The Equations of Structure of an N-Linear Connection in the Bundle of Accelerations

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Abstract

The study of higher order Lagrange spaces founded on the notion of bundle of velocities of order k has been recently given by Radu Miron and author in [2]-[5]. The bundle of acceleration correspond in this study to $k = 2$.

In this paper we shall give the equations of structure of an N-linear connection in the bundle of accelerations.

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1 The bundle of accelerations ([2],[3])

Let M be a real n-dimensional C^∞-manifold and (Osc^2M, π, M) its 2-osculator bundle, or the bundle of accelerations. The canonical local coordinates on the total space $E = Osc^2M$ are denoted by $(x^i, y^{(1)i}, y^{(2)i})$. A coordinate transformation $(x^i, y^{(1)i}, y^{(2)i}) \to (\tilde{x}^i, \tilde{y}^{(1)i}, \tilde{y}^{(2)i})$ on E is given by

$$\text{(1.1)} \qquad \begin{cases} \tilde{x}^i &= \tilde{x}^i(x^1, ..., x^n), \quad \text{rank} \parallel \frac{\partial \tilde{x}^i}{\partial x^j} \parallel = n, \\ \tilde{y}^{(1)i} &= \frac{\partial \tilde{x}^i}{\partial x^j}\, y^{(1)j} \\ 2\tilde{y}^{(2)i} &= \frac{\partial \tilde{y}^{(1)i}}{\partial x^j} y^{(1)j} + 2\frac{\partial \tilde{y}^{(1)i}}{\partial y^{(1)j}} y^{(2)j} \end{cases} .$$

If N is a nonlinear connection on E and J is the tangent structure of second order [2], then $N_0 = N$, $N_1 = J(N_0)$ are two distributions geometrically defined on E, everyone of local dimension n. Let us consider the distribution V_2 on E locally generated by the vector fields $\{\frac{\partial}{\partial y^{(2)i}}\}$. Consequently, the tangent space to E at a point $u \in E$ is given by a direct sum of the vector spaces:

$$\text{(1.2)} \qquad T_u(E) = N_0(u) \oplus N_1(u) \oplus V_2(u), \quad \forall u \in E.$$

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An adapted basis to the direct decomposition (1.2) is given by

$$\left\{\frac{\delta}{\delta x^i},\quad \frac{\delta}{\delta y^{(1)i}},\quad \frac{\partial}{\partial y^{(2)i}}\right\}\quad (i=1,...,n), \tag{1.3}$$

where

$$\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-\underset{(1)}{N}{}^r{}_i\frac{\partial}{\partial y^{(1)r}}-\underset{(2)}{N}{}^r{}_i\frac{\partial}{\partial y^{(2)r}} \tag{1.4}$$

and

$$\frac{\delta}{\delta y^{(1)i}}=\frac{\partial}{\partial y^{(1)i}}-N^r_{(1)i}\frac{\partial}{\partial y^{(2)r}}. \tag{1.4'}$$

The systems of functions $\underset{(1)}{N}{}^i_j$, $\underset{(2)}{N}{}^i_j$ are called the *coefficients* of the nonlinear connection N.

If we consider the projectors h, v_1, v_2 determined by (1.2) and denoting $v_\alpha X = X^{v_\alpha}\quad(\alpha=1,2)$, we can uniquely write

$$X=X^H+X^{v_1}+X^{v_2},\quad (\forall)X\in\mathcal{X}(\mathcal{E}). \tag{1.5}$$

Thus, we have

$$X^H=X^{(0)i}\frac{\delta}{\delta x^i},\quad X^{v_1}=X^{(1)i}\frac{\delta}{\delta y^{(1)i}},\quad X^{v_2}=X^{(2)i}\frac{\partial}{\partial y^{(2)i}}. \tag{1.5'}$$

The coordinates $X^{(\alpha)i},\quad (\alpha=0,1,2)$, change under (1.1) as follows:

$$\tilde{X}^{(\alpha)i}=\frac{\partial\tilde{x}^i}{\partial x^j}X^{(\alpha)j},\quad (\alpha=0,1,2). \tag{1.5''}$$

Each one of them is called a *distinguished vector field*, shortly a *d-vector field.*

Let us consider the dual basis of (1.3):

$$\{dx^i,\delta y^{(1)i},\delta y^{(2)i}\}\quad (i=1,...,n). \tag{1.3'}$$

Then for a field of 1-form ω on E, we can put

$$\omega=\omega^H+\omega^{v_1}+\omega^{v_2}, \tag{1.6}$$

where

$$\omega^H=\omega^{(0)}_i dx^i,\quad \omega^{v_1}=\omega^{(1)}_i\delta y^{(1)i},\quad \omega^{v_2}=\omega^{(2)}_i\delta y^{(2)i}, \tag{1.6'}$$

and with respect to (1.1) we have

$$\omega^{(\alpha)}_i=\frac{\partial\tilde{x}^j}{\partial x^i}\tilde{\omega}^{(\alpha)}_j,\quad (\alpha=0,1,2). \tag{1.6''}$$

Each one of them is called a *distinguished covector field*, shortly a *d-covector field.*

Analogously, we can define a *distinguished tensor field* on E of type (r,s) (shortly, a *d-tensor field*).

