at u (SPC).

B (Pseudoconcavities at a point. The function f is said to be ρ-*pseudoconcave* at u if the function $-f$ is ρ-pseudoconvex at u. If $\rho > 0,\ \rho = 0$ or $\rho < 0$ then f is called *strongly pseudoconcave, pseudoconcave* or *weakly pseudoconcave* at u. The function f is said to be *strictly pseudoconcave* at u if the function $-f$ is strictly pseudoconvex at u.

Definition 2.3. A(Quasiconcavities at a point). The function f is said to be ρ-*quasiconvex* at $u \in A$ (ρQC) if

$$(\rho QC) \quad \forall x \in A,\ \forall \lambda \in [0,1] : f(x) \leq f(u) \Rightarrow f(\lambda x + (1-\lambda)u) + \rho\lambda\|x-u\|^2 \leq f(u).$$

If

(3′a) $\rho > 0$ the function f is called *strongly quasiconvex* at u (SgQC);

(3′b) $\rho = 0$ the function f is called *quasiconvex* at u (QC);

(3′c) $\rho < 0$ the function f is called *weakly quasiconvex* at u (WQC);

(3′d) $\forall x \in A,\ x \neq u,\ \forall \lambda \in (0,1) : f(x) \leq f(u) \Rightarrow f(\lambda x + (1-\lambda)u) < f(u)$ then f is called *strictly quasiconvex* at u (SQC).

(3′e) $\forall x \in A,\ x \neq u,\ \forall \lambda \in (0,1) : f(x) < f(u) \Rightarrow f(\lambda x + (1-\lambda)u) < f(u)$ then f is called *semistrictly quasiconvex* at u (SSQC).

B (Quasiconcavities at a point). The function f is said to be ρ-*quasiconcave* at u if the function $-f$ is ρ-quasiconvex at u. If $\rho > 0,\ \rho = 0$ or $\rho < 0$ then f is called *strongly quasiconcave, quasiconcave* or *weakly quasiconcave* at u. The function f is said to be *strictly quasiconcave* at u if the function $-f$ is strictly quasiconvex at u and f is called *semistrictly quasiconcave* at u if the function $-f$ is semistrictly quasiconvex at u.

3 Global minimum properties for (generalized) invex functions

In this section we show that for all (striclty) invex, (striclty) pseudoinvex and (striclty) quasiinvex functions every (strict) local minimum point is a (strict) global minimum point. The global minimum property for these functions make them indispensable in the optimization theory. For a semistrictly quasiinvex function every local minimum point is one of absolutely minimum point.

Theorem 3.1. *If a is a (strict) local minimum point of the function f in A and f is (striclty) invex at a, then a is a (strict) global minimum point of f on A.*

Proof. We suppose that a is a local minimum point and that f is invex at a in respect to $\eta : A \times A \to R^n$. Then there is a neighbourhood N of a such that $f(x) \geq f(a),\ \forall x \in N \cap A$. Let now an arbitrary point $t \in A$. Then exists a $\lambda > 0$, enough small, such that $a + \lambda\eta(t,a) \in N \cap A$. We have $f(a) \leq f(a + \lambda\eta(t,a))$ and by the invexity of f at a it results $f(a) \leq f(a) + \lambda[f(t) - f(a)]$. From this inequality one obtains $f(t) \geq f(a)$, $\forall t \in A$. The proof of the strict variant of the theorem is obtained on the same text.

Theorem 3.2. *If $a \in A$ is a (strict) local minimum point of the function f in A and f is (strictly) pseudoinvex at a, then a is a (strict) global minimum point of f on A.*

Proof. We suppose that f is pseudoinvex at a in respect to $\eta : A \times A \to R^n$ and that a is a local minimum point of f in A. Then there is a neighbourhood N of a such that

$$f(x) \geq f(a), \quad \forall x \in N \cap A. \tag{9}$$

Let now t be an arbitrary point in A. Then there is a $\lambda > 0$, enough small, such that

$$a + \lambda\eta(t, a) \in N \cap A.$$

According to (9) we have $f(a + \lambda\eta(t, a)) \geq f(a)$. But f is pseudoinvex at a and then $f(a + \lambda\eta(t, a)) \geq f(a)$ implies $f(t) \geq f(a)$, $\forall t \in A$. The strict variant of the theorem is obtained in the same manner.

