The Geometrical Barbilian's Work from a Modern Point of View

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Abstract

We comment–from a modern point of view the Barbilian's geometrical work, emphasizing his results in the study of the geometries over rings. Recent progresses are reported here and open problems are listed.

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Dan BARBILIAN was one of the most impressive personality of Romanian mathematics. He was born in 1895 and studied mathematics at the Universities of Bucharest and Göttingen. He obtained important results in projective and differential geometry, in algebra, and in number theory. At the same time he was an outstanding Romanian poet.

Three years ago, in 1993, in the frame of the FIRST INTERNATIONAL WORKSHOP ON DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS (Bucharest, July, 25 - 30), I have organized a BARBILIAN SPECIAL SESSION celebrating a half a century from the Barbilian's studies on projective planes over arbitrary rings. On that occasion, Prof. J. R. Faulkner, an outstanding personality in this field, talked about current results on Barbilian planes (see FAULKNER [7d]). My talk was devoted to the rôle played by Jordan structures as unifying framework for Barbilian planes and differential geometry (see IORDANESCU [12b]), while Dr. V. Boskoff talked about his recent results on metric Barbilian planes (see BOSKOFF [5]).

In 1995 we have celebrated 100 years from Barbilian's birth. The Romanian Academy devoted two of its public meetings to the Barbilian's lyrical poetry and to his mathematical thinking (in March and in May, respectively). On the occasion of the SECOND INTERNATIONAL WORKSHOP ON DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS (Constantza, September, 25–28, 1995) I gave the present text to all participants.

I like to make here - from a modern point of view - some comments on Barbilian's geometrical work, especially on his contribution to the study of the geometries over rings.

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A systematic study of projective planes over large classes of associative rings was initiated by BARBILIAN in his very general approach [2b]. There he proved that the rings which can be underlying rings for projective geometries are (with a few exceptions) rings with a unit element in which any one-sided inverse is a two-sided inverse. BARBILIAN [2b, I] called these rings "Z-rings" (from "Zweiseitigsinguläre Ringe") and gave a set of 11 axioms of projective geometry over a certain type of Z-ring (see [2b, II]).

For more than thirty years no development of Barbilian's study succeded. Beginning with 1975, outstanding mathematicians like W. Leissner in Germany, F.D. Veldkamp in Holland, and J.R. Faulkner in U.S.A., developed Barbilian's ideas, and so, notions as "Barbilian domains", "Barbilian planes", "Barbilian spaces", or "Barbilian geometry" appeared.

Note. I would like to mention here the fact that projective geometries over more general algebraic structures (as Jordan algebras, for instance) like that considered by Barbilian, were studied. For an information untill 1990, including Barbilian structures, can be used the second part of [12a] by IORDANESCU.

LEISSNER [14a] developed a plane geometry over an arbitrary Z-ring R, in which a point is an element of $R \times R$ and a line is a set of the form

$$\{(x + ra,\ y + rb)\ \mid\ (x, y) \in R \times R,\ \ r \in \mathbf{R},\ (a, b) \in B\},$$

where B is a "Barbilian domain", i.e. a set of unmodular pairs from $R \times R$ satisfying certain axioms.

Note. Let us mention in this context that LANTZ extended in [13] the results of BENZ from [3] by showing that large classes of commutative rings admit only one Barbilian domain.

RADÓ [16] extended LEISSNER's results [14a] to affine Barbilian planes over an arbitrary ring with a unit element and investigated the corresponding affine Barbilian structures and translation Barbilian planes. Corresponding to the algebraic representation of affine Barbilian spaces as affine geometries over unitary free modules, LEISSNER [14b], characterized algebraic properties of the underlying ring R, respectively module M_R, respectively Barbilian domain $B \subset M_R$ by geometric properties of the affine Barbilian space and viceversa.

VELDKAMP [21a] gave an axiomatic description of plane geometries of the kind considered by BINGEN [4]. A most satisfactory situation is reached by extending the class of rings used for coordinatization from semiprimary rings, which Bingen used, to rings of stable rank 2. These rings have played a rôle in algebraic K-theory, and seem to form a natural framework for many geometric problems.

Recently, VELDKAMP refined in [21f] the notion of Barbilian domain to n-Barbilian domain in a free module of rank n. This leads to results that bear on n-dimensional affine ring geometry. The case of infinite rank is also considered by VELDKAMP in [21f].

Note. For a good survey on the theory of projective planes over rings of stable rank 2, see VELDKAMP [21d]. Such a plane is described as a structure of points and lines together with an incidence relation and a neighbor relation and which has to satisfy two groups of axioms. The axioms in the first group express elementary relations between points and lines such as, e.g., the existence of a unique line joining any two non-neighboring points, and define what is called a Barbilian plane. In the second group of axioms the existence of sufficiently many transvections, dilatations, and generalizations of the latter, the affine dilatations and their duals, is required.