Now, we consider the 2-tangent structure J on $Osc^2M = E$ and a nonlinear connection N on E.

We define an *N-linear connection* on E as a linear connection D on E which preserves by parallelism the horizontal distribution N and which is compatible with the structure $J(i.e., D_X J = 0, \quad \forall X \in \mathcal{X}(\mathcal{E}))$.

In the adapted basis (1.3) it is sufficient to give

$$D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta y^{(\alpha)i}} = L^m_{ij} \frac{\delta}{\delta y^{(\alpha)m}}, \quad D_{\frac{\delta}{\delta y^{(\beta)j}}} \frac{\delta}{\delta y^{(\alpha)i}} = \underset{(\beta)}{C}{}^m_{ij} \frac{\delta}{\delta y^{(\alpha)m}} \tag{1.7}$$

$$(\alpha = 0,1,2, \quad \beta = 1,2 \quad \text{and} \quad y^{(0)i} = x^i)$$

in order to obtain all the coefficients

$$D\Gamma(N) = \left(L^i_{jm}, \underset{(1)}{C}{}^i_{jm}, \underset{(2)}{C}{}^i_{jm} \right)$$

of an N-linear connection D.

In the algebra of the d-tensor field generated by

$$\left\{ 1, \frac{\delta}{\delta x^i}, \quad \frac{\delta}{\delta y^{(1)i}}, \quad \frac{\partial}{\partial y^{(2)i}}, \quad dx^i, \quad \delta y^{(1)i}, \quad \delta y^{(2)i} \right\},$$

the h-covariant derivates will be noted with " $|$ " and the v_α-covariant derivatives will be noted with " $\overset{(\alpha)}{|}$ ", $\alpha = 1,2$.

Applying (1.4) and (1.4′) we obtain

Theorem 1.1. *If N is a nonlinear connection on E with the coefficients $\underset{(1)}{N}{}^i{}_j$, $\underset{(2)}{N}{}^i{}_j$, then the following relations hold*

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right] = \sum_{(i,j)} \left\{ \left(\frac{\partial \underset{(1)}{N}{}^m_i}{\partial x^j} + \underset{(1)}{N}{}^r_i \frac{\partial \underset{(1)}{N}{}^m_j}{\partial y^{(1)r}} + \underset{(2)}{N}{}^r_i \frac{\partial \underset{(1)}{N}{}^m_j}{\partial y^{(2)r}} \right) \frac{\partial}{\partial y^{(1)m}} + \right. \tag{1.8}$$

$$\left. + \left(\frac{\partial \underset{(2)}{N}{}^m_i}{\partial x^j} + \underset{(1)}{N}{}^r_i \frac{\partial \underset{(2)}{N}{}^m_j}{\partial y^{(1)r}} + \underset{(2)}{N}{}^r_i \frac{\partial \underset{(2)}{N}{}^m_j}{\partial y^{(2)r}} \right) \frac{\partial}{\partial y^{(2)m}} \right\},$$

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)j}} \right] = \left(\frac{\partial \underset{(1)}{N}{}^m_i}{\partial y^{(1)j}} - \underset{(1)}{N}{}^r_j \frac{\partial \underset{(1)}{N}{}^m_j}{\partial y^{(2)r}} \right) \frac{\partial}{\partial y^{(1)m}} + \tag{1.9}$$

$$+ \left[\left(\frac{\partial \underset{(2)}{N}{}^m_i}{\partial y^{(1)j}} - \frac{\partial \underset{(1)}{N}{}^m_j}{\partial x^i} \right) + \underset{(1)}{N}{}^r_i \frac{\partial \underset{(1)}{N}{}^m_j}{\partial y^{(1)r}} + \underset{(2)}{N}{}^r_i \frac{\partial \underset{(1)}{N}{}^m_j}{\partial y^{(2)r}} - \underset{(1)}{N}{}^r_j \frac{\partial \underset{(2)}{N}{}^m_i}{\partial y^{(2)r}} \right] \frac{\partial}{\partial y^{(2)m}},$$