Theorem 3.3. *If $a \in A$ is a strict local minimum point of the function f in A and f is strictly quasiinvex at a, then a is a strict global minimum point of f on A.*

Proof. We suppose that f is (SQI) at $a \in A$ in respect to $\eta : A \times A \to R^n$. Then, equivalently, we have:

$$\forall x \in A,\ x \neq a,\ \forall \lambda in (0, 1) : f(a + \lambda\eta(x, a)) \geq f(a) \Rightarrow f(x) > f(a). \tag{10}$$

If a is a strict local minimum point in A then there is a neighbourhood N of a such that

$$f(x) > f(a), \quad \forall x \in N \cap A \setminus a. \tag{11}$$

At consequence there is an $\varepsilon > 0$ such that $a + \varepsilon\eta(t, a) \in N \cap A \setminus a$, where t is arbitrary in A. Then, according to (11), we have

$$f(a + \varepsilon\eta(t, a)) > f(a), \quad \forall \lambda \in (0, \varepsilon], \quad \forall t \in A \setminus a. \tag{12}$$

Taking now into account relations (10) and (12) we obtain $f(t) > f(a), \quad \forall t \in A \setminus a$ and so, a is the strict global minimum point of f on A.

Theorem 3.4.. *Let f be a semistrictly quasiinvex function at the point $a \in A$. If a is a local minimum point of f in A, then a is an absolut minimum point of f on A.*

Proof. If f is (SSQI) at a in respect to $\eta : A \times A \to R^n$ then, equivalently, one has

$$\forall x \in A,\ x \neq a,\ \forall \lambda in (0, 1) : f(a + \lambda\eta(x, a)) \geq f(a) \Rightarrow f(x) \geq f(a). \tag{13}$$

The point a is one of local minimum of f and then there exists a neighbourhood N of a, such that $f(x) \geq f(a), \quad \forall x \in N \cap A$. Particulary, there is an $\varepsilon > 0$ such that

$$f(a + \lambda\eta(x, a)) \geq f(a),\ \forall x \in A, \quad \forall \lambda \in (0, \varepsilon]. \tag{14}$$

Combining relations (13) and (14) it results $f(x) \geq f(a)$, $\forall x \in A$.

Definition 3.1. Let V and S be two nonempty subset of A. V is said to be a "neighbourhood" of S if $S \subset V$ and (Fr S) $\cap$ (Fr V) $= \emptyset$ or otherwise
$\emptyset \neq$ (Fr S) $\cap$ (Fr V)$\subset$ Fr A.

Definition 3.2. Let $f : A \to R$ be where the set A has a nonempty relative interior.

a) The nonempty set $S \subset A$ is said to be a *local minimum subdomain* of the function f in A if f is constant on S and there is a "neighbourhood" $V \subset A$ of S such that $f(y) > f(x)$, $\forall x \in S$, $\forall y \in V$.

b) If M is the single local minimum subdomain of the function f in A then M is called the *global minimum subdomain* of f on A.

Theorem 3.5. *Let f be a quasiinvex function on A. Then any local minimum subdomain of f in A is a global minimum subdomain of f on A.*

Proof. Let $M \subset A$ be a local minimum subdomain of f in A. We suppose, ad absurdum, that the quasiinvex function f admits in A another local minimum subdomain S, different from M. Evidently $M \cap S = \emptyset$. Let $u \in M$ and $a \in S$ be arbitrary choosed and we suppose

$$f(a) \leq f(u). \tag{15}$$

Let V be a "neighbourhood" of M. We ascertain that

$$B = \{u + \lambda\eta(x,u) \in A | \forall x \in A,\ \forall\lambda \in (0,\bar{\lambda})\} \cap V \neq \phi$$

where

$$\bar{\lambda} = \min\{1, \sup\{\lambda > 0 | u + \lambda\eta(x,u) \in A\}\}.$$

Let y be an arbitrary point in $B \setminus M$. Then there are $x' \in A$ and $\lambda' \in (0,\bar{\lambda})$ such that $y = u + \lambda'\eta(x',u)$. We can choose y in $B \setminus M$ such that $f(x') < f(u)$ because, according to (15), u is not an absolutely minimum point. Then, the quasiinvexity of f at u imply

$$f(u + \lambda'\eta(x',u)) \leq f(u). \tag{16}$$

But $y = u + \lambda'\eta(x',u)$ in V implies

$$f(y) = f(u + \lambda'\eta(x',u)) > f(u). \tag{17}$$

Relations (16) and (17) are contradictory. Then it follows that the supposition above maked, that M is not the single local minimum subdomain of f in A, is false.

Remark 3. The preceding theory can be translated in the Riemannian language using the ideas of Udrişte [10].

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