In 1987, VELDKAMP [21e] extended all this above mentioned results to arbitrary finite dimension. Basic objects in the axioms are points and hyperplanes, by analogy with the selfdual set-up for clasical projective spaces over skew fields given by ESSER [6]. As basic relations again serve incidence and the neighbour relation. The self-dual approach is quite natural since incidence and the neighbour relation between points and hyperplanes have a simple algebraic description in coordinates. Homomorphisms are more or less the same as in the plane case, things becoming a bit more complicated because Veldkamp included homomorphism between spaces of unequal dimension.

Note. Veldkamp confine himself to full homomorphisms, which can only increase the dimension or leave it the same. Thus he excluded homomorphisms which lower the dimension, an example of which was given by FRITSCH and PRESTEL [9].

Recently, FAULKNER [7c] defined and studied the so-called F-*planes* which generalize the projective planes. Planes considered by BARBILIAN in the Zusatz to [2b] are connected F-planes in Faulkner's setting. Besides extending the class of coordinate ring, FAULKNER's work [7e] introduces some new concepts, techniques, and connections with other areas. These include a theory of covering planes and homotopy although there is no topology, a theory of tangent bundle planes and their sections although there is no differential or algebraic geometry, a purely geometric and coordinate-free construction of the Lie ring of the group generated by transvections, and connections to the K-theory of the coordinate ring.

For some fundamental properties of full homomorphisms between Barbilian spaces as well as for their algebraic description we refer the reader to VELDKAMP [21e, I].

FAULKNER and FERRAR [8] surveyed the development which leads from classical Desarguesian projective plane *via* Moufang planes Moufang-Veldkamp planes. They first sketched inhomogeneous and homogeneous coordinates in the real and projective planes and in ring planes, the Jordan algebra construction of Moufang planes, and the representation of all these planes as homogeneous spaces for their groups of transvections. Then attention is focussed on *Moufang-Veldkamp planes*, i.e. projective Barbilian planes in which all possible transvections exist and which satisfy the little quadrangle section condition for quadrangles in general position. As coordinates for the affine plane one easily obtains an alternative ring of stable rank 2. Unfortunately, the Bruck and Kleinfeld theorem for alternative division rings does not carry over to alternative rings in general, i.e. such a ring need neither be associative nor be an octonion algebra. Therefore, to coordinatize the whole projective plane one cannot rely on either homogeneous coordinates (as in the associative case) or the Jordan algebra construction (as in the octonion case). In this case, one has to follow a more complicated way, namely: first to construct a certain Jordan pair from the given alternative ring, then to define a group of transformations of this Jordan pair and, finally, to represent the projective plane as a homogeneous space for that group.

FAULKNER [7a] proved that for a *connected Barbilian transvection plane* P (i.e., a plane with incidence and neighbouring generalizing Moufang projective planes) one can construct a connected Barbilian plane $T(P)$ called *tangent bundle plane*. This construction agrees with the usual tangent bundle when such exists. If $T(P)$ is also a transvection plane, then the set S of sections of $T(P)$ is a Lie ring. The group G generated by all transvections of P acts on S. Since S is isomorphic to the Koecher-Tits Lie ring constructed from the Jordan pair $(M_{12}(R),\ M_{21}(R))$, where R is the associated alternative ring, one can determine G and thereby P from R.

In 1987, SPANICCIATI [17] introduced *near Barbilian planes (NBP)* and *strong near Barbilian planes (SNBP)* as a variation of Barbilian planes. Recently, HANSSENS and van MALDEGHEM [11] showed that a *NBP* is an *SNBP*, and classified all *NBP* up to the classification of linear spaces (many examples follows as a result of a universal construction). They also showed that only *NBP*s that are also *BP*s are those mentioned in [17], namely the projective planes.

In 1984, ALLISON and FAULKNER [1] have given an algebraic construction of degree 3 Jordan algebras (including the exceptional one) as trace zero elements in a degree 4 Jordan algebra. Recently, FAULKNER [7b] translated this algebraic construction to give a geometric construction of Barbilian planes coordinatized by composition algebras (including the Moufang plane) as skew polar line pairs and points on the quadratic surface determined by a polarity of projective 3-space over a smaller composition algebra.

By making use of the talk [7e] given by FAULKNER at the Conference on Jordan Algebras in Oberwolfach (1992), let us end by mentioning other new results as well as some open problems:

1. The theory of higher dimensional geometries for (associative) nondivison rings has been done for stable range two rings by VELDKAMP [21a, b, c, e] and certain two-sided inverse rings by MAGNUS [15]. An open problem is to know whether the fundamental group for the geometries studied by Magnus stabilizes as dimensions increase. There are also questions about when a geometry embeds in a higher dimension. These are being examined by Magnus.

2. A theory of n-gon Barbilian planes for $n = 6$ due to TORRENCE [19] is largely complete, but the case $n = 4$ is completely open. Also, a general theory of Barbilian buildings is a completely open problem.

3. Another open problem is a general study of polarities and conics for Barbilian planes.

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