$$\left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^{(2)j}}\right] = \frac{\partial \underset{(1)}{N}{}_i^m}{\partial y^{(2)j}} \frac{\partial}{\partial y^{(1)m}} + \frac{\partial \underset{(2)}{N}{}_i^m}{\partial y^{(2)j}} \frac{\partial}{\partial y^{(2)m}}, \tag{1.10}$$

$$\left[\frac{\delta}{\delta y^{(1)i}}, \frac{\delta}{\delta y^{(1)j}}\right] = \sum_{(i,j)} \left\{ \left(\frac{\partial \underset{(1)}{N}{}_i^m}{\partial y^{(1)j}} + \underset{(1)}{N}{}_i^r \frac{\partial \underset{(1)}{N}{}_j^m}{\partial y^{(2)r}} \right) \frac{\partial}{\partial y^{(2)m}} \right\}, \tag{1.11}$$

$$\left[\frac{\delta}{\delta y^{(1)i}}, \frac{\partial}{\partial y^{(2)j}}\right] = \frac{\partial \underset{(1)}{N}{}_i^m}{\partial y^{(2)j}} \frac{\partial}{\partial y^{(2)m}}, \tag{1.12}$$

where $\sum_{(i,j)}$ is the symbol of alternate sum.

2 The torsion and curvature d-tensor fields

The torsion tensor of the N-linear connection D on $E = Osc^2M$,

$$T(X,Y) = D_X Y - D_Y X - [X,Y], \quad \forall X, Y \in \mathcal{X}(\mathcal{E}) \tag{2.1}$$

has a number of horizontal and vertical components corresponding to D^H, D^{v_1} and D^{v_2}.

We put

$$\begin{aligned}
&T\left(\tfrac{\delta}{\delta x^j}, \tfrac{\delta}{\delta x^i}\right) = \underset{(0)}{\overset{(0)}{T}}{}^m{}_{ij} \tfrac{\delta}{\delta x^m} + \underset{(0)}{\overset{(1)}{T}}{}^m{}_{ij} \tfrac{\delta}{\delta y^{(1)m}} + \underset{(0)}{\overset{(2)}{T}}{}^m{}_{ij} \tfrac{\partial}{\partial y^{(2)m}} \\
&T\left(\tfrac{\delta}{\delta y^{(1)j}}, \tfrac{\delta}{\delta x^i}\right) = \underset{(1)}{\overset{(0)}{P}}{}^m{}_{ij} \tfrac{\delta}{\delta x^m} + \underset{(1)}{\overset{(1)}{P}}{}^m{}_{ij} \tfrac{\delta}{\delta y^{(1)m}} + \underset{(1)}{\overset{(2)}{P}}{}^m{}_{ij} \tfrac{\partial}{\partial y^{(2)m}} \\
&T\left(\tfrac{\partial}{\partial y^{(2)j}}, \tfrac{\delta}{\delta x^i}\right) = \underset{(2)}{\overset{(0)}{P}}{}^m{}_{ij} \tfrac{\delta}{\delta x^m} + \underset{(2)}{\overset{(1)}{P}}{}^m{}_{ij} \tfrac{\delta}{\delta y^{(1)m}} + \underset{(2)}{\overset{(2)}{P}}{}^m{}_{ij} \tfrac{\partial}{\partial y^{(2)m}} \\
&T\left(\tfrac{\partial}{\partial y^{(2)j}}, \tfrac{\delta}{\delta y^{(1)i}}\right) = \underset{(12)}{\overset{(0)}{P}}{}^m{}_{ij} \tfrac{\delta}{\delta x^m} + \underset{(12)}{\overset{(1)}{P}}{}^m{}_{ij} \tfrac{\delta}{\delta y^{(1)m}} + \underset{(12)}{\overset{(2)}{P}}{}^m{}_{ij} \tfrac{\partial}{\partial y^{(2)m}} \\
&T\left(\tfrac{\delta}{\delta y^{(1)j}}, \tfrac{\delta}{\delta y^{(1)i}}\right) = \underset{(1)}{\overset{(0)}{S}}{}^m{}_{ij} \tfrac{\delta}{\delta x^m} + \underset{(1)}{\overset{(1)}{S}}{}^m{}_{ij} \tfrac{\delta}{\delta y^{(1)m}} + \underset{(1)}{\overset{(2)}{S}}{}^m{}_{ij} \tfrac{\partial}{\partial y^{(2)m}} \\
&T\left(\tfrac{\partial}{\partial y^{(2)j}}, \tfrac{\partial}{\partial y^{(2)i}}\right) = \underset{(2)}{\overset{(0)}{S}}{}^m{}_{ij} \tfrac{\delta}{\delta x^m} + \underset{(2)}{\overset{(1)}{S}}{}^m{}_{ij} \tfrac{\delta}{\delta y^{(1)m}} + \underset{(2)}{\overset{(2)}{S}}{}^m{}_{ij} \tfrac{\partial}{\partial y^{(2)m}}
\end{aligned} \tag{2.2}$$

Then, by (2.1), (1.7) and Theorem 1.1, we have

Theorem 2.1. *The torsion tensor of an N-linear connection $D\Gamma(N) = \left(L^m_{ij}, \underset{(1)}{C}{}^m_{ij}, \underset{(2)}{C}{}^m_{ij}\right)$ is characterized by the d-tensor fields with local components*:

$$(2.3)\qquad \begin{cases} \underset{(0)}{\overset{(0)}{T}}{}^{m}{}_{ij} &= L^m_{ij} - L^m_{ji}, \underset{(1)}{\overset{(0)}{P}}{}^{m}{}_{ij} = \underset{(1)}{C}{}^m_{ij}, \underset{(2)}{\overset{(0)}{P}}{}^m_{ij} = \underset{(2)}{C}{}^m{}_{ij} \\ \underset{(0)}{\overset{(1)}{T}}{}^{m}{}_{ij} &= \dfrac{\delta \underset{(1)}{N}{}^m{}_i}{\delta x^j} - \dfrac{\delta \underset{(1)}{N}{}^m{}_j}{\delta x^i}, \\ \underset{(0)}{\overset{(2)}{T}}{}^{m}{}_{ij} &= \dfrac{\delta \underset{(2)}{N}{}^m{}_i}{\delta x^j} - \dfrac{\delta \underset{(2)}{N}{}^m{}_j}{\delta x^i} + \underset{(1)}{N}{}^m{}_r \left(\dfrac{\delta \underset{(1)}{N}{}^r{}_i}{\delta x^j} - \dfrac{\delta \underset{(1)}{N}{}^r{}_j}{\delta x^i} \right), \end{cases}$$

$$(2.4)\qquad \begin{cases} \underset{(1)}{\overset{(1)}{P}}{}^{m}{}_{ij} &= \dfrac{\delta \underset{(1)}{N}{}^m{}_i}{\delta y^{(1)j}} - L^m_{ji}, \quad \underset{(1)}{\overset{(2)}{P}}{}^{m}{}_{ij} = \dfrac{\delta \underset{(2)}{N}{}^m{}_i}{\delta y^{(1)j}} + \underset{(1)}{N}{}^m{}_r \dfrac{\delta \underset{(1)}{N}{}^r{}_i}{\delta y^{(1)j}} - \dfrac{\delta \underset{(1)}{N}{}^m{}_j}{\delta y^{(1)i}}, \\ \underset{(2)}{\overset{(1)}{P}}{}^{m}{}_{ij} &= \dfrac{\partial \underset{(1)}{N}{}^m{}_i}{\partial y^{(2)j}}, \quad \underset{(2)}{\overset{(2)}{P}}{}^{m}{}_{ij} = \dfrac{\partial \underset{(2)}{N}{}^m{}_i}{\partial y^{(2)j}} + \underset{(1)}{N}{}^m{}_r \dfrac{\partial \underset{(1)}{N}{}^r{}_i}{\partial y^{(2)j}} - L^m_{ji}, \\ \underset{(12)}{\overset{(0)}{P}}{}^{m}{}_{ij} &= 0, \quad \underset{(12)}{\overset{(1)}{P}}{}^{m}{}_{ij} = \underset{(2)}{C}{}^m{}_{ij}, \quad \underset{(12)}{\overset{(2)}{P}}{}^{m}{}_{ij} = \dfrac{\partial \underset{(1)}{N}{}^m{}_i}{\partial y^{(2)j}} - \underset{(1)}{C}{}^m_{ji}, \end{cases}$$

$$(2.5)\qquad \begin{cases} \underset{(1)}{\overset{(0)}{S}}{}^{m}{}_{ij} = 0, \quad \underset{(1)}{\overset{(1)}{S}}{}^{m}{}_{ij} = \underset{(1)}{C}{}^m{}_{ij} - \underset{(1)}{C}{}^m{}_{ji}, \quad S^{(2)}_{(1)}{}^{m}{}_{ij} = \dfrac{\delta \underset{(1)}{N}{}^m{}_i}{\delta y^{(1)j}} - \dfrac{\delta \underset{(1)}{N}{}^m{}_j}{\delta y^{(1)i}}, \\ \underset{(2)}{\overset{(0)}{S}}{}^{m}{}_{ij} = 0, \quad \underset{(2)}{\overset{(1)}{S}}{}^{m}{}_{ij} = 0, \quad \underset{(2)}{\overset{(2)}{S}}{}^{m}{}_{ij} = \underset{(2)}{C}{}^m{}_{ij} - \underset{(2)}{C}{}^m{}_{ji} \end{cases}$$

Also, we can use the notations

$$\underset{(0)}{\overset{(0)}{T}}{}^{m}{}_{ij} = T^m{}_{ij}, \quad \underset{(0)}{\overset{(1)}{T}}{}^{m}{}_{ij} = \underset{(0)}{\overset{(1)}{R}}{}^{m}{}_{ij}, \quad \underset{(0)}{\overset{(2)}{T}}{}^{m}{}_{ij} = \underset{(0)}{\overset{(2)}{R}}{}^{m}_{ij},$$

$$\underset{(1)}{\overset{(1)}{S}}{}^{m}{}_{ij} = \underset{(1)}{S}{}^m{}_{ij}, \quad \underset{(1)}{\overset{(2)}{S}}{}^{m}{}_{ij} = \underset{(1)}{\overset{(1)}{R}}{}^{m}{}_{ij}, \quad \underset{(2)}{\overset{(2)}{S}}{}^{m}{}_{ij} = S_{(2)}{}^m_{ij}$$

Theorem 2.2. *An N-linear connection $D\Gamma(N) = (L^m_{ij}\ \underset{(1)}{C}{}^m{}_{ij},\ \underset{(2)}{C}{}^m{}_{ij})$ is without torsion if and only if*

$$(2.6)\qquad \begin{cases} L^m_{ij} = L^m_{ji}, \quad \underset{(\gamma)}{C}{}^m{}_{ij} = 0, \quad \underset{(0)}{\overset{(\gamma)}{R}}{}^{m}{}_{ij} = 0, \\ \underset{(\beta)}{\overset{(\gamma)}{P}}{}^{m}{}_{ij} = 0, \quad \underset{(\alpha\beta)}{\overset{(\gamma)}{P}}{}^{m}{}_{ij} = 0, \quad \underset{(\alpha)}{\overset{(\gamma)}{S}}{}^{m}{}_{ij} = 0, \quad \alpha < \beta;\ \alpha, \beta, \gamma = 1, 2. \end{cases}$$

It is to notice the fact that an N-linear connection D is called semi-symmetric if

$$(2.7)\qquad [T(X^H, Y^H)]^H = X^H \eta(Y^H) - Y^H \eta(X^H)$$

$$[T(X^{v_\alpha}, Y^{v_\alpha})]^{v_\alpha} = X^{v_\alpha} \sigma_\alpha(Y^{v_\alpha}) - Y^{v_\alpha} \sigma_\alpha(X^{v_\alpha}),$$
$$\forall X, Y \in \mathcal{X}(\mathcal{E}), \quad \eta, \sigma_\alpha \in \mathcal{X}^\star(\mathcal{E}), \quad \alpha = \infty, \in.$$

Denoting by

$$R\left(\frac{\delta}{\delta x^q},\frac{\delta}{\delta x^p}\right)\frac{\delta}{\delta x^r}=R_r{}^m{}_{pq}\frac{\delta}{\delta x^m},\quad R\left(\frac{\delta}{\delta y^{(\beta)q}},\frac{\delta}{\delta x^p}\right)\frac{\delta}{\delta x^r}=\underset{(\beta)}{P}{}_r{}^m{}_{pq}\frac{\delta}{\delta x^m},$$
$$R\left(\frac{\delta}{\delta y^{(\beta)q}},\frac{\delta}{\delta y^{(\alpha)p}}\right)\frac{\delta}{\delta x^r}=\underset{(\alpha\beta)}{P}{}_r{}^m{}_{pq}\frac{\delta}{\delta x^m},\quad R\left(\frac{\delta}{\delta y^{(\alpha)q}},\frac{\delta}{\delta y^{(\alpha)p}}\right)\frac{\delta}{\delta x^r}=S_{(\alpha)r}{}^m{}_{pq}\frac{\delta}{\delta x^m}$$

we have
Theorem 2.3. *The curvature tensor field R of an N-linear connection $D\Gamma(N)=(L^m_{ij},\underset{(1)}{C}{}^m_{ij},\underset{(2)}{C}{}^m_{ij})$ is characterized by the following d-tensor fields on E:*

$$R_r{}^m{}_{pq}=\frac{\delta L^m_{rp}}{\delta x^q}-\frac{\delta L^m_{rq}}{\delta x^p}+L^t_{rp}L^m_{tq}-L^t_{rq}L^m_{tp}+ \tag{2.8}$$
$$+\underset{(0)}{\overset{(1)}{R}}{}^t{}_{pq}\underset{(1)}{C}{}^m_{rt}+\underset{(0)}{\overset{(2)}{R}}{}^t_{pq}\underset{(2)}{C}{}^m{}_{rt},$$

$$\underset{(1)}{P}{}_r{}^m{}_{pq}=\frac{\delta L^m_{rp}}{\delta y^{(1)q}}-\underset{(1)}{C}{}^m_{rq|p}+\underset{(1)}{\overset{(1)}{P}}{}^t{}_{pq}\underset{(1)}{C}{}^m_{rt}+\underset{(1)}{\overset{(2)}{P}}{}^t_{pq}\underset{(2)}{C}{}^m_{rt}, \tag{2.9}$$

$$\underset{(2)}{P}{}_r{}^m{}_{pq}=\frac{\partial L^m_{rp}}{\partial y^{(2)q}}-\underset{(2)}{C}{}^m_{rq|p}+\underset{(2)}{\overset{(1)}{P}}{}^t{}_{pq}\underset{(1)}{C}{}^m_{rt}+\underset{(2)}{\overset{(2)}{P}}{}^t_{pq}\underset{(2)}{C}{}^m_{rt}, \tag{2.10}$$

$$\underset{(12)}{P}{}_r{}^m{}_{pq}=\frac{\partial \underset{(1)}{C}{}^m_{rp}}{\partial y^{(2)q}}-\underset{(2)}{C}{}^m_{rq\overset{(1)}{|}_p}+\underset{(12)}{\overset{(2)}{P}}{}^t{}_{pq}\underset{(2)}{C}{}^m_{rt}, \tag{2.11}$$

$$\underset{(1)}{S}{}_r{}^m{}_{pq}=\frac{\delta \underset{(1)}{C}{}^m_{rp}}{\delta y^{(1)q}}-\frac{\delta \underset{(1)}{C}{}^m_{rq}}{\delta y^{(1)p}}+\underset{(1)}{C}{}^t_{rp}\underset{(1)}{C}{}^m{}_{tq}-\underset{(1)}{C}{}^t_{rq}\underset{(1)}{C}{}^m_{tp}+\underset{(1)}{\overset{(1)}{R}}{}^t{}_{pq}\underset{(1)}{C}{}^m_{rt}, \tag{2.12}$$

$$\underset{(2)}{S}{}_r{}^m{}_{pq}=\frac{\partial \underset{(2)}{C}{}^m_{rp}}{\partial y^{(2)q}}-\frac{\partial \underset{(2)}{C}{}^m_{rq}}{\partial y^{(2)p}}+\underset{(2)}{C}{}^t_{rp}\underset{(2)}{C}{}^m{}_{tq}-\underset{(2)}{C}{}^t_{rq}\underset{(2)}{C}{}^m_{tp}. \tag{2.13}$$

Theorem 2.4. *The curvature tensor field R of an N-linear connection D becomes zero if and only if*

$$R_r{}^m{}_{pq}=\underset{(\alpha)}{P}{}_r{}^m{}_{pq}=\underset{(12)}{P}{}_r{}^m{}_{pq}=\underset{(\alpha)}{S}{}_r{}^m{}_{pq}=0,\quad \alpha=1,2. \tag{2.14}$$

3 The equations of structure

Let (C,c), $c:I\to Osc^2M$, $C=Im\,c$ be a smooth parametrized curve on Osc^2M and let $\dot c$ be the tangent vector field

$$\dot c=\dot c^H+\dot c^{v_1}+\dot c^{v_2} \tag{3.1}$$

We consider the vector field $dc = \dot{c}dt$ on the curve c.

According to (3.1) we have

$$dc = (dc)^H + (dc)^{v_1} + (dc)^{v_2} \tag{3.2}$$

and by the adapted bases, we get

$$dc = dx^i \frac{\delta}{\delta x^i} + \delta y^{(1)i} \frac{\delta}{\delta y^{(1)i}} + \delta y^{(2)i} \frac{\partial}{\partial y^{(2)i}}. \tag{3.2'}$$

Let D be an N-linear connection and $Y \in \mathcal{X}(\mathcal{O}\int\rfloor^{\in}\mathcal{M})$. We denote $D_{dc}Y$ with DY. DY is *the covariant differential* of the vector field Y on the curve c.

We put

$$D_{(dc)^H}Y = D^H Y, \quad D_{(dc)^{v_1}} = D^{v_1}Y, \quad D_{(dc)^{v_2}} = D^{v_2}Y$$

and we can write the covariant differential of Y in the form

$$DY = D^H Y + D^{v_1}Y + D^{v_2}T. \tag{3.3}$$

Now, we take $Y = Y^H = Y^i \frac{\delta}{\delta x^i}$ and we obtain

$$DY = (Y^i{}_{|m} dx^m + Y^i \overset{(1)}{|}_m \delta y^{(1)m} + Y^i \overset{(2)}{|}_m \delta y^{(2)m}) \frac{\delta}{\delta x^i}. \tag{3.4}$$

The equality (3.4) is changes correspondingly if $Y = Y^{v_1}$ or $Y = Y^{v_2}$.

If we consider the h- and v_α-covariant derivatives of Y^i, we have

$$DY = (dY^i + Y^j \omega^i{}_j) \frac{\delta}{\delta x^i} \tag{3.5}$$

where dY^i is the usual differential of the functions $Y^i(x, y^{(1)}, y^{(2)})$ in the adapted bases

$$dY^i = \frac{\delta Y^i}{\delta x^m} dx^m + \frac{\delta Y^i}{\delta y^{(1)m}} \delta y^{(1)m} + \frac{\partial Y^i}{\partial y^{(2)m}} \delta y^{(2)m}, \tag{3.6}$$

and $\omega^i{}_j$ are the notations of the following covector fields:

$$\omega^i{}_j = L^i_{jm} dx^m + \underset{(1)}{C}{}^i_{jm} \delta y^{(1)m} + \underset{(2)}{C}{}^i_{jm} \delta y^{(2)m} \tag{3.7}$$

The covector fields $\omega^i{}_j$ are not dependent on the choice of the vector field $Y = Y^H$ or $Y = Y^{v_\alpha}$ $(\alpha = 1, 2)$. They are determined by the N-linear connection, only.

We shall call $\omega^i{}_j$, the *connection forms* of $D\Gamma(N)$.

To deduce the equations of structure of an N-linear connection $D\Gamma(N) = (L^i_{jm}, \underset{(1)}{C}{}^i_{jm}, \underset{(2)}{C}{}^i_{jm})$ we consider the exterior differentials of the 1-form fields $dx^i, \delta y^{(1)i}, \delta y^{(2)i}$ and of $\omega^i{}_j$ in the adapted bases $dx^i, \delta y^{(1)i}, \delta y^{(2)i}$.

Firstly, we obtain:

Theorem 3.1. *The exterior differentials of* $\delta y^{(1)i}, \delta Y^{(2)i}$ *are given by*

$$(3.8)\qquad d(\delta y^{(1)m}) = -\frac{1}{2}\overset{(1)}{\underset{(0)}{R}}{}^m{}_{ij}dx^i\wedge dx^j - (\overset{(1)}{\underset{(1)}{P}}{}^m{}_{ij} + L^m_{ij})dx^i\wedge\delta y^{(1)j} -$$

$$-\overset{(1)}{\underset{(2)}{P}}{}^m{}_{ij}dx^i\wedge\delta y^{(2)j},$$

$$(3.9)\qquad d(\delta y^{(2)m}) = -\frac{1}{2}\overset{(2)}{\underset{(0)}{R}}{}^m{}_{ij}dx^i\wedge dx^j - \overset{(2)}{\underset{(1)}{P}}{}^m{}_{ij}dx^i\wedge\delta y^{(1)j} -$$

$$(\overset{(2)}{\underset{(2)}{P}}{}^m{}_{ij}) + L^m_{ji})dx^i\wedge\delta y^{(2)j} - -\frac{1}{2}\overset{(1)}{\underset{(1)}{R}}{}^m{}_{ij}\delta y^{(1)i}\wedge\delta y^{(1)j} - \overset{(1)}{\underset{(2)}{P}}{}^m{}_{ij}\delta y^{(1)i}\wedge\delta y^{(2)j}.$$

Using (3.8), (3.9) and (3.7) we have

Theorem 3.2. *The equations of structure of an N-linear connection $D\Gamma(N)$ are given by*

$$(3.10)\qquad \begin{cases} d(dx^m) - dx^i\wedge\omega^m{}_j = -\Omega^{(0)m},\\ d(\delta y^{(1)m}) - \delta y^{(1)j}\wedge\omega^m{}_j = -\Omega^{(1)m},\\ d(\delta y^{(2)m}) - \delta y^{(2)j}\wedge\omega^m{}_j = -\Omega^{(2)m},\end{cases}$$

and by

$$(3.11)\qquad d\omega^m{}_r - \omega^s{}_r\wedge\omega^m{}_s = -\Omega^m{}_r,$$

where the 2-forms of torsion $\overset{(0)}{\Omega}{}^m, \overset{(1)}{\Omega}{}^m, \overset{(2)}{\Omega}{}^m$ are given by

$$(3.12)\qquad \overset{(0)}{\Omega}{}^m = \frac{1}{2}T^m{}_{ij}dx^i\wedge dx^j + \underset{(1)}{C}{}^m_{ij}dx^i\wedge\delta y^{(1)j} + \underset{(2)}{C}{}^m_{ij}dx^i\wedge\delta y^{(2)j},$$

$$(3.13)\qquad \overset{(1)}{\Omega}{}^m = \frac{1}{2}\overset{(1)}{\underset{(0)}{R}}{}^m{}_{ij}dx^i\wedge dx^j + \overset{(1)}{\underset{(1)}{P}}{}^m{}_{ij}dx^i\wedge\delta y^{(1)j} +$$

$$+\overset{(1)}{\underset{(2)}{P}}{}^m{}_{ij}dx^i\wedge\delta y^{(2)j} + \frac{1}{2}S_{(1)}{}^m{}_{ij}\delta y^{(1)i}\wedge\delta y^{(1)j} + \underset{(2)}{C}{}^m_{ij}\delta y^{(1)i}\wedge\delta y^{(2)j},$$

$$(3.14)\qquad \overset{(2)}{\Omega}{}^m = \frac{1}{2}\overset{(2)}{\underset{(0)}{R}}{}^m{}_{ij}dx^i\wedge dx^j + \overset{(2)}{\underset{(1)}{P}}{}^m{}_{ij}dx^i\wedge\delta y^{(1)j} +$$

$$+\overset{(2)}{\underset{(2)}{P}}{}^m{}_{ij}dx^i\wedge\delta y^{(2)j} + \tfrac{1}{2}\overset{(1)}{\underset{(1)}{R}}{}^m{}_{ij}\delta y^{(1)i}\wedge\delta y^{(1)j} +$$
$$+\overset{(2)}{\underset{(12)}{P}}{}^m{}_{ij}\delta y^{(1)i}\wedge\delta y^{(2)j} + \tfrac{1}{2}S_{(2)}{}^m{}_{ij}\delta y^{(2)i}\wedge\delta y^{(2)j},$$

and the 2-form of curvature $\Omega^m{}_r$ is given by

$$(3.15)\qquad \Omega^m{}_r = \frac{1}{2}R_r{}^m{}_{pq}dx^p\wedge dx^q + P_{(1)r}{}^m{}_{pq}dx^p\wedge\delta y^{(1)q} +$$

$$+P_{(2)r}{}^m{}_{pq}dx^p\wedge\delta y^{(2)q} + \tfrac{1}{2}S_{(1)r}{}^m{}_{pq}\delta y^{(1)p}\wedge\delta y^{(1)q} +$$
$$+P_{(12)r}{}^m{}_{pq}\delta y^{(1)p}\wedge\delta y^{(2)q} + \tfrac{1}{2}S_{(2)r}{}^m{}_{pq}\delta y^{(2)p}\wedge\delta y^{(2)q}.$$

The equations of structure of $D\Gamma(N)$ allow us to get some remarkable geometrical interpretations for the torsion and curvature d-tensor fields of the N-linear connection $D\Gamma(N)$.

References

[1] C.T.J. Dodson and M.S.Radivoiovici, *Tangent and frame bundles of order two*, Anal. Şt. Univ. "Al.I.Cuza" Iaşi, 28(1982), 65-71.

[2] R.Miron and Gh.Atanasiu, *Lagrange Geometry of Second Order*, Math. Comput. Modelling, Pergamon, 20, 4/5 (1994), 41-56.

[3] R.Miron and Gh.Atanasiu, *Differential geometry of the k-osculator bundle*, Rev.Roumaine Math.Pures Appl., 41, 3/4 (1996), 205-236.

[4] R.Miron and Gh.Atanasiu, *Prolongation of Riemannian, Finslerian and Lagrangian structures*, Rev.Roumaine Math.Pures.Appl., 41, 3/4(1996), 237-249.

[5] R.Miron and Gh.Atanasiu, *Higher-order Lagrange spaces*, Rev.Roumaine Math.Pures Appl., 41, 3/4(1996), 251-262.